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ON THE L_p^γ-BOUNDEDNESS OF THE ANISOTROPIC FOURIER-BESSEL SINGULAR INTEGRALS

Abstract

In this work the anisotropic Fourier-Bessel singular integrals are introduced and the boundedness of these singular integrals in $L_p^\gamma(R_+^n)$ space are proved.

The anisotropic Fourier-Bessel singular integral (B_n anisotropic singular integral) is introduced and L_p^γ boundedness is proved.

Note that in the isotropic case Fourier-Bessel's singular integral is introduced in [1], and its L_p^γ boundedness is also proved. The weighted L_p^γ boundedness of the isotropic Fourier-Bessels singular integral is proved in [2].

Let R^n be the n-dimensional Euclidean space of points

$$x=(x_1,\dots,x_n),\ |x|=\left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}},\ R_+^n=\{x\in R^n;\ x=(x_1,\dots,x_n),x_n>0\},\ |x|_a=\max_{1\le i\le n}|x_i|^{1/a_i},$$

where $a=(a_1,a_2,\dots,a_n)$, $a_i\ge 1$, $i=1,2,\dots,n$, $S_+=\{x\in R_+^n:|x|_a=1\}$. For $\gamma>0$ and $1\le p\le\infty$ denote a space of all measurable functions f, with the finite norm

$$\|f\|_{L_p^\gamma(R_+^n)}=\|f\|_{p,\gamma}=\left[\int_{R_+^n}|f(x)|^p x_n^\gamma dx\right]^{\frac{1}{p}},\quad 1\le p<\infty,$$

$$\|f\|_{L_\infty^\gamma(R_+^n)}=vrai\sup_{x\in R_+^n}|f(x)|,\quad p=\infty.$$

Such functional spaces are adjusted to work with a general shift of the form (see [3])

$$T^y f(x)=C_\gamma\int_0^\pi f\left(x'-y',\sqrt{x_n^2-2x_ny_n\cos\alpha+y_n^2}\right)\sin^{\gamma-1}\alpha d\alpha, \tag{1}$$

where $x'=(x_1,\dots,x_{n-1})$, $y'=(y_1,\dots,y_{n-1})$, $C_\gamma=\Gamma\left(\frac{\gamma+1}{2}\right)\Big/\left(\Gamma\left(\frac{\gamma}{2}\right)\Gamma\left(\frac{1}{2}\right)\right)$.

Note that this shift is closely connected with Bessel's differential operator $B_n=\frac{\partial^2}{\partial x_n^2}+\frac{\gamma}{x_n}\frac{\partial}{\partial x_n}$ and therefore we call it B_n-shift.

On the basis of the shift (1) a generalized convolution (of B_n-convolution) of two functions

$$(f*g)_\gamma(x)=\int_{R_+^n}T^y f(x)g(y)y_n^\gamma dy \tag{2}$$

is introduced.

Using the property of B_n-shift, it is easy to show that $(f*g)_\gamma=(g*f)_\gamma$.

By $L_0^\gamma\left(R_+^n\right)$ we denote a space of functions $f \in L_1^\gamma\left(R_+^n\right)$ with a compact support in R_+^n.

Fourier-Bessel's transformation is determined by the formula

$$F_B\varphi(x) \equiv \hat{\varphi}(x) = \int_{R_+^n} e^{i\pi(x',y')} j_{\frac{\gamma-1}{2}}(\pi x_n y_n)\varphi(y) y_n^\gamma dy,$$

where $j_{\frac{\gamma-1}{2}}(x)$ is a Bessel's function

$$j_{\frac{\gamma-1}{2}}(x) = \left(\Gamma((\gamma+1)/2)/\sqrt{\pi}\,\Gamma(\gamma/2)\right)\int_{-1}^{1} e^{ixt}\left(1-t^2\right)^{\frac{\gamma}{2}-1} dt.$$

Note that for the Fourier-Bessel's transformation it is valid the Parseval equality (see. [4])

$$\left\|\hat{f}\right\|_{L_2^\gamma(R_+^n)} = 2^{(\gamma-1)/2}\Gamma((\gamma+1)/2)\|f\|_{L_2^\gamma(R_+^n)}.$$

It is valid

Theorem 1. *Let* $1<p<\infty$ *and the kernel* $K \in L_{1,loc}^\gamma\left(R_+^n\right)$ *satisfy the following conditions*

$$\left|\int_{\{x\in R_+^n:\varepsilon<|x|_a<r\}} K(x) x_n^\gamma dx\right| \le C,\ \ 0<\varepsilon<r<\infty, \tag{3}$$

$$\int_{\{x\in R_+^n: r<|x|_a<4r\}} |K(x)| x_n^\gamma dx \le C,\ \ 0<r<\infty, \tag{4}$$

$$\int_{\{x\in R_+^n:|x|_a\ge 4|y|_a\}} \left|T^y K(x) - K(x)\right| x_n^\gamma dx \le C, \quad y \in R_+^n. \tag{5}$$

Let also $f \in L_p^\gamma\left(R_+^n\right)$ *and for* $\varepsilon>0$

$$A_\varepsilon f(x) = \int_{\{y:|y|_a>\varepsilon\}} T^y f(x) K(y) y_n^\gamma dy.$$

Then, the operator $A_\varepsilon f = \left(K_\varepsilon * f\right)_\gamma$ *acts boundedly from* $L_p^\gamma\left(R_+^n\right)$ *in* $L_p^\gamma\left(R_+^n\right)$ *and it is valid the inequality*

$$\left\|A_\varepsilon f\right\|_{L_p^\gamma(R_+^n)} \le C_{p,\gamma}\|f\|_{L_p^\gamma(R_+^n)}. \tag{6}$$

In the sense of convergence in $L_p^\gamma\left(R_+^n\right)$ *there exists*

$$Af(x) = \lim_{\varepsilon\to 0+} A_\varepsilon f(x)$$

and the operator Af *defined in such a way is bounded in* $L_p^\gamma\left(R_+^n\right)$.

First prove the theorem in case when $p=2$. We need some auxiliary facts.

Consider the following function

$$h(x) = K_{\varepsilon,\gamma}(x) = \begin{cases} K(x); & \varepsilon \le |x|_a \le r, \\ 0; & \varepsilon > |x|_a,\ |x|_a > r. \end{cases}$$

By making substitution $x_i = \eta^{a_i}\xi_i$, we get

$$J_1=\int\limits_{0\le|x|_\alpha\le b}|x|_\alpha h(x)x_n^\gamma dx=\int\limits_0^b\int\limits_{S+}\eta^{-|\alpha|-\alpha_n\gamma_\eta\gamma^{\alpha_n}}\eta h(\xi)\xi_n^\gamma d\sigma(\xi)d\eta=$$

$$=\int\limits_0^b d\eta\int\limits_{S+}h(\xi)\xi_n^\gamma d\sigma(\xi),$$

and hence

$$|J_1|\le Cb. \tag{7}$$

Analogously for

$$J_2=\int\limits_{b\le|x|_\alpha\le 4b}K(x)x_n^\gamma dx=\int\limits_b^{4b}\eta^{-1}d\eta\int\limits_{S+}K(\xi)\xi_n^\gamma d\sigma(\xi),$$

we get the estimate

$$|J_2|\le C. \tag{8}$$

Lemma 1. *Let* $f\in L_2^\gamma\left(R_+^n\right)$, *and the kernel* $K\in L_{1,loc}^\gamma\left(R_+^n\right)$ *satisfy the conditions (3), (4), (5). Then*

$$\left|\hat{h}(x)\right|\le C,\quad for\quad x\in S_+, \tag{9}$$

where C *independed on function* h.

Proof. For $x\in S_+$ by virtue of the properties of the generalized shift we have

$$\int\limits_{R_+^n}e^{i\pi(x',y')}j_{\frac{\gamma-1}{2}}(\pi x_n y_n)T^x h(y)y_n^\gamma dy=$$

$$=\int\limits_{R_+^n}T^x\left[e^{i\pi(x',y')}j\frac{\gamma-1}{2}(\pi x_n y_n)\right]h(y)y_n^\gamma dy=$$

$$=\int\limits_{R_+^n}e^{i\pi(x',y'-x')}T^{x_n}j_{\frac{\gamma-1}{2}}(\pi x_n y_n)h(y)y_n^\gamma dy$$

and since

$$e^{i\pi(x',y'-x')}T^{x_n}j_{\frac{\gamma-1}{2}}(\pi x_n y_n)=e^{i\pi(x',y')}e^{-i\pi|x'|^2}j_{\frac{\gamma-1}{2}}(\pi x_n y_n)j_{\frac{\gamma-1}{2}}\left(\pi x_n^2\right)=$$

$$=-e^{i\pi|x_n|^2}j_{\frac{\gamma-1}{2}}\left(\pi x_n^2\right)e^{i\pi(x',y')}j_{\frac{\gamma-1}{2}}(\pi x_n y_n),$$

Then

$$\int\limits_{R_+^n}e^{i\pi(x',y')}j_{\frac{\gamma-1}{2}}(\pi x_n y_n)T^x h(y)y_n^\gamma dy=$$

$$=-e^{i\pi x_n^2}j_{\frac{\gamma-1}{2}}\left(\pi x_n^2\right)\int\limits_{R_+^n}e^{i\pi(x',y')}j_{\frac{\gamma-1}{2}}(\pi x_n y_n)h(y)y_n^\gamma dy=$$

$$=-e^{i\pi x_n^2}j_{\frac{\gamma-1}{2}}\left(\pi x_n^2\right)\hat{h}(x).$$

Consequently,

$$\left[1+e^{i\pi x_n^2}j_{\frac{\gamma-1}{2}}\left(\pi x_n^2\right)\right]\hat{h}(x)=\int\limits_{R_+^n}e^{i\pi(x',y')}j_{\frac{\gamma-1}{2}}(\pi x_n y_n)\left[h(y)-T^x h(y)\right]y_n^\gamma dy.$$

Further

$$\int_{R_+^n} e^{i\pi(x',y')} j_{\frac{\gamma-1}{2}}(\pi x_n y_n)\left[h(y)-T^x h(y)\right]y_n^\gamma dy =$$

$$= \int_{|y|_a\ge 4} e^{i\pi(x',y')} j_{\frac{\gamma-1}{2}}(\pi x_n y_n)\left[h(y)-T^x h(y)\right]y_n^\gamma dy +$$

$$+ \int_{|y|_a<4} e^{i\pi(x',y')} j_{\frac{\gamma-1}{2}}(\pi x_n y_n)\left[h(y)-T^x h(y)\right]y_n^\gamma dy = I_1 + I_2.$$

Estimate I_1 and I_2. First calculate I_2.

$$I_2 = \int_{|y|_a<4}\left[e^{i\pi(x',y')} j_{\frac{\gamma-1}{2}}(\pi x_n y_n)-1\right]h(y)y_n^\gamma dy -$$

$$- \int_{|y|_a<4}\left[e^{i\pi(x',y')} j_{\frac{\gamma-1}{2}}(\pi x_n y_n)+1\right]T^x h(y)y_n^\gamma dy +$$

$$+ \int_{|y|_a<4} h(y)y_n^\gamma dy + \int_{|y|_a<4} T^x h(y)y_n^\gamma dy = L_1 - L_2 + L_3 + L_4.$$

Now estimate L_1. Note that

$$\left|e^{i\pi(x',y')} j_{\frac{\gamma-1}{2}}(\pi x_n y_n)-1\right| \le$$

$$\le \left|j_{\frac{\gamma-1}{2}}(\pi x_n y_n)\right|\left|e^{i\pi(x',y')}-1\right| + \left|j_{\frac{\gamma-1}{2}}(\pi x_n y_n)-1\right| \le$$

$$\le \left|e^{i\pi(x',y')}-1\right| + C_\gamma \int_{-1}^{1}\left|e^{i\pi x_n y_n t}-1\right|\left(1-t^2\right)^{\frac{\gamma-2}{2}} dt,$$

where $C_\gamma = \left(\Gamma((\gamma+1)/2)/\sqrt{\pi}\,\Gamma(\gamma/2)\right) = \int_{-1}^{1}\left(1-t^2\right)^{\frac{\gamma-2}{2}} dt$.

Since $|x|_a = 1$ and $|t| \le 1$

$$\left|e^{i\pi(x',y')}-1\right| \le B|y'| \text{ and } \left|e^{i\pi x_n y_n t}-1\right| \le B|y_n|,$$

then

$$\left|e^{i\pi(x',y')} j_{\frac{\gamma-1}{2}}(\pi x_n y_n)-1\right| \le B|y'| + B|y_n| = B|y| \le B|y|_a.$$

Therefore

$$|L_1| \le C\cdot B.$$

To estimate L_3 use the inequality (7),

$$|L_3| \le \int_{|x|_a<4} |K(y)| y_n^\gamma dy \le C.$$

Further

$$L_2 = \int_{|y|_a}\left[e^{i\pi(x',y')} j_{\frac{\gamma-1}{2}}(\pi x_n y_n)+1\right]T^x h(y)y_n^\gamma dy =$$

$$= \int_{R_+^n} T^x \left[\chi_{\{|\cdot|_a<4\}}(y) \left(e^{i\pi(x',y')} j_{\frac{\gamma-1}{2}}(\pi x_n y_n) + 1 \right) \right] h(y) y_n^\gamma dy.$$

Note that

$$\left| T^x \chi_{\{|\cdot|_a<4\}}(y) \right| \le$$

$$\le C_\gamma \int_0^\pi \left| \chi_{\{|\cdot|_a<4\}} \left(y'-x', \sqrt{x_n^2 - 2x_n y_n \cos\alpha + y_n^2} \right) \right| \sin^{\gamma-1} \alpha d\alpha \le 1.$$

If $\left| y'-x', \sqrt{x_n^2 - 2x_n y_n \cos\alpha + y_n^2} \right|_a \ge 4$ for $\forall \alpha \in (0,\pi)$, then $T^x \chi_{\{|\cdot|_a<4\}}(y) = 0$.

Denote $D_x = \left\{ y \in R_+^n : \exists \alpha \in (0,\pi), \left| y'-x', \sqrt{x_n^2 - 2x_n y_n \cos\alpha + y_n^2} \right|_a < 4 \right\}$.

Estimate L_2.

$$|L_2| \le \int_{D_x} \left| T^x \left[e^{i\pi(x',y')} j_{\frac{\gamma-1}{2}}(\pi x_n y_n) + 1 \right] \right| |h(y)| y_n^\gamma dy \le$$

$$\le \int_{D_x} \left| e^{i\pi(x',y'-x')} T^{x_n} j_{\frac{\gamma-1}{2}}(\pi x_n y_n) + 1 \right| |h(y)| y_n^\gamma dy \le$$

$$\le \int_{D_x} \left| e^{i\pi x_n^2} j_{\frac{\gamma-1}{2}}(\pi x_n^2) e^{i\pi(x',y')} j_{\frac{\gamma-1}{2}}(\pi x_n y_n) - 1 \right| |h(y)| y_n^\gamma dy \le$$

$$\le \int_{D_x} \left| e^{i\pi x_n^2} j_{\frac{\gamma-1}{2}}(\pi x_n^2) \right| \left| e^{i\pi(x',y')} j_{\frac{\gamma-1}{2}}(\pi x_n y_n) - 1 \right| |h(y)| y_n^\gamma dy +$$

$$+ \int_{D_x} \left| e^{i\pi x_n^2} j_{\frac{\gamma-1}{2}}(\pi x_n^2) - 1 \right| |h(y)| y_n^\gamma dy = |L_5| + |L_6|.$$

On the set D_x we obviously have:

$$|y|_a \le |y-x|_a + |x|_a \le \left| y'-x', \sqrt{x_n^2 - 2x_n y_n \cos\alpha + y_n^2} \right|_a + |x|_a < 4+1=5.$$

Therefore

$$|L_5| = \int_{D_x} \left| e^{i\pi x_n^2} j_{\frac{\gamma-1}{2}}(\pi x_n^2) \right| \left| e^{i\pi(x',y')} j_{\frac{\gamma-1}{2}}(\pi x_n y_n) - 1 \right| |h(y)| y_n^\gamma dy \le$$

$$\le C \int_{D_x} |y|_a |h(y)| y_n^\gamma dy \le C \int_{|y|_a<5} |y|_a |h(y)| y_n^\gamma dy \le C_1,$$

and also

$$|L_6| = \int_{D_x} \left| e^{i\pi x_n^2} j_{\frac{\gamma-1}{2}}(\pi x_n^2) - 1 \right| |h(y)| y_n^\gamma dy \le$$

$$\le C_2 \int_{D_x} |h(y)| y_n^\gamma dy \le C_2 \int_{|y|_a<5} |h(y)| y_n^\gamma dy \le C_3,$$

It is analogously shown that $|I_1| \le C_4$..

Considering that $\beta := \left|1 + e^{i\pi x_n^2} j_{\frac{\gamma-1}{2}}\left(\pi x_n^2\right)\right| > 0$, then we get

$$\left|\hat{h}(x)\right| \le C_5 \beta^{-1}.$$

Lemma 1 is proved.

Since for the Fourier-Bessel transformation it is fulfilled the relation $\widehat{Kf} = \hat{K}\hat{f}$, then by means of Parsevall equality we conclude that

$$\left\|(K * f)_\gamma\right\|_{L_2^\gamma(R_+^n)} \le C \sup_{x \in S_+} \left|\hat{K}(x)\right| \cdot \|f\|_{L_2^\gamma(R_+^n)}, \quad f \in L_0^\gamma.$$

Later on we shall use the weighted variance of the covering lemma (the weight analogy of the Calderon-Zygmund decomposition).

Lemma 2. *Let $f \in L_1^\gamma\left(R_+^n\right)$ and t be some positive number. Then there exists the expression $R_+^n = F^+ \cup \Omega^+$, $F^+ \cap \Omega^+ = \varnothing$ such that*

1) $|f(x)| < t$ almost everywhere on F^+

2) Ω^+ is the join of non-intersecting parallelepipeds $\Omega^+ = \bigcup_k \Omega_k^+$, and the function

$$v(x) = \begin{cases} \dfrac{1}{\left|\Omega_k^+\right|_\gamma} \displaystyle\int_{\Omega_k^+} f(x) x_n^\gamma dx, & x \in \Omega_k^+, \\ f(x), & x \in F^+ \end{cases}$$

satisfying the inequality

$$t \le v(x) \le 2^{|\alpha| + \alpha_n \gamma} t, \quad x \in \Omega_k^+;$$

3) $f(x) = v(x) + \sum_K \omega_k(x)$, where $\omega_k \in L_1^\gamma\left(R_+^n\right)$, $\int_{R_+^n} \omega_k(x) x_n^\gamma dx = 0, \omega_k(x) \ne 0$ for $x \notin \Omega_k^+$;

4) $\|v\|_{L_1^\gamma(R_+^n)} + \sum_k \|\omega_k\|_{L_1^\gamma(R_+^n)} \le C\|f\|_{L_1^\gamma(R_+^n)}$;

5) $\left|\Omega^+\right|_\gamma \le t^{-1} \|f\|_{L_{L_1^\gamma(R_+^n)}}$.

It is valid the following

Theorem 2. *Let $K \in L_{1,loc}^\gamma\left(R_+^n\right)$ and there exist numbers $B > 0$, $C > 0$ that*

$$\int_{x \in R_+^n; |x|_a > B} \left|T^y K(x) - K(x)\right| x_n^\gamma dx \le C, \quad |y|_a < \frac{1}{B}. \tag{10}$$

Let $Af(x) = (f * K)_\gamma$, $f \in L_0^\gamma\left(R_+^n\right)$.

Let also

$$\|Af\|_{L_2^\gamma(R_+^n)} \le C \|f\|_{L_2^\gamma}\left(R_+^n\right). \tag{11}$$

Then for some constant C_1

$$\left|\left\{x \in R_+^n : |Af(x)| > s\right\}\right|_\gamma \le \frac{C_1}{s} \int_{R_+^n} |f(x)| x_n^\gamma dx, \tag{12}$$

where $|E|_\gamma = \int_E x_n^\gamma dx$ *for* $E \subset R_+^n$.

Proof. We shall use the covering lemma by which for any number $s > 0$ and $f \in L_0^\gamma(R_+^n)$ we can write

$$f(x) = v(x) + \omega(x) = v(x) + \sum_k \omega_k(x),$$

$$|v| \le Cs, \ \|v\|_{L_1^\gamma(R_+^n)} \le C\|f\|_{L_1^\gamma(R_+^n)},$$

and in addition

$$\left|\left\{x \in R_+^n : |Af(x)| > t\right\}\right|_\gamma \le$$

$$\le \left|\left\{x \in R_+^n : |Av(x)| > t/2\right\}\right|_\gamma + \left|\left\{x \in R_+^n : |A\omega(x)| > t/2\right\}\right|_\gamma$$

Estimate Av. We note, that

$$\|v\|_{L_2^\gamma(R_+^n)} \le C_1 s^{\frac{1}{2}} \|v\|_{L_1^\gamma(R_+^n)}^{1/2} \le C_1 s^{\frac{1}{2}} \|f\|_{L_1^\gamma(R_+^n)}^{1/2}.$$

Hence and from the condition that the operator A is the $(2,2)_\gamma$ type operator see (11), we have:

$$\|Av\|_{L_2^\gamma(R_+^n)} \le C_1 \|v\|_{L_2^\gamma(R_+^n)} \le C_1 s^{\frac{1}{2}} \|f\|_{L_1^\gamma(R_+^n)}^{1/2}.$$

From the $(2,2)_\gamma$ type condition for the operator A, the weak type condition $(1,1)_\gamma$ follows, i.e. we have the inequality:

$$\left|\left\{x \in R_+^n : |Av(x)| > t/2\right\}\right|_\gamma = \frac{4}{t^2} \int_{\{x \in R_+^n : |Av(x)| > t/2\}} \left(\frac{t}{2}\right)^2 x_n^\gamma dx \le$$

$$\le \frac{4}{t^2} \int_{R_+^n} |Av(x)|^2 x_n^\gamma dx \le \frac{4}{t^2} \int_{R_+^n} |v(x)|^2 x_n^\gamma dx = \frac{4}{t^2} \|v\|_{L_2^\gamma(R_+^n)} \le C \frac{s}{t^2} \|f\|_{L_1^\gamma(R_+^n)}.$$

Thus,

$$\left|\left\{x \in R_+^n : |Av(x)| > t/2\right\}\right|_\gamma \le C \frac{s}{t^2} \|f\|_{L_1^\gamma(R_+^n)}. \qquad (13)$$

Estimate $A\omega$.

First consider the function $\omega_1(x)$, concentrated in the parallelepiped Ω_1 with a center in the origin of coordinates:

$$\Omega_1 = \left\{x, \ |x|_a = \max_{i=1,n} |x_i|^{1/a_i} < 1/B\right\}$$

with an integer equal to zero (extending the arguments to the whole space R^n, we used parity of functions in the variable x_n). The part of the parallelepiped Ω_1 belonging to R_+^n, denote by Ω_1^+. Let the parallelepiped $Q_1 = \{x, |x|_a < B\}$ be constructed from the parallelepiped Ω_1 by expanding from the origin of coordinates B^2 times, and Ω_1^+ be its corresponding part, belonging to R_+^n. We have

$$A\omega_1(x) = \int_{\Omega_1^+} T^y K(x) \omega_1(y) y_n^\gamma dy = \int_{\Omega_1^+} \left[T^y K(x) - K(x)\right] \omega_1(y) y_n^\gamma dy.$$

Applying Minkowskii inequality, and then the condition (10), we get

$$\left[\int_{CQ_1^+}|A\omega_1(x)|^2 x_n^\gamma dx\right]^{1/2}=\left\{\int_{CQ_1^+}\left|\int_{\Omega_1^+}\left[T^y K(x)-K(x)\right]\omega_1(y)y_n^\gamma dy\right|^2 x_n^\gamma dx\right\}^{1/2}\le$$

$$\le\int_{\Omega_1^+}|\omega_1(y)|\left[\int_{CQ_1^+}\left|T^y k(x)-K(x)\right|^2 x_n^\gamma dx\right]^{1/2} y_n^\gamma dy\le$$

$$\le C_1\|\omega_1\|_{L_1^\gamma(R_+^n)}\le C_1\|f\|_{L_1^\gamma(R_+^n)}.$$

Since for each function $\omega(x)\in L_2^\gamma\left(R_+^n\right)$

$$\left|\left\{x\in R_+^n:\ |\omega(x)|>t\right\}\right|_\gamma\le\frac{C_1}{t^2}\|\omega\|^2_{L_2^\gamma(R_+^n)},\tag{14}$$

then denoting by $\chi_{Q_1^+}(x)$ the characteristic function of the set Q_1^+, we have

$$\left|\left\{x\in R_+^n:\left|\left(1-\chi_{Q_1^+}(x)\right)A\omega(x)\right|>t\right\}\right|_\gamma\le$$

$$\le\frac{C_1}{t^2}\left\|\left(1-\chi_{Q_1^+}(x)\right)A\omega(x)\right\|^2_{L_2^\gamma(R_+^n)}\le\frac{C_1}{t^2}\|f\|_{L_1^\gamma(R_+^n)}.$$

On the other hand,

$$\left|\left\{x\in R_+^n:\left|A\omega_1(x)\right|>t\right\}\right|_\gamma\le$$

$$\le\left|\left\{x\in R_+^n:\left|\left(1-\chi_{Q_1^+}(x)\right)A\omega_1(x)\right|>t\right\}\right|_\gamma+\left|Q_1^+\right|_\gamma\le$$

$$\le\frac{C_1}{t^2}\|f\|_{L_1^\gamma(R_+^n)}+\left|Q_1^+\right|_\gamma.$$

The weight degrees of the parallelepipeds Q_1^+ and Ω_1^+ are connected by the inequality

$$\left|Q_1^+\right|_\gamma\le C\left|\Omega_1^+\right|_\gamma,$$

therefore

$$\left|\left\{x\in R_+^n:\ \left|A\omega_1(x)\right|>t\right\}\right|_\gamma\le C_1\left(t^{-1}\|f\|_{L_1^\gamma(R_+^n)}+s^{-1}|f|_{L_1^\gamma(R_+^n)}\right).\tag{15}$$

Now let the support of the function ω_k be concentrated in the parallelepiped $\Omega_k=\left\{x,\left[x-x^{(k)}\right]_a=\max_{i=1,n}\left|x_i-x_i^{(k)}\right|^{1/a_i}<1/B\right\}$ with a center at the point $x^{(k)}\in R_+^n$. As in the first case, we get

$$\int_{CQ_k^+}\left|\int_{\Omega_k^+}T^y K(x)\omega_k(y)y_n^\gamma dy\right|x_n^\gamma dx=$$

$$= \int_{CQ_k^+} \left| \int_{\Omega_k^+} \left[T^y K(x) - T^{x^{(k)}} K(x) \right] \omega_k(y) y_n^\gamma dy \right| x_n^\gamma dx \le$$

$$\le \int_{\Omega_k^+} |\omega_k(y)| \left| \int_{CQ_k^+} \left[T^y K(x) - T^{x^{(k)}} K(x) \right] x_n^\gamma dx \right| y_n^\gamma dy$$

where $CQ_k^+ = \left\{ x, x \in R_+^n, \min_{i=\overline{1,n}} \left| x_i - x_i^{(k)} \right|^{1/a_i} > B \right\}$.

Consider the inner integral separately. By substituting $z_n = x_n \cos\alpha$, $z_{n+1} = x_n \sin\alpha, 0 \le \alpha \le \pi$, transform this integral to the form

$$J = \int_{CB_k^+} \left| K\left(x' - y', \sqrt{(z_n - y_n)^2 + z_{n+1}^2} \right) - \right.$$

$$\left. - K\left(x' - \left(x^{(k)}\right)', \sqrt{\left(z_n - x_n^{(k)}\right)^2 + z_{n+1}^2} \right) \right| z_{n+1}^{\gamma-1} dz,$$

where $z = (x', z_n, z_{n+1}) \in R_+^{n+1} = \{z; z_{n+1} > 0\}$

$$CB_k^+ = \left\{ z \in R_+^{n+}; \sqrt{z_n^2 + z_{n+1}^2} > B, \left| x_i - x_i^{(k)} \right|^{1/a_i} > B, i = \overline{1, n-1} \right\}.$$

It is convenient to interpret the domain B_k^+ as a domain obtained by rotating the parallelepiped Q_k^+ about the angle π around the hyperaxis $z_n = 0$, $z_{n+1} = 0$ on each pair of variable z_n, z_{n+1}.

The shift $\xi_n = z_n - x_n^{(k)}, \xi' = x' - \left(x^{(k)}\right)'$ on the hyperplane $E^n = \left\{ z; z \in \overline{R_+^{n+1}}; \right.$ $\left. z_{n+1} = 0 \right\}$ (not touching upon the weight variable) reduces this integral to the view

$$J = \int_{\left(CB_k^+\right)'} \left| K\left(\xi' + \eta', \sqrt{(\xi_n + \eta_n)^2 + z_{n+1}^2} \right) - K\left(\xi', \sqrt{\xi_n^2 + z_{n+1}^2} \right) \right| z_{n+1}^{\gamma-1} d\xi,$$

where

$$\eta = (\eta', \eta_n, 0) \in \overline{R_+^{n+1}},\ \eta_n = y_n - x_n,\ \eta' = y' - \left(x^{(k)}\right)',\ \left(CB_k^+\right)' = \left\{ (\xi', \xi_n, z_{n+1}) \in R_+^{n+1}, \right.$$

$$\left. \left| \sqrt{\left(\xi_{n+1} + x_n^{(k)}\right)^2 + z_{n+1}^2} - x_n^{(k)} \right|^{1/a_n} > B,\ \left| \xi_i - x_i^{(k)} \right|^{1/a_i} > B,\ i = 1, 2, \ldots, n-1 \right\}.$$

The shift obtained in this representation of inner integral again is reduced to the generalized transformation $\xi_n = x_n \cos\alpha, z_{n+1} = x_n \sin\alpha, x_n > 0$ (passage to polar coordinates in the domain of rotation around new axes obtained by the parallel transfer of the old one in the hyperplane $x_{n+1} = 0$). From the inequality

$$B<\left|\sqrt{\left(\xi_n+x_n^{(k)}\right)^2+z_{n+1}^2}-x_n^{(k)}\right|^{\frac{1}{a_n}}=\left|\sqrt{x_n^2+\left(x_n^{(k)}\right)^2+2x_nx_n^{(k)}\cos\alpha}-x_n^{(k)}\right|^{\frac{1}{a_k}}\le$$

$$\le\left|\sqrt{\left(x_n+x_n^{(k)}\right)^2}-x_n^{(k)}\right|^{\frac{1}{a_k}}=\left|x_n\right|^{\frac{1}{a_n}}=x_n^{1/a_n}$$

it follows

$$|J|\le\int_{x\in R_+^n,|x|_a>B}\left|T^\eta K(x)-K(x)\right|x_n^\gamma dx.$$

Here $\eta=y-x^{(k)}$, and $y\in\Omega_k^+$, therefore $[\eta]_a<1/B$. Use the inequality (10) that gives

$$\int_{C\Omega_k^+}\left|A\omega_k(x)\right|x_n^\gamma dx\le C\|\omega_k\|_{L_1^\gamma(R_+^n)}.$$

Now, as at the beginning of the proof, we get (15) for all ω_k. For $\omega=\sum_k\omega_k$ we have

$$\left|\left\{x\in R_+^n:\left|A\omega(x)\right|>t\right\}\right|_\gamma\le C_1\left(t^{-1}\|f\|_{L_1^\gamma(R_+^n)}+s^{-1}\|f\|_{L_1^\gamma(R_+^n)}\right). \qquad (16)$$

Considering (13), (16) we find that

$$\left|\left\{x\in R_+^n:\left|A\omega(x)\right|>t\right\}\right|_\gamma\le C_1\left(t^{-1}\|f\|_{L_1^\gamma(R_+^n)}+s^{-1}\|f\|_{L_1^\gamma(R_+^n)}+\frac{s}{t^2}\|f\|_{L_1^\gamma(R_+^n)}\right).$$

By choosing $s>0$ to minimize the expression at the right hand side of this inequality, we get

$$\left|\left\{x\in R_+^n:\left|Af(x)\right|>t\right\}\right|_\gamma\le\frac{C_1}{t}\|f\|_{L_1^\gamma(R_+^n)},$$

and hence the proof of theorem 2 follows. Full proof of theorem 1 follows from Martsinkevitch's theorem and theorem 2.

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