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INVERSE SINGULAR PERIODIC PROBLEM OF STURM-LIOUVILLE

Abstract

Inverse spectral problems for differential operator of Sturm-Liouville on finite interval with non-integrable singularity at one of the ends of interval were investigated. The necessary and sufficient conditions and procedure of solution of inverse periodic problem were obtained.

Consider boundary value problems, generated on interval $(0;\pi)$ by equation

$$-y''(x)+q(x)y(x)=\mu\, y(x) \tag{1}$$

(μ is a complex parameter) with real potential

$$q(x)=\sum_{i=1}^{m}\frac{A_i}{x^{p_i}}+q_0(x), \tag{2}$$

where $p_i\in\left(1;\frac{5}{4}\right)$ and $A_i\ \left(i=\overline{1,m}\right)$ are real numbers, $q_0(x)\in L_2[0,\pi]$ and seperated boundary conditions

$$y(0)=0,\quad y(\pi)=0 \tag{3}$$

or,

$$y(0)=0,\quad y'(\pi)=0, \tag{4}$$

and also periodic

$$y(0)-y(\pi)=y'(0)-y'(\pi)=0$$

and antiperiodic

$$y(0)+y(\pi)=y'(0)+y'(\pi)=0$$

boundary conditions.

By virtue of singularity of potential, the derivative of one of linear independent solution diverges at the neighborhood of point $x=0$ (see [1,2]).

At paper [1] it was shown that eigenvalues (EV) of problem (1), (3) are the roots of equation $s(\pi,\lambda)=0$, (EV) of the problem (1), (4) are roots of equation $s'(\pi,\lambda)=0$; and (EV) of periodic problem are roots of system

$$\begin{cases} s(\pi,\lambda)=0 \\ s'(\pi,\lambda)=1 \end{cases},$$

(EV) of antiperiodic are roots of system

$$\begin{cases} s(\pi,\lambda)=0 \\ s'(\pi,\lambda)=-1 \end{cases},$$

where $s(x,\lambda)$ is solution of equation (1) with initial conditions $s(0,\lambda)=0$, $s'(0,\lambda)=1$, and satisfying to the following asymptotic formulas for $|\lambda|\to\infty$ by virtue of [1, 2]:

$$s(x,\lambda)=\frac{\sin\lambda x}{\lambda}+\frac{1}{\lambda^2}\int_0^x\sin\lambda(x-t)\sin\lambda t\, q(t)dt+e^{|\mathrm{Im}\,\lambda|x}O\left[\frac{R^2(\lambda)}{|\lambda|}\right], \tag{5}$$

$$s'(x,\lambda)=\cos\lambda x+\frac{1}{\lambda}\int_0^x\cos\lambda(x-t)\sin\lambda t\cdot q(t)dt+e^{|\mathrm{Im}\,\lambda|x}O\left[R^2(\lambda)\right], \tag{6}$$

where

$$R(\lambda)=\int_0^{1/|\lambda|} t|q(t)|dt+\frac{1}{|\lambda|}\int_{1/|\lambda|}^{\pi}|q(t)|dt. \tag{7}$$

Using conclusions from [1-3], asymptotic formulas (5-7), we obtain, that (EV) of considered boundary value problems satisfies to the following asymptotic formulas:

$$\sqrt{\lambda_k}=k+\frac{1}{\pi k}\int_0^{\pi}\sin^2 kt\cdot q(t)dt+\alpha_k, \tag{8}$$

$$\sqrt{\lambda_k}=k-\frac{1}{2}+\frac{1}{\pi(k-1/2)}\int_0^{\pi}\sin^2(k-1/2)t\cdot q(t)dt+\beta_k, \tag{9}$$

$$\sqrt{\mu^{\pm}}=k+\frac{1}{\pi k}\int_0^{\pi}\sin^2 kt\cdot q(t)dt+\varepsilon_k^{\pm}, \tag{10}$$

where $\alpha_k,\beta_k,\varepsilon_k^{\pm}=O\left[R^2(k)\right]$ and $\sum_{k=1}^{\infty}|\alpha_k|^2+|\beta_k|^2+|\varepsilon_k^{\pm}|^2<\infty$, λ_k are (EV) of problem (1), (3), ν_k are (EV) of problem (1), (4), $\mu_{2k}^{\pm}$ are (EV) of periodic, $\mu_{2k+1}^{\pm}$ are (EV) of antiperiodic problems.

All considered boundary value problems are self-adjoint, its (EV) are real, EV of problems (1), (3) and (1), (4) are simple, and (EV) of periodic and antiperiodic problems could be multiple. From classical oscillation theorems for solutions of equation with real potential it follows, that mutual arrangement of these (EV) is:

$$-\infty<\nu_1<\lambda_1<\nu_2<\lambda_2<\nu_3<\cdots,$$

$$-\infty<\mu_0<\mu_1^-\le\lambda_1\le\mu_1^+<\mu_2^-\le\lambda_2\le\mu_2^+<\cdots.$$

At paper [4] inverse problem was solved by two spectrums in case of separated boundary value problems (1), (3) and (1), (4) with considered potential (2) in terms of necessary and sufficient conditions, which formulated by the following way:

Theorem 1. *For two sequences of real numbers $\{\lambda_k\}$, $\{\nu_k\}$ $(k=1,2,\ldots)$ to be spectrums of boundary value problems, generated by the same Sturm-Liouville equation (1) and with real potential (2), it is necessary and sufficient for these sequences to alternate and satisfies to asymptotic formulas:*

$$\lambda_k=k^2+\frac{2}{\pi}\sum_{i=1}^{m}A_iC_{p_i}k^{p_i-1}-2A+a_k, \tag{11}$$

$$\nu_k=(k-1/2)^2+\frac{2}{\pi}\sum_{i=1}^{m}A_iC_{p_i}(k-1/2)^{p_i-1}-2A+b_k, \tag{12}$$

where $C_{p_i}=\int_0^{\pi}\frac{\sin^2\xi}{\xi^{p_i}}d\xi=\frac{2^{p_i-3}\pi}{(p_i-1)\Gamma(p_i-1)\sin\left[\frac{\pi(p_i-1)}{2}\right]}$, *$A$ are arbitrary real numbers and* $\sum_{k=1}^{\infty}|a_k|^2+|b_k|^2<\infty$.

The method of proving of this theorem is analogous to [3], we reduce it to the solution of inverse scattering problem on halfplane $0\le x<\infty$ generated by Sturm-Liouville equation (1) and boundary condition $y(0)=0$, which have

1) $q(x)=\begin{cases} q_1(x)+q_0(x), & 0\le x\le\pi \\ 0, & x>\pi\end{cases}$

where $q_1(x)$ is such that $\int_0^{\pi} x^{1/4}|q_1(x)|dx < \infty$, $q_0(x) \in L_2[0;\pi]$;

2) discrete spectrum is absent.

The total solution has shown has shown at paper [5], at paper [4] was noted, that function of scattering $S(\lambda)$ have form:

$$S(\lambda) = \frac{e(-\lambda,0)}{e(\lambda,0)}, \quad \text{where} \quad e(\lambda,0) = e^{i\lambda\pi}[s'(\lambda,\pi) - i\lambda s(\lambda,\pi)].$$

By similar way from [6] we can state

Theorem 2. *For the function $u(z)$, $\vartheta(z)$ to allow the representation*

$$u(z) = \sin\pi z + \int_0^{\pi} \frac{\sin z(\pi - t)}{z} q(t)\sin zt dt + A\pi\frac{4z\cos\pi z}{4z^2 - 1} + \frac{f(z)}{z},$$

$$\vartheta(z) = \cos\pi z + \int_0^{\pi} \frac{\sin z(\pi - t)}{z} q(t)\sin zt dt - B\pi\frac{\sin\pi z}{z} + \frac{g(z)}{z},$$

where $q(t) = \sum_{i=1}^{m} \frac{A}{t^{p_i}} + q_0(t)$, *$A$, B, A_i are real constant numbers,* $p_i \in (1;5/4)$, $q_0(t) \in$ $\in L_2[0,\pi]$, $f(z) = \int_0^{\pi} \tilde{f}(t)\cos zt dt$, $\tilde{f}(t) \in L_2[0;\pi]$, $\int_0^{\pi} \tilde{f}(t)dt = 0$, $g(z) = \int_0^{\pi} \tilde{g}(t)\sin zt dt$, $\tilde{g}(t) \in L_2[0;\pi]$, *it is necessary and sufficient that*

$$u(z) = \pi z \prod_{k=1}^{\infty} \frac{u_k^2 - z^2}{k^2}, \quad \vartheta(z) = \prod_{k=1}^{\infty} \frac{v_k^2 - z^2}{(k - 1/2)^2},$$

where

$$u_k = k + \frac{1}{\pi}\sum_{i=1}^{m} \frac{A_i C_{p_i}}{k^{2-p_i}} - \frac{1}{k}\left[A + \sum_{i=1}^{m} \frac{A_i}{2(p_i - 1)\pi^{p_i}}\right] + \frac{a_k}{k}, \tag{13}$$

$$\vartheta_k = k - \frac{1}{2} + \frac{1}{\pi}\sum_{i=1}^{m} \frac{A_i C_{p_i}}{(k - 1/2)^{2-p_i}} - \frac{1}{k - 1/2}\left[B + \sum_{i=1}^{m} \frac{A_i}{2(p_i - 1)\pi^{p_i}}\right] + \frac{b_k}{k - 1/2} \tag{14}$$

and moreover, $\sum_{k=1}^{\infty} |a_k|^2 + |b_k|^2 < \infty$.

In the statement of this theorem, functions $u(z)$ and $\vartheta(z)$ are not less, than functions $zs(z,\pi)$ and $s'(z,\pi)$.

Now we will consider inverse periodic singular problem of Sturm-Liouville on the finite interval. It can be stated by the following way: to find out necessary and sufficient conditions, to which should satisfy two sequences of real numbers to be spectral periodic and antiperiodic boundary value problems, generated by the same equation with potential of the form (2); and to find out method of construction of all such equations.

The sequence $-\infty < \mu_0 < \mu_1^- \le \mu_1^- < \mu_2^+ \le \mu_3^- < \mu_3^+ < \cdots$ consists of (EV) $\{\mu_{2k}^{\pm}\}$ of periodic and $\{\mu_{2k-1}^{\pm}\}$ of antiperiodic boundary value problems, generated by equation (1) with real potential of the form (2). Then sequence $\infty < 0 < (\mu_1^- - \mu_0) \le (\mu_1^+ - \mu_0) < (\mu_2^- - \mu_0) \le (\mu_2^+ - \mu_0) < \cdots$ consists of (EV) of periodic $\{\mu_{2k}^{\pm} - \mu_0\}$ and antiperiodic $\{\mu_{2k-1}^{\pm} - \mu_0\}$ boundary value problems, generated by equation

$$-y''(x)+[q(x)-\mu_0]y(x)=\mu y(x)$$

with real potential $q(x)-\mu_0$ from the class of potential of the form (2) as well. Therefore, without lose of generality, we can suppose $\mu_0=0$.

Substituting potential of the form (2) to the formula (10), using conclusions similar to that was done at paper [1,2], we obtain following asymptotic formulas:

$$\mu_{2k}^{\pm}=(2k)^2+\frac{2}{\pi}\sum_{i=1}^{m}A_iC_{p_i}(2k)^{p_i-1}-2A+a_{2k}^{\pm}, \tag{15}$$

$$\mu_{2k-1}^{\pm}=(2k-1)^2+\frac{2}{\pi}\sum_{i=1}^{m}A_iC_{p_i}(2k-1)^{p_i-1}-2A+a_{2k+1}^{\pm}. \tag{16}$$

If we make correspondence $\mu_{2k}^{\pm}\to\mu_k^{+}(\mu_k^{-})$ and $\mu_{2k-1}^{\pm}\to\mu_{k-1/2}^{+}(\mu_{k-1/2}^{-})$, and supposing $\mu_k^{+}\overset{def}{=}\lambda_k'$, and $\mu_{k-1/2}^{+}\overset{def}{=}\nu_k'$, from asymptotic formulas (15), (16) we obtain that sequences $\{\lambda_k'\}$ and $\{\nu_k'\}$ satisfies to asymptotic formulas (11) and (12), and therefore to (13), (14). Really,

$$\sqrt{\lambda_k'}=\left[k^2+\frac{2}{\pi}\sum_{i=1}^{m}A_iC_{p_i}k^{p_i-1}-2A+a_k\right]^{1/2}=k+\frac{1}{\pi}\sum_{i=1}^{m}\frac{A_iC_{p_i}}{k^{2-p_i}}-\frac{A}{k}+\frac{a_k^{+}}{2k}+$$

$$+O\left[\frac{1}{k^{5-2k_i-p_i}}\right]=k+\frac{1}{\pi}\sum_{i=1}^{m}\frac{A_iC_{p_i}}{k^{2-p_i}}-\frac{A}{k}+\frac{\tilde{a}_k}{k}$$

and by analogous way, we obtain

$$\sqrt{\nu_k'}=k-\frac{1}{2}+\frac{1}{\pi}\sum_{i=1}^{m}\frac{A_iC_{p_i}}{(k-1/2)^{2-p_i}}-\frac{A}{k-1/2}+\frac{\tilde{b}_k}{k-1/2},$$

where $\sum_{k=1}^{\infty}|\tilde{a}_k|^2+|\tilde{b}_k|^2<\infty$.

Thus, we obtain, that by sequences $\{\mu_{2k}^{\pm}\}$ and $\{\mu_{2k-1}^{\pm}\}$ we can determine functions $s(\lambda;\pi)$ and $s'(\lambda;\pi)$ by virtue of Theorem 2, and this is enough for Theorem 1 for reconstruction of potential.

Therefore, we state theorem

Theorem 3. *For sequence* $-\infty<\mu_0<\mu_1^{-}\le\mu_1^{+}<\mu_2^{-}\le\mu_2^{+}<\cdots$ *to consists of spectrums of periodic* $(\mu_0,\mu_2^{-},\mu_2^{+},\mu_4^{-},\mu_4^{+},\ldots)$ *and antiperiodic* $(\mu_1^{-},\mu_1^{+},\mu_3^{-},\mu_3^{+},\ldots)$ *boundary value problems, generated on the interval* $(0,\pi)$ *by the same equation (1) with potential of form (2), it is necessary and sufficient for this sequence to be represented in the form*

$$\mu_k^{\pm}=k^2+\frac{2}{\pi}\sum_{i=1}^{m}A_iC_{p_i}k^{p_i-1}-2A+\varepsilon_k^{\pm},$$

where $p_i\in(1;5/4)$, A_i, A *are arbitrary real numbers,* $C_{p_i}=\int_0^{\infty}\frac{\sin^2\xi}{\xi^{p_i}}d\xi$ *and*

$$\sum_{i=1}^{\infty}|\varepsilon_k^{+}|^2+|\varepsilon_k^{-}|^2<\infty.$$

Remark 1. At paper [1] it was proved, that asymptotic formulas (5), (6) take place in case of singular points at the ends of interval in case of potential, which satisfies to condition $\int_0^\pi x(\pi - x)|q(x)|dx < \infty$ with the only difference, that in remainder member $R(\lambda)$ in formula (7) will be function

$$N(\lambda) = \int_0^{1/|\lambda|} t|q(t)|dt + \frac{1}{|\lambda|}\int_{1/|\lambda|}^{\pi-1/|\lambda|}|q(t)|dt + \int_{\pi-1/|\lambda|}^{\pi}(\pi - t)|q(t)|dt.$$

But asymptotic formulas (8)-(10) take place for potentials from class $x|q(x)| \in L_1(0,\pi]$ and condition $\sum_{k=1}^{\infty}|\alpha_k|^2 + |\beta_k|^2 + |\varepsilon_k^{\pm}|^2 < \infty$ holds for potentials from class $x^{1/2}|q(x)| \in L_1[0;\pi]$.

Remark 2. All described results on solution of inverse problems in regular case were of excellent studying and were stated on paper (see, for example, [3]).

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