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ON UNIFORM APPROXIMATION BY BLEIMANN, BUTZER AND HAHN OPERATORS ON ALL POSITIVE SEMIAXIS

Abstract

In this paper the authors present a Korovkins type theorem on uniform approximation of some subclass of continuos and bounded functions by linear positive operators on all positive semiaxis, by using the Test functions $\left(\frac{x}{x+1}\right)^{\nu}$, $\nu = 0,1,2$. *As an application of this general result, a simple proof of corresponding statement for Bleimann, Butzer and Hahn operators has been given.*

1. Introduction and Preliminaries

Bernstein type linear positive operators

$$L_n(f;x) \equiv L_n(f(t);x) = \frac{1}{(1+x)^n}\sum_{k=0}^{n} f\left(\frac{k}{n-k+1}\right)\binom{n}{k}x^k, \quad x \geq 0, n \in N \qquad (1.1)$$

were introduced by G. Bleimann, P.L. Butzer and L.Hahn [1] and then investigated in different papers (see, for example, [2]-[5]). In [1] the theorem on pointwise convergence of operators (1.1) on $[0,\infty)$ and on uniform convergence on any finite interval of positive semiaxis were proved, some results concerning with the probabilitic interpretation may be found in [2] and generalizations of operators (1.1) were given in [3], [4], the properties of monotonicity of (1.1) and its Lipschitz properties were studied in [5].

The aim of this note is to present a simple proof of theorem on uniform convergence of operators (1.1) on semiaxis $[0,\infty)$ on some subspace of bounded and continuous functions. Firstly we define this class of functions.

Let ω be a function of the type of modulus of continuity. The principal properties of these type functions are the following:

a) ω is a non-negative increasing function on $[0,\infty)$,

b) $\omega(\delta_1+\delta_2) \leq \omega(\delta_1)+\omega(\delta_2)$,

c) $\lim_{\delta\to 0}(\delta) = 0$.

Let H_ω be the space of all real-valued functions f defined on semiaxis $[0,\infty)$ and satisfying the following condition: for any $x, y \in [0,\infty)$

$$|f(x)-f(y)| \leq \omega\left(\left|\frac{x}{1+x}-\frac{y}{1-y}\right|\right). \qquad (1.2)$$

Also we denote the space of functions f, which is bounded and continuous on $[0,\infty)$ by $C_B[0,\infty)$. A norm in $C_B[0,\infty)$ may be defined by

$$\|f\|_{C_B} = \sup_{x\geq 0}|f(x)|.$$

It is obvious, by c) that any function in H_ω is continuous on $[0,\infty)$.

Moreover, any function $f \in H_\omega$ satisfies the inequality

$$|f(x)| \leq |f(0)| + \omega(1), \qquad x \geq 0$$

and therefore is bounded on $[0,\infty)$. So $H_\omega \subset C_B[0,\infty)$.

Some examples of functions, belonging to H_ω are the following:

$$f_1(x) = \sum_{k=0}^{\infty} C_k \left(\frac{x}{1+x}\right)^k,$$

where $\sum_{k=1}^{\infty} k|C_k| < \infty$ with $\omega(t) = 2t^\alpha \sum_{k=1}^{\infty} k|C_k|, \quad 0 < \alpha \leq 1$

$$f_2(x) = \frac{1+2x}{1+x}$$

with $\omega(t) = t$.

In the case of $\omega(t) = Mt^\alpha, \quad 0 < \alpha \leq 1$ we denote H_ω by H_α. In this case it follows from (1.2) that

$$|f(x) - f(y)| \leq M \frac{|x-y|^\alpha}{(1+x)^\alpha (1+t)^\alpha}$$

and therefore $H_\alpha \subset Lip_M \alpha$.

2. Korovkin's type theorem in H_ω

Our first result is the Korovkin type theorem on the conditions of convergence of the sequences of linear positive operators to functions in H_ω.

Note that this type of theorems can not be obtained neither from the classical Korovkin's theorem nor from the Korovkin's theorem conserning Chebyshevs system since both of them are devoted to the problem of approximation by positive operators on finite intervals, [6]. It can not also be obtained from weighted Korovkin's type theorem (see [7] and [8]) since all test functions in those theorems connected with the weight functions $\rho(x) \geq 1$.

Our theorem is the following

Theorem 2.1. *Let A_n be the sequence of linear positive operators, acting from H_ω to $C_B[0,\infty)$ and satisfying three conditions*

$$\lim_{n\to\infty} \left\| A_n\left(\left(\frac{t}{1+t}\right)^\nu; x\right) - \left(\frac{x}{1+x}\right)^\nu \right\|_{C_B} = 0, \quad \nu = 0,1,2. \tag{2.1}$$

Then for any function $f \in H_\omega$

$$\lim_{n\to\infty} \|A_n f - f\|_{C_B} = 0. \tag{2.2}$$

Proof. Let $f \in H_\omega$. Then from (1.2) it follows that for any $\varepsilon > 0$ there exists a number δ such that

$$|f(t) - f(x)| < \varepsilon \quad \text{if} \quad \left|\frac{t}{1+t} - \frac{x}{1+x}\right| < \delta.$$

Also, since f is bounded, there exists a positive constant M such that

$$\left|f(t)-f(x)\right|<\frac{2M}{\delta^2}\left[\frac{t-x}{(1+t)(1+x)}\right]^2 \quad \text{if} \quad \left|\frac{t}{1+t}-\frac{x}{1+x}\right|\geq\delta .$$

Therefore for all $t,x\in[0,\infty)$

$$\left|f(t)-f(x)\right|<\varepsilon+\frac{2M}{\delta^2}\left[\frac{t-x}{(1+t)(1+x)}\right]^2 . \tag{2.3}$$

Now from the conditions (2.1) we can write

$$\begin{gathered}\left\|A_n(1;x)-1\right\|_{C_B}<\varepsilon_n\\ \left\|A_n\left(\frac{t}{1+t};x\right)-\frac{x}{1+x}\right\|_{C_B}<\varepsilon_n\\ \left\|A_n\left(\left(\frac{t}{1+t}\right)^2;x\right)-\left(\frac{x}{1+x}\right)^2\right\|_{C_B}<\varepsilon_n\end{gathered} \tag{2.4}$$

respectively, where $\varepsilon_n\to 0$ as $n\to\infty$.

This gives

$$\left\|A_n\left(\left[\frac{t-x}{(1+t)(1+x)}\right]^2;x\right)\right\|_{C_B}<C\varepsilon_n , \tag{2.5}$$

where C is a constant independent of n.

Writing

$$\left\|A_n(f;x)-f(x)\right\|_{C_B}\leq\left\|A_n\left(\left|f(t)-f(x)\right|;x\right)\right\|_{C_B}+\left\|f(x)\right\|_{C_B}\left\|A_n(1;x)-1\right\|_{C_B}=I_n'+I_n''$$

we can see that

$$\lim_{n\to\infty} I_n''=0$$

since

$$\|f\|_{C_B}\leq M .$$

Let us consider the term I_n'. By (2.3) and (2.5) we obtain

$$I_n'<\varepsilon\left\|A_n(1;x)\right\|_{C_B}+\frac{2M}{\delta^2}\left\|A_n\left(\left[\frac{t-x}{(1+t)(1+x)}\right]^2;x\right)\right\|_{C_B}\leq\varepsilon(1+\varepsilon_n)+\frac{2M}{\delta^2}C\varepsilon_n .$$

Therefore

$$\lim_{n\to\infty} I_n'=0$$

and the proof is completed.

3. Main Result

We can obtain our main result on convergence of Bleimann, Butzer, Hahn operator (1.1)by using Theorem 2.1. Since the operators (1.1) are acting from C_B to C_B, these are also the operators from H_ω to C_B because $H_\omega\subset C_B$.

Theorem 3.1. *Let L_n be the sequence of linear positive operators defined in (1.1). Then for any $f \in H_\omega$*

$$\lim_{n\to\infty}\left\|L_n f - f\right\|_{C_B} = 0 .$$

Proof. By Theorem 2.1 it is sufficient to verify the conditions (2.1). Obviously

$$L_n(1;x) = 1, \tag{3.1}$$

$$L_n\left(\frac{t}{1+t};x\right) = (1+x)^{-n}\sum_{k=0}^{n}\frac{\frac{k}{n-k+1}}{1+\frac{k}{n-k+1}}\binom{n}{k}x^k = (1+x)^{-n}\sum_{k=0}^{n}\frac{k}{n+1}\binom{n}{k}x^k =$$

$$= \frac{n}{n+1}(1+x)^{-n}\sum_{k=0}^{n}\binom{n-1}{k}x^{k+1} = \frac{n}{n+1}x(1+x)^{-n}(1+x)^{n-1} = \frac{n}{n+1}\left(\frac{x}{x+1}\right).$$

Therefore,

$$\left\|L_n\left(\frac{t}{1+t};x\right) - \frac{x}{x+1}\right\|_{C_B} \le \frac{1}{n+1}. \tag{3.2}$$

Finally,

$$L_n\left(\left(\frac{t}{1+t}\right)^2;x\right) = (1+x)^{-n}\sum_{k=1}^{n}\frac{k^2}{(n+1)^2}\binom{n}{k}x^k = (1+x)^{-n}\sum_{k=2}^{n}\frac{k(k-1)}{(n+1)^2}\binom{n}{k}x^k +$$

$$+ (1+x)^{-n}\sum_{k=1}^{n}\frac{k}{(n+1)^2}\binom{n}{k}x^k = (1+x)^{-n}\frac{n(n-1)}{(n+1)^2}\sum_{k=0}^{n-2}\binom{n-2}{k}x^{k+2} +$$

$$+ (1+x)^{-n}\frac{n}{(n+1)^2}\sum_{k=0}^{n-1}\binom{n-1}{k}x^{k+1} = \frac{n(n-1)}{(n+1)^2}\frac{x^2}{(1+x)^n}(1+x)^{n-2} +$$

$$+ \frac{n}{(n+1)^2}\frac{x}{1+x} = \frac{n(n-1)}{(n+1)^2}\left(\frac{x}{1+x}\right)^2 + \frac{n}{(n+1)^2}\frac{x}{1+x},$$

and consequently

$$\left\|L_n\left(\left(\frac{t}{1+t}\right)^2;x\right) - \left(\frac{x}{x+1}\right)^2\right\|_{C_B} \le \frac{3n+2}{(n+1)^2}. \tag{3.3}$$

The inequalities (3.2), (3.3) and the equality (3.1) show that conditions (2.1) are satisfied and the theorem is proved.

Corollary 3.2. *For any $f \in H_\alpha$, $0 < \alpha \le 1$,*

$$\lim_{n\to\infty}\left\|L_n f - f\right\|_{C_B} = 0 .$$

Remark. As mentioned in [1]: «There exists a connection between L_n and the Bernstein polynomials B_n, which is basically given by the rational transformation $h(u) = \frac{v}{v+1}$, $u \in [0,\infty)$ and its inverse $h^{-1}(v) = \frac{v}{v+1}$, $v \in [0,1)$. Since this transformation is rational, one can hardly expect to carry over well-known results of the operator B_n to the transformed operators L_n.» This means that the convergence theorem for operators (1.1) can not be obtained by using above transformation and the classical Korovkin's theorem [6].

In conclusion we will show that the general approximation theorem 2.1 not holds in whole space $C_B[0,\infty)$. Namely we have the following

Theorem 3.3. *There exist a sequence of linear positive operators* A_n, *acting from* C_B *to* C_B *and satisfying the conditions (2.1) and there exist a function* $f^* \in C_B$, *for which*

$$\lim_{n\to\infty} \sup \left\| A_n f^* - f^* \right\|_{C_B} > 0 .$$

Proof. The sequence of operators

$$A_n(f;x) = f(x) + \frac{1-\frac{x}{n}}{n+1}\left[\left(x+\frac{3}{2}\right)f\left(x+\frac{1}{2}\right) - (x+1)f(x)\right], \text{ if } 0 \le x \le n,$$

$$A_n(f;x) = f(x), \quad \text{if} \quad x \ge n$$

obviously satisfies all conditions of the theorem. At that time, for a function

$$f^*(x) = \cos 2\pi x$$

$$\left\| A_n f^* - f^* \right\|_{C_B} \ge \sup_{0\le x\le n} \frac{1-\frac{x}{n}}{n+1}\left(2x+\frac{5}{2}\right)\left|\cos 2\pi x\right| \ge \frac{2n+5}{n+1}\left|\cos \pi n\right|$$

which gives a proof.

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