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DIRECT AND INVERSE PROBLEMS OF BIG DEFORMATIONS OF THREADS UNDER NORMAL LOADING

Abstract

The analytical solutions of the direct and inverse nonlinear boundary valued problems of large (of unit order and more) elastic deformations of the plane threads under action of the normal loading are obtained, also different particular cases are considered.

The direct and inverse problems of large (equal to unit or more) axial symmetric elastic deformations of plane threads with constant cross-section whose ends are rigidly fixed are considered. The threads are under normal loading axial applied along all length; the own weight of the thread is not considered.

The similar direct problems for the threads, direct and different inverse problems for diaphragms and without moment shells were considered in [1]-[10].

The main dependencies and equations. Let refer the thread to the rectangle system of coordinates XOY. Let the axis OX passes cross the points of fixing of thread's ends and the axis OY passes cross the means of the line connecting the last ones and is perpendicular to this line, T_* is tension of the thread, $q_0(x)$ is intensity of the normal loading, acting on the unit of the length of the thread, ξ, x are the abscissas, ζ, y are the ordinates of some point of the thread, F_0, F are the squares of cross-section, ψ, φ are the angles between the tangential to the thread and the positive direction of the axis OX in some point before and after deformation, l is half of the distance between the points of fixing of the thread, E is modulus of elasticity of thread is material, ν is Poisson's coefficient.

Let

$$\varsigma = \varsigma(\xi), \quad Y = Y(X) \tag{1}$$

be the equations of the thread before and after deformation.

Projecting on the axes OX and OY the external and internal forces acting to the element of the thread we obtain

$$\frac{d}{dX}\left[T_*\left(1+Y'^2\right)^{-1/2}\right] - q_0(X)\cdot Y' = 0; \quad \frac{d}{dX}\left[T_*\cdot Y'\left(1+Y'^2\right)^{-1/2}\right] - q_0(X) = 0. \tag{2}$$

According to [4] and [5] for thread's material between the real stress and logarithmic deformations the linear dependence is taken in the form:

$$\begin{aligned} \varepsilon_1 &= \frac{1}{E}\sigma = \frac{1}{E}\cdot\frac{T_*}{F}; \\ \varepsilon_2 &= -\frac{\nu}{E}\sigma = -\frac{\nu}{E}\cdot\frac{T_*}{F} = -\nu\varepsilon_1; \\ \varepsilon_3 &= -\frac{\nu}{E}\sigma = -\frac{\nu}{E}\cdot\frac{T_*}{F} = -\nu\varepsilon_1, \end{aligned} \tag{3}$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are the main logarithmic deformations; ε_1 is the longitudinal strain, and $\varepsilon_2, \varepsilon_3$ is the radial and circle deformations, if the thread with the circle cross-section, transverse deformations, if the thread with square or rectangular cross-section, σ is the real normal stress.

Using (3) for the thread with circle, square and rectangular cross-sections, correspondingly, we can write

$$\ln\frac{F}{F_0}=\left\{\begin{array}{c}\ln\frac{\pi\rho^2}{\pi\rho_0^2}=2\ln\frac{\rho}{\rho_0}=2\varepsilon_2\\ \ln\frac{a^2}{a_0^2}=2\ln\frac{a}{a_0}=2\varepsilon_2\\ \ln\frac{\alpha\cdot\beta}{\alpha_0\cdot\beta_0}=\ln\frac{\alpha}{\alpha_0}+\ln\frac{\beta}{\beta_0}=\varepsilon_2+\varepsilon_3\end{array}\right\}=-2\nu\varepsilon_1, \tag{4}$$

where ρ and ρ_0, a_0 and a, α_0, β_0 and α, β are the radiuses with circle cross-section, the lengths of the square and rectangular cross-sections of the thread, correspondingly, before and after deformation.

From (4) we obtain

$$F=F_0\exp(-2\nu\varepsilon_1): \tag{5}$$

Let introduce the dimensionless variables and quantities

$$r=\frac{\xi}{l};\quad \eta=\frac{\varsigma}{l};\quad x=\frac{X}{l};\quad y=\frac{Y}{l};\quad T=\frac{T_*}{EF_0};\quad q(x)=\frac{q_0(X)\cdot l}{EF_0}. \tag{6}$$

Taking into account from (1) and (2) we obtain

$$\eta=\eta(r),\quad y=y(x), \tag{7}$$

$$\begin{gathered}\frac{d}{dx}\left[T\left(1+y'^2\right)^{-1/2}\right]-q(x)\cdot y'=0,\\ \frac{d}{dx}\left[T\cdot y'\left(1+y'^2\right)^{-1/2}\right]-q(x)=0.\end{gathered} \tag{8}$$

The equations of the tangential to the thread before and deformation will be:

$$\frac{d\eta}{dr}=tg\psi,\quad \frac{dy}{dx}=tg\varphi. \tag{9}$$

Using (5) and (6) from (3) we will obtain

$$T=\varepsilon_1\cdot\exp(-2\nu\varepsilon_1). \tag{10}$$

For the longitudinal deformation ε_1 we have:

$$\varepsilon_1=\ln\frac{dx\cdot\cos\psi}{dr\cos\varphi}=\ln\left(\frac{dx}{dr}\right)\left(\frac{1+y'^2}{1+\eta'^2}\right)^{1/2}. \tag{11}$$

Taking into account the axial symmetry and rigid fixing of thread's ends the boundary conditions we can write as:

$$\begin{gathered}x=0,\ \ y=y_0,\ \ \eta=\eta_0,\ \ y'=0,\ \ \eta'=0,\ \ \varphi=0,\ \ \psi=0,\ \ T=T_0\ \text{ for } r=0;\\ x=1,\ \ y=0,\ \ \eta=0,\ \ y'=y_1',\ \ \eta'=\eta_1',\ \ \varphi=\varphi_1,\ \ \psi=\psi_1,\ \ T=T_1\ \text{ for } r=1,\end{gathered} \tag{12}$$

where η_0, y_0 are the deflections in the center of the thread before and after deformation; y_0, y_1', φ_1 are for the direct problem; T_0, T_1 are the unknown values for the direct and inverse problem, they are found at the solving process.

The correlations (3), (5) and (7)-(11) joint with boundary conditions (12) are the complete system for solution of direct and inverse problems of large axial of plane threads which are under action of the normal loading with constant and variable intensity.

From (8) we obtain

$$\frac{dT}{dX}=0,\quad T=const, \tag{13}$$

$$\frac{d}{dx}\left[\left(1+y'^2\right)^{-1/2}\right]+\frac{q(x)}{T}\cdot y'=0,$$
$$\frac{d}{dx}\left[y'\cdot\left(1+y'^2\right)^{-1/2}\right]-\frac{q(x)}{T}=0. \tag{14}$$

From (13) it follows that for the axial symmetric deformation of the thread under action of the normal loading with constant and variable intensity, independently tension is constant anywhere.

Taking (10) and (13) into account we can write (11) in the form:

$$\varepsilon_1=\ln\frac{\int_0^x\left(1+y'^2\right)^{1/2}dx}{\int_0^r\left(1+\eta'^2\right)^{1/2}dr}=\ln\frac{\int_0^1\left(1+y'^2\right)^{1/2}dx}{\int_0^1\left(1+\eta'^2\right)^{1/2}dr}=const. \tag{15}$$

From (15) we obtain

$$\int_0^x\left(1+y'^2\right)^{1/2}dx=\left(\frac{\int_0^1\left(1+y'^2\right)^{1/2}dx}{\int_0^1\left(1+\eta'^2\right)^{1/2}dr}\right)\int_0^r\left(1+\eta'^2\right)^{1/2}dr. \tag{16}$$

Note that for known $y=y(x)$ and $\eta=\eta(r)$ form (16) the dependence between x and r is determined.

The direct problem. For solution of the direct problem the initial form of the thread and the intensity of the normal loading are given and the longitudinal and transverse deformations, the form of the thread after deformation, tension (stress), change of the square of the cross-section of the thread at deformation process are seeked; also the critic value of tension and intensity of loading, longitudinal and transverse deformations are found for which increase the equilibrium of the stretched thread developed from the initial becomes unstable; for further post-critic increase of deformation it corresponds not increase of tension and intensity of loading but their decrease [4]-[7], [9].

Let consider the case when $q=const$. Then from the first or the second equation of (14) for the boundary conditions (12) we obtain

$$y''\left(1+y'^2\right)^{-3/2}=\frac{1}{R};\quad y'\left(1+y'^2\right)^{-1/2}=\frac{x}{R};\quad y'=\pm\frac{x}{\sqrt{R^2-x^2}}. \tag{17}$$

$$y\pm\sqrt{R^2-1}\mp\sqrt{R^2-x^2}, \tag{18}$$

$$R=\frac{T}{q}. \tag{19}$$

From (18) it follows that at the axial symmetric deformation the plane thread deforming under action of the normal loading with constant intensity takes the form of arc of the circumference with radius R with the center in the point with the coordinates

$$x_0=0,\quad y_0=\pm\sqrt{R^2-1}. \tag{20}$$

Using (10), (17) and (19) from (15) we obtain

$$q=\varepsilon_1\left[\exp(-2\nu\varepsilon_1)\right]\sin\left\{\frac{q}{\varepsilon_1}\left[\exp((1+2\nu)\varepsilon_1)\right]\cdot\int_0^1\left(1+\eta'^2\right)^{1/2}dr\right\}, \tag{21}$$

$$x=\frac{\varepsilon_1}{q}[\exp(-2\nu\varepsilon_1)]\sin\left\{\frac{q}{\varepsilon_1}[\exp((1+2\nu)\varepsilon_1)]\cdot\int_0^r\left(1+\eta'^2\right)^{1/2}dr\right\}. \qquad (22)$$

For solution of the direct problem q and $\eta=\eta(r)$ are given and from (21) ε_1 is determined, further knowing ε_1 and $\eta=\eta(r)$ from (10) and (22) T and the dependence $x=x(r)$ are found, then from (3), (5), (19) and (18), subsequently, $\varepsilon_2,\varepsilon_3,F,R$ and the dependence $y=y(x)$ are determined, knowing $\eta=\eta(r)$ and $y=y(x)$ from (9) the dependencies $\psi=\psi(r)$ and $\varphi=\varphi(x)$ are found.

Let note that q must not be given arbitrary, it is the bounded quantity. Let's determine the maximal value of q, following [4] and [5] from [10] we obtain

$$\varepsilon_1=\varepsilon_1^*=\frac{1}{2\nu},\quad T=T^*=\frac{1}{2e\nu}, \qquad (23)$$

where ε_1^*,T^* are the maximal values of ε_1 and T.

Using (23) from (3), (5) and (21), (22) we obtain

$$\varepsilon_2^*=\varepsilon_3^*=-\frac{1}{2}, \qquad (24)$$

$$F^*=\frac{F_0}{e}, \qquad (25)$$

$$q^*=\frac{1}{2\nu e}\cdot\sin\left[2\nu\left(\exp\left(1+\frac{1}{2\nu}\right)\right)q^*\cdot\int_0^1\left(1+\eta^2\right)^{1/2}dr\right], \qquad (26)$$

$$x=\frac{1}{2\nu eq^*}\cdot\sin\left[2\nu\left(\exp\left(1+\frac{1}{2\nu}\right)\right)q^*\cdot\int_0^r\left(1+\eta^2\right)^{1/2}dr\right], \qquad (27)$$

where q^* is maximal value of q, and ε_2^*, ε_3^* and F^* are minimal values of ε_2, ε_2 and F.

Since we know T^* and q^* from (19) and (18) we can determine R^*, i.e. the value of R corresponding to the values of T and q.

1. The strength thread $\eta=\eta(r)=0$, $\eta'=0$ passes to the arc of the circumference (18).

Then from (21) and (22) we obtain

$$q=\varepsilon_1[\exp(-2\nu\varepsilon_1)]\sin\left\{\frac{q}{\varepsilon_1}[\exp((1+2\nu)\varepsilon_1)]\right\}, \qquad (28)$$

$$x=\frac{\varepsilon_1}{q}[\exp(-2\nu\varepsilon_1)]\sin\left\{\frac{q}{\varepsilon_1}[\exp((1+2\nu)\varepsilon_1)]r\right\}. \qquad (29)$$

For $\varepsilon_1=\varepsilon_1^*=\frac{1}{2\nu}$ from (28) and (29) we obtain

$$q^*=\frac{1}{2\nu e}\cdot\sin\left[2\nu\cdot q^*\exp\left(1+\frac{1}{2\nu}\right)\right], \qquad (30)$$

$$x=\frac{1}{2\nu eq^*}\cdot\sin\left\{\left[2\nu\cdot q^*\exp\left(1+\frac{1}{2\nu}\right)\right]r\right\}. \qquad (31)$$

For $\nu=0{,}5$ from (23) and (30), (31) we obtain

$$\varepsilon_1^* = 1, \quad T^* = \frac{1}{e} \approx 0{,}3679, \tag{32}$$

$$q^* = \frac{1}{e} \cdot \sin\left(e^2 \cdot q^*\right), \tag{33}$$

$$x = \frac{1}{eq^*} \sin\left(e^2 \cdot q^* \cdot r\right). \tag{34}$$

From (33) it was found

$$q^* = 0{,}2976. \tag{35}$$

Using (32) and (35) from (19) and (20) we obtain

$$R^* = \frac{T^*}{q^*} \approx \frac{0{,}3679}{0{,}2976} \approx 1{,}2362, \quad y_0 = -\sqrt{R^{*2} - 1} \approx -0{,}7266. \tag{36}$$

It should be noted that for solution of the direct problem for $q = const$ the value of q can be given only on the segment $\left[0; q^*\right]$.

The inverse problem. For solution of the inverse problem the initial and the finite forms of the plane thread are given and tension (stress), longitudinal and transverse deformations and the law of intensity change of the normal loading under which action beforehand given formchange of the thread happens; also the critic value of the parameter characterizing the geometry of the so as this parameter comes into the analytic expression of the loading then taking in that expression as the value of the parameter its critic value, and by this way the analytic expression for the critic loading will be determined.

From the first or the second equations of the system (14) taking into account (13), (10) and (15) we have

$$q(x) = \frac{y''}{\left(1 + y'^2\right)^{3/2}} \cdot \left(\frac{\int_0^1 \left(1 + \eta'^2\right)^{1/2} dr}{\int_0^1 \left(1 + y'^2\right)^{1/2} dx} \right)^{2\nu} \ln \frac{\int_0^1 \left(1 + y'^2\right)^{1/2} dx}{\int_0^1 \left(1 + \eta'^2\right)^{1/2} dr}, \tag{37}$$

$q(x)$ is the intensity of the normal loading, under which action the plane thread of the initial form given by the equation $\eta = \eta(r)$ passes into the finite form given by the equation $y = y(x)$.

2. Let the plane thread with the initial form $\eta = \eta(r)$ after deformation takes the form of parabola

$$y = y(x) = ax^2 - a. \tag{38}$$

Then using (38) from (37) we obtain

$$q(x) = \frac{2a}{\left[1 + (2ax)^2\right]^{3/2}} \left(\frac{\int_0^1 \left(1 + \eta'^2\right)^{1/2} dr}{f(a)} \right)^{2\nu} \cdot \ln \frac{f(a)}{\int_0^1 \left(1 + \eta'^2\right)^{1/2} dr}, \tag{39}$$

where

$$f(a) = \int_0^1 \left(1 + y'^2\right)^{1/2} dx = \int_0^1 \left[1 + (2ax)^2\right]^{1/2} dx =$$

$$=\frac{1}{4a}\left\{2a\left[1+(2a)^2\right]^{1/2}+\ln\left(2a+\left[1+(2a)^2\right]^{1/2}\right)\right\}. \qquad (40)$$

For $\eta=\eta(r)=0$, $\eta'=0$, $\nu=0{,}5$, $\varepsilon_1=\varepsilon_1^*=\frac{1}{2\nu}=1$, $T=T^*=\frac{1}{2\nu e}\approx 0{,}3679$ from (40) we obtain

$$f(a)=f(a^*)=\frac{1}{4a^*}\left\{2a^*\left[1+(2a^*)^2\right]^{1/2}+\ln\left(2a^*+\left[1+(2a^*)^2\right]^{1/2}\right)\right\}=e. \qquad (41)$$

From (41) we obtain

$$a^*\approx 2{,}433, \qquad (42)$$

where a^* is the maximal value of parameter a

$$q^*(x)=\frac{2a^*}{e\left[1+(2a^*x)^2\right]^{3/2}}\approx\frac{1{,}7901}{\left[1+(4{,}866x)^2\right]^{3/2}};$$
$$q^*(0)=\frac{2a^*}{e}\approx 1{,}7901;\quad q^*(1)\approx\frac{1{,}7901}{122{,}53}\approx 0{,}0146. \qquad (43)$$

3. Let the strength thread-strip after deformation take the form of parabola

$$y=ax^2-a=0{,}5x^2-0{,}5;\quad a=0{,}5 \qquad (44)$$

with the focus in the origin of coordinates.

Then for $\nu=0{,}5$ we obtain

$$f_1(a)=f(0{,}5)=\int_0^1\left(1+y'^2\right)^{1/2}dx=\int_0^1\left(1+x^2\right)^{1/2}dx=$$
$$=\frac{1}{2}\left\{x\left(1+x^2\right)^{1/2}+\ln\left[x+\left(1+x^2\right)^{1/2}\right]\right\}_0^1\approx 1{,}1478;$$
$$\varepsilon_1=\ln f_1(a)=\ln f_1(0{,}5)=\ln 1{,}1478\approx 0{,}1373; \qquad (45)$$
$$\varepsilon_2=\varepsilon_3=-\nu\varepsilon_1\approx -0{,}06865;\quad T=\varepsilon_1\exp(-2\nu\varepsilon_1)\approx 0{,}1196;$$
$$F=F_0\exp(-2\nu\varepsilon_1)\approx 0{,}8713F_0;\quad q(x)\approx\frac{0{,}1196}{\left(1+x^2\right)^{3/2}};$$
$$q(0)\approx 0{,}1196;\quad q(1)\approx 0{,}0423.$$

4. Let the Strength thread-strip $(\eta=\eta(r)=0;\eta'=0)$ after deformation take the form of arc of the circumference

$$y=\frac{R}{2}-\sqrt{R^2-x^2};\quad R=\frac{2}{\sqrt{3}}>1 \qquad (46)$$

with the focus in the origin of coordinates.

Substituting (46) into (36) we obtain

$$q(x)=\frac{1}{R}\left(\frac{1}{f_2(R)}\right)^{2\nu}\cdot\ln f_2(R), \qquad (47)$$

where

$$f_2(R)=\int_0^1\left(1+y'^2\right)^{1/2}dx=R\arcsin\frac{1}{R}\approx 1{,}2091. \qquad (48)$$

For $\nu=0{,}5$ from (47), (10), (15), (3), (5), (16) we obtain

$$q(x)=\frac{1}{R}\cdot\frac{\ln f_2(R)}{f_2(R)}\approx 0.1350;$$

$$T=Rq=\frac{\ln f_1(R)}{f_2(R)}\approx 0{,}1559; \qquad (49)$$

$$\varepsilon_1=\ln f_2(R)\approx 0{,}1885; \quad \varepsilon_2=\varepsilon_3=-\nu\ln f_2(R)\approx -0{,}09425;$$

$$F=\frac{F_0}{f_2(R)}\approx 0{,}8271\cdot F_0;$$

$$x=R\sin\left[\left(\arcsin\frac{1}{R}\right)r\right]=\frac{2}{\sqrt{3}}\sin\left[\left(\arcsin\frac{\sqrt{3}}{2}\right)r\right].$$

The surface of strips obtained from the thread-strip deforming by the point out in points 3 and 4 ways are used in sun planes and in antenna parts of radio-telescopes; they concentrate falling on them parallel to the axis of symmetry the sun energy, radio-signals and other radiations to the focuses [12], [13].

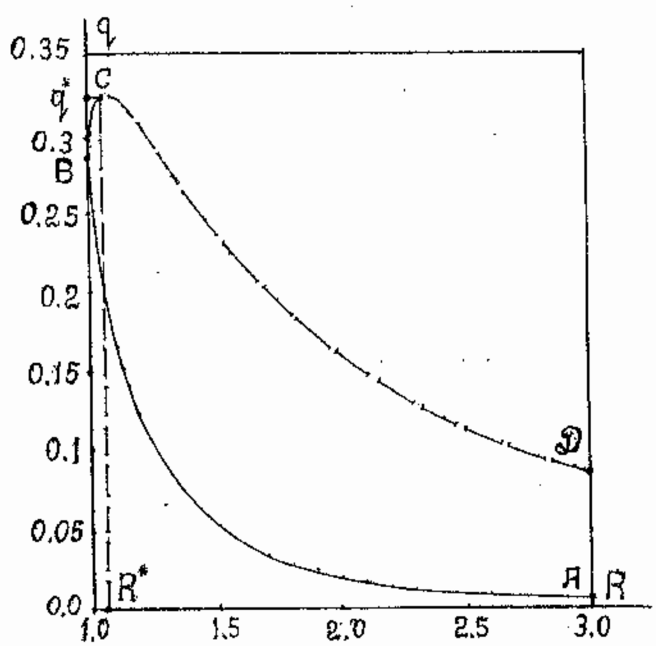

Fig. 1.

In fig. 1 for $\nu=0{,}5$, $\eta=\eta(r)=0$, $\eta'=0$ using (18) and the formula

$$q=\frac{1}{R}\cdot\frac{\ln\int_0^1\left(1+y'^2\right)^{1/2}dx}{\int_0^1\left(1+y'^2\right)^{1/2}dx} \qquad (50)$$

the graphic of the dependence between q -intensity of the uniformly distributed loading and R - radius of the arc of the circumference of the deformed thread has been constructed. Moreover, for the branch AB $\int_0^1 (1+y'^2)^{1/2} dx = R\arcsin\frac{1}{R}$, and for the branches BC and CD $\int_0^1 (1+y'^2)^{1/2} dx = R\left(\pi - \arcsin\frac{1}{R}\right)$ and for all branches it is taken the least value of $R\arcsin\frac{1}{R}$. The dash line CD refers to the unstable condition of equilibrium [4]-[7]; q^* and R^* are the maximal values of q and R at which increase either the equilibrium of the deformed thread becomes unstable (line CD) or the problem has not got any solution [11].

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