ALI-ZADEH A.N.

ON THE APPLICATION OF COMPULSION MEASURES TO ELASTICITY THEORY PROBLEMS

Abstract

A differential variational principle of elasticity theory based on the compulsion measure and on Hu-Vasidzy's variational principle is suggested in the paper. Lagrange-Gauss principle is constructed on this principle. It is shown that the functional of this principle achieves its minimum for real values.

Introduction. Dynamical problems of elasticity theory conditionally may be divided into two groups. To the first group belong the problems for the solution of which at the moments $t=t_0$ and $t=t_1$ the displacements of the points of the body are given, where t_0 and t_1 are correspondingly the beginning and the end of the time interval on which the motion process in studied. To the second group belong the problems of which at the initial time moment the displacements and velocities of the arbitrary point of the body are known. The construction method of variational principles for solving problems belonging to the first group is known. Corresponding functionals are analogous to the Hamilton functional. By the constructing variational principles for solving problems belonging to the second group there arises some difficulties related with inclusion of corresponding functional both the equations of elastic body motion and initial conditions to Euler's equation system. In many cases we can overcome these difficulties by introducing the convolution: in Hamilton type functionals, the multiplication is replaced by the convolution and additional terms are introduced. Numerical realization of such variational principles leads to some difficulties. Therefore, it is necessary to construct other functionals. The aim of this paper is the formulation of variational principles for dynamical problems of elasticity theory when initial conditions are given.

1. Differential variational principles of elasticity theory.

Variational principles of mechanics are divided into differential and integral ones. The first principles give the real motion criteria for the given fixed time moment, but the second ones - at the final time interval [1]. Among differential variational principles of mechanics we distinguish the Gauss principle (the least compulsion principle) for a mass points system. This principle says: among comparatively kinematics possible motions, the real motion is distinguished by the fact that the compulsion measure Z is minimal for it, where [1]:

$$Z=\frac{1}{2}\sum_{i=1}^{N} m_i\left(\vec{w}_i-\frac{\vec{F}_i}{m_i}\right)^2 \qquad (1)$$

m_i is a point mass, $\vec{w}_i$ is the acceleration, $\vec{F}_i$ is the resultant force effecting to the chosen point, N is the number of points contained in the system. Hence we have that the quantity Z considered as a function of possible accelerations is minimal at the values of accelerations of the points of the system corresponding to the real motion.

The choice of the Gauss principle among the differential variational principles of mechanics in explained by the fact that the motion equations of elastic body points depend on acceleration. The system of motion equations of the elastic body points take as [2]:

$$\sigma^{ij}_{,j} = \rho \frac{\partial^2 u^i}{\partial t^2}; \quad \sigma^{ij} = A^{ijkl} e_{kl}; \quad e_{kl} = \frac{1}{2}\left(u_{k,l} + u_{l,k}\right), \quad x \in V$$
$$u_i = \bar{u}_i, \quad x \in S_u; \quad \bar{T}^i = \sigma^{ij} n_j, \quad x \in S_\sigma \ , \tag{2}$$

where σ^{ij} are the components of stress tensor, ρ is the density, u^i are the components of a displacement vector, t is the time, A^{ijkl} are elasticity modules, e_{kl} are the components of strain tensor, the comma means a covariant differentiation with respect to the coordinate, S_u is a part of the considered body on which the displacements $\bar{u}_i$ are given, S_σ is the remaining part of the surface on which surface efforts $\bar{T}^i$ are given, n_i are the components of the normal vector. We take the initial conditions for solving the system (2) as follows:

$$for \quad t = t_0 \quad u_i = u_i^0; \quad \frac{\partial u_i}{\partial t} = \upsilon_i^0, \quad x \in V. \tag{3}$$

Rewrite the system (2) as:

$$\sigma^{ij}_{,j} = \rho w^i; \quad \sigma^{ij} = A^{ijkl} e_{kl}; \quad \ddot{e}_{kl} = \frac{1}{2}\left(w_{k,l} + w_{l,k}\right), \quad x \in V$$
$$w_i = \bar{w}_i, \quad x \in S_u; \quad \bar{T}^i = \sigma^{ij} n_j, \quad x \in S_\sigma; \quad w_i = \frac{\partial^2 u_i}{\partial t^2} \ , \tag{4}$$

where the point means the differentiation in time. The quantities $\bar{w}_i$ are defined from the following equations:

$$\bar{w}^i = \frac{1}{\rho}\sigma^{0ij}_{,j}; \quad \sigma^{0ij} = A^{ijkl} e^0_{kl}; \quad e^0_{kl} = \frac{1}{2}\left(u^0_{k,l} + u^0_{l,k}\right).$$

In addition it was supposed that the quantities u_i^0 have necessary number of derivatives. To obtain the solution of the system (2) from the solution of the system (4) the Cauchy relations, and boundary conditions twice must be integrated in time. By integrating boundary conditions it is necessary to use the initial conditions (3), namely

$$for \quad x \in S_u \quad \left.\frac{\partial u_i}{\partial t}\right|_{t=t_0} = \upsilon_i^0; \quad u_i\big|_{t=t_0} = u_i^0. \tag{5}$$

By integrating Cauchy relations it is necessary to use the following equalities:

$$\dot{e}^0_{kl}\big|_{t=t_0} = \frac{1}{2}\left(\vartheta^0_{k,l} + \vartheta^0_{l,k}\right), \quad e_{kl}\big|_{t=t_0} = \frac{1}{2}\left(u^0_{k,l} + u^0_{l,k}\right) \tag{6}$$

obtained from the equalities (3). Thus, it is shown that the system (4) with initial conditions (3), (5), (6) is equivalent to the system (2) with initial conditions (3).

To solve the equation system (4) we suggest a variational principle. The suggested principle is based on the compulsion measures (1) and on one of the variational principles of the elasticity theory. If , in particular, we take Hu-Vasidzy's variational principle [3], we have:

$$J = \int_V \left\{ \sigma^{ij}\left[\ddot{e}_{ij} - \frac{1}{2}\left(w_{i,j} + w_{j,i}\right)\right] - A^{ijkl} e_{ij} \ddot{e}_{kl} - \frac{1}{2}\rho w^i w_i \right\} dV +$$
$$+ \int_{S_u} \sigma^{ij} n_j \left(w_i - \bar{w}_i\right) dS + \int_{S_\sigma} \bar{T}^i w_i dS \tag{7}$$

where varying quantities are $\sigma^{ij}, \ddot{e}_{ij}, w_i$ (the quantities e_{ij} are not varied). Considering that variation operator functions only on indicated quantities, we find the expression of the first variation. It has the view:

$$\delta J = \int_V \left\{ \delta\sigma^{ij}\left[\ddot{e}_{ij} - \frac{1}{2}\left(w_{i,j} + w_{j,i}\right)\right] + \delta\ddot{e}_{ij}\left(\sigma^{ij} - A^{ijkl}e_{kl}\right) - \sigma^{ij}\delta w_{i,j} - \rho w^i \delta w_i \right\} dV +$$
$$+ \int_{S_u}\left[\delta\sigma^{ij} n_j\left(w_i - \overline{w}_i\right) + \sigma^{ij} n_j \delta w_i\right] dS + \int_{S_\sigma} \overline{T}^i \delta w_i dS.$$

Apply the Gauss theorem to the third term of δJ and by grouping δJ with respect to the variation of independent quantities, we get:

$$\delta J = \int_V \left\{ \delta\sigma^{ij}\left[\ddot{e}_{ij} - \frac{1}{2}\left(w_{i,j} + w_{j,i}\right)\right] + \delta\ddot{e}_{ij}\left(\sigma^{ij} - A^{ijkl}e_{kl}\right) - \delta w_i\left(-\rho w^i + \sigma^{ij}_{,j}\right) \right\} dV -$$
$$- \int_{S_u} \sigma^{ij} n_j \delta w_i dS - \int_{S_\sigma} \sigma^{ij} n_j \delta w_i dS + \int_{S_u}\left[\delta\sigma^{ij} n_j\left(w_i - \overline{w}_i\right) + \sigma^{ij} n_j \delta w_i\right] dS + \int_{S_\sigma} \overline{T}^i \delta w_i dS.$$

By writing the above equality it was considered that the surface of the body consists of two surfaces S_σ and S_u, and they don't intersect. From the stationary state condition of the functional (7) $\delta J = 0$, by equaling the coefficients to zero by variation of independent quantities we get that Euler's equations system of the functional J coincides with the equation system (4). Thus, we can show that the solution of the equation system (4). Thus, we can show that the solution of the equations system (4) is equivalent to the finding of stationary value of the functional (7). Since the system (4) is equivalent to the system (2), we can formulate the following principle by Hu-Vasidzy-Gauss; real motion of the elastic body points differs from all other ones by the fact that for them the functional (7) accepts a stationary value. Note that this principle is admissible both for the first group of dynamical problems of the elasticity theory, and for the second one, when integral variational principles may be applied only for the first group of problems.

2. Extremal property of differential variational principles of elasticity theory.

By analogy with the functional (7) we can construct many functionals corresponding to differential variational principles of the elasticity theory based on variational principles of elasticity theory. In particular based on the Lagrange functional and compulsion measure we construct the following functional similar to (7):

$$J = \int_V \left[\frac{1}{4} A^{ijkl}\left(u_{i,j} + u_{j,i}\right)\left(w_{k,l} + w_{l,k}\right) + \frac{1}{2}\rho w^i w_i\right] dV - \int_{S_\sigma} \overline{T}^i w_i dS, \qquad (8)$$

where independent quantities are w^i and the quantities u_i are not varied. It is clear that the Euler equations found from the stationary state condition (8) coincide with the system (2) under corresponding transformations. Note that the functional (7) is complete and the functional (8) may be obtained from (7) by the corresponding transformation. Therefore, (7) may be considered partial by analogy with (7),we may call it the Lagrange-Gauss functional.

Analyze the stationary value of the functional (8). In this functional, with respect to the chosen varying quantity, all terms, besides one, are linear. Therefore, the second variation has the form:

$$\delta^2 J = \delta^2 \int_V \frac{1}{2} \rho w^i w_i dV = \int_V \rho \delta w^i \delta w_i dV .$$

Quadratic form appearing under the integral is positively defined, since $p > 0$. Hence it follows that for the solution of the system (4), the functional accepts the extremal value, namely the minimal value. Note, that the extremal property of the Lagrangian functional has been proved only for the problems of statics. The extremal property of Lagrange-Gauss functional distinguishes it among similar functionals by numerical investigation of solution of a dynamic problem of the elasticity theory, when initial conditions are given.

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Ali-zadeh A.N.
Azerbaijan Technical University,
25, H.Javid av., 370073, Baku, Azerbaijan.

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