SOLTANOV K.N.

SOME IMBEDDING THEOREMS AND NONLINEAR DIFFERENTIAL EQUATIONS

Abstract

This work is devoted to the study of some pn-spaces and non-linear equations. Besides anisotropic pn-spaces are studied, some imbedding and compactness theorems are proved.

Using the obtained results for pn-spaces it is proved the solvability of the boundary value problem for a class of elliptic-parabolic equations. The considered class of equations, in particular, contains the unstable filtration equations type equations.

The research of boundary value problems often leads to the study of functional spaces directly related with considered problems. In other words, while studying boundary value problems there arise spaces being the domain of definition of operators generated by the considered problems, for instance we can say that Sobolev spaces and their different generalizations arises namely so while studying boundary value problems for linear differential equations (see [1]).

Unlike linear boundary value problems, in the case of nonlinear boundary value problems, the sets generated by these problems, i.e. the domain of definitions of corresponding operators generally speaking are the subsets of linear spaces not possessing the linear structure. In particular, these sets are the subsets of Sobolev spaces (see [2,3]). Note, for example, the subsets arises in papers [4,5] and called pn-spaces proceeding from the structure that they possess. (Note that these spaces possess and the metric spaces structure, i.e. they are metric spaces).

In the given paper, the classes of pn-spaces, related with nonlinear boundary value problems, and some nonlinear boundary value problems are studied. The following mixed value problem and related spaces are also studied:

$$\frac{\partial |u|^{\rho} u}{\partial t} - \sum_{i=1}^{n} D_i\left[A_i(x,t,u,Du) + B_i(x,t,u)\right] + f(x,t,u) = 0, \quad (x,t) \in Q, \tag{1}$$

$$u(x,0) = u_0(x), \ x \in \Omega \subset R^n, \ n \geq 1, \ \rho \geq 0, \tag{2}$$

$$u|_{\Gamma} = \psi(x',t), \ (x',t) \in \partial\Omega \times [0,T] \equiv \Gamma, \ Q \equiv \Omega \times (0,T), \tag{3}$$

where $\Omega \subset R^n$ is a domain with sufficiently smooth boundary $\partial\Omega$ and $Q \equiv \Omega \times (0,T)$, and $A_i(x,t,\xi,\eta)$, $B_i(x,t,\xi)$, $f(x,t,\xi)$, $u_0(x)$, $\psi(x',t)$, $\left(i = \overline{1,n}\right)$ are given functions, $D \equiv (D_1, D_2, \ldots, D_n)$, $D_i \equiv \frac{\partial}{\partial x_i}$, $i = 1,2,\ldots,n$, $\eta \equiv (\eta_1, \eta_2, \ldots, \eta_n)$.

In particular, the problems are studied where the equation (1) has the following form:

$$\frac{\partial |u|^{\rho} u}{\partial t} - \sum_{i=1}^{n} D_i\left[a_i(x,t,u)|u|^{p_0-2} D_i u + b_i(x,t,u,Du)|D_i u|^{p_1-2} D_i u\right] + f(x,t,u) = 0 \tag{4}$$

$\rho > -1, \ p_0 \geq 2, \ p_1 > 1$

or

$$\frac{\partial |u|^{\rho}u}{\partial t}-\sum_{i=1}^{n}D_i\left[a_i(x,t,u)|u|^{p_0-2}D_iu+b_i(x,t,u,u)|u|^{\mu}|D_iu|^{\nu}D_iu\right]+f(x,t,u)=0\,, \tag{5}$$

$\rho>0,\ p_0\geq 2,\ \mu>0,\nu>-1$.

The (1)-(3) type problems under different conditions on the function $A_i(x,t,\xi,\eta)$, $B_i(x,t,\xi)$ and on the domain Q (i.e. in cases, when Q is a free boundary or a fixed boundary domain) have been considered in many papers (see [5,6] and their references). In these papers, the problem is studied under such conditions that the elliptic part of the problem in the most general case may generate a variational calculation operator ([8]). In the given paper the problem (1)-(3) is investigated in more general conditions, in particular, as in papers [4,5], here a homogeneous problem is studied for the equation of the form (1), in particular, the equations (4) and also (5). Besides, here the problem with inhomogeneous conditions is considered under more general conditions, and the method in the indicated papers may not be directly applied to this problem. Therefore, the method used in papers [4,5] is somewhat modified, and this requires more detailed study of corresponding pn-spaces. Thus, at first some class of pn-spaces (p.1) is investigated both in isotropic and in anisotropic cases, and next the solvability of the problem (1)-(3) and more general problems (p.2) is proved.

We are to note that the (1) type equations (see (4) and (5)) arise in filtration theory, diffusion and other processes, in particular, the equation (1) describes the processes related with fluid in a porous medium (see references of papers [4,5] and others).

1. Some pn-spaces and imbedding theorems.

Let $\Omega\subset R^n$ $(n\geq 1)$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$, $Q\equiv\Omega\times(0,T)$. Consider the following classes of functions: $\mathrm{u}:\Omega\to\mathrm{R}^1$, $\mathrm{u}:Q\to\mathrm{R}^1$

$$S_{m,\mu,\nu}(\Omega)\equiv\left\{u(x)\in L_1(\Omega)\middle|[u]_{S_{m,\mu,\nu}}^{\mu+\nu}\equiv\sum_{|\alpha|\leq m}\left(\int_{\Omega}|u|^{\mu}|D^{\alpha}u|^{\nu}dx\right)<+\infty\right\}, \tag{6}$$

$$S_{m,\bar\mu,\bar\nu}(\Omega)\equiv\left\{u(x)\in L_1(\Omega)\middle|[u]_{S_{\bar m}}\equiv\sum_{|\alpha|\leq m}\left(\int_{\Omega}|u|^{\mu_\alpha}|D^{\alpha}u|^{\nu_\alpha}dx\right)^{\frac{1}{\mu_\alpha+\nu_\alpha}}<+\infty\right\}, \tag{7}$$

$$L_p(0,T;S_{m,\bar\mu,\bar\nu}(\Omega))\equiv\left\{u(x,t)\in L_1(Q)\middle|[u]_{L_p(S_{\bar m})}^{p}\equiv\int_0^T[u]_{S_{\bar m}}^{p}dt<+\infty,\quad p\geq 1\right\}, \tag{8}$$

$$L_p(\Omega;S_{1,\mu,\nu}(0,T))\equiv\left\{u(x,t)\in L_1(Q)\middle|[u]_{L_p(S_1)}^{p}\equiv\int_0[u]_{S_1}^{p}dx<+\infty,\quad p\geq 1\right\}, \tag{9}$$

$$\overset{0}{S}_{m,\bar\mu,\bar\nu}(\Omega)\equiv S_{m,\bar\mu,\bar\nu}(\Omega)\cap\left\{u(x)\middle|u|_{\partial\Omega}=0\right\},\ \mu_\alpha\geq 0,\ \nu_\alpha\geq 1,\ m\geq 1,\ |\alpha|\leq m\,, \tag{10}$$

$$\bar S^{l}{}_{m,\bar\mu,\bar\nu}(\Omega)\equiv\left\{u(x)\in L_1(\Omega)\middle|D^{\beta}u\in S_{m,\bar\mu,\bar\nu}(\Omega),\ |\beta|=l\right\}, \tag{11}$$

$$\begin{gathered}S^{k}_{m,\bar\mu,\bar\nu}(\Omega)\equiv\bigcap_{l=0}^{k}\bar S^{\,l}{}_{m,\bar\mu,\bar\nu}(\Omega),\quad k\geq 0\\ \overset{0}{S}{}^{k}_{m,\bar\mu,\bar\nu}(\Omega)\equiv S^{k}_{m,\bar\mu,\bar\nu}(\Omega)\cap\left\{u(x)\middle|D^{\beta}u\big|_{\partial\Omega}=0,\ |\beta|\leq k-1\right\},\quad k\geq 1\end{gathered} \tag{12}$$

here and everywhere later on, the zero from above means that this is a class of functions with zero boundary values (may be together with derivatives up to corresponding order as in Sobolev spaces case), and the zero from below means the same, only on the part of the boundary. Further, $\mu_\alpha \geq 0, \nu_\alpha \geq 0, m \geq 1$ are numbers, $\overline{\mu}, \overline{\nu}$ are vectors, the other vectors are also determined, $\alpha = (\alpha_1, ..., \alpha_n)$ are multi-indices, and the other spaces of type S of functions $u : \Omega \to R^1$ are defined analogously. We are to note that some special cases of reduced spaces were studied in papers [4,5,7] and others.

We also consider the following pn-spaces of functions $v : \Omega \to R^1$

$$S_{1,\mu,\nu}(\Omega) \equiv \left\{ v(x) \in L_1(\Omega) \middle| [v]_S \equiv \sum_{i=1}^{n} \left(\int_\Omega |v|^{\mu_i} |D_i v|^{\nu_i} dx \right)^{\frac{1}{(\mu_i + \nu_i)}} + \|v\|_{L_{\mu+\nu}(\Omega)} < +\infty \right\}. \quad (13)$$

where $\mu_i \geq 0,\ \nu_i \geq 1,\ i = \overline{1,n}$ are numbers, these spaces are equivalent to the spaces of (7) type corresponding to the case $m = 1$. Note that in this case these spaces actually are anisotropic (see [10,12]), in the sense that the derivatives at different directions have the different integrability exponent on the whole domain.

In connection with these spaces we cite a class of anisotropic Sobolev spaces that they are closely connected to. Namely, we shall consider spaces looked like the spaces determined as follows:

$$W_{\overline{p}}^1(\Omega) \equiv \left\{ u(x) \middle| u(x), \frac{\partial u}{\partial x_i} \in L_{p_i}(\Omega), i = \overline{1,n} \right\}, \quad \overline{p} = (p_1, p_2, ..., p_n)$$

(note that such type pn-spaces in this case is for instance, the space determined in (7) for $m = 1$). It is easy to see that the inclusion:

$$W_{\overline{p}}^1(\Omega) \subseteq W_{p_0}^1(\Omega), \ p_0 = \min\left\{ p_i \middle| i = \overline{1,n} \right\}, \ 0 < p_i < \infty$$

holds.

Such spaces have been investigated in papers [10,12] and others, where the imbedding theorem nave been proved and multiplicative inequalities have been obtained, in particular, it is proved the following.

Theorem ([12]). *For each function* $u \in \overset{0}{C}{}^1(\Omega)$ *it holds the inequality*

$$\|u\|_{L_s(\Omega)} \leq c_1 \prod_{i=1}^{n} \left\| \frac{\partial u}{\partial x_i} \right\|_{L_{p_i}(\Omega)}^{1/n}, \quad s = \widetilde{p}^* = \frac{n\widetilde{p}}{n - \widetilde{p}}, \ \ if \ \ \widetilde{p} < n; \frac{1}{\widetilde{p}} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{p_i}, c_1 = c(n, p_i),$$

and if $\widetilde{p} \geq n$ *the inequality holds for each*

$$s \in [1, \infty) \ \ and \ \ c_1 = c(s, mes\Omega).$$

By virtue of known results on the compactness theorems from [10], arguing as in [10], we obtain from this theorem that the following imbedding is compact

$$W_{\overline{p}}^1(\Omega) \subset L_S(\Omega), s < \widetilde{p}^* = \frac{n\widetilde{p}}{n - \widetilde{p}}, \quad for \quad \widetilde{p} > n;$$

other imbedding are formulated analogously.

As we know (see for instance [6,9]), the introduced spaces (6)-(10) and (2) by virtue of general definition are weakly complete pn-spaces with introduced in them p-norms, since the functionals $[\cdot]_s$ determined on them satisfy the corresponding conditions from [6,9], unlike to them, the space (11) is not the same. This space is a kn-space. For completeness reduce the definitions of $kn(pn)$-spaces from paper [5].

Definition 1. *The set S called a quasi-pseudonormed (kn)-space, if S is a topological space and we can associate to each element $x \in S$ the number $[x]_s$ such that*
$N_1)$ $[x]_s \geq 0, \forall x \in S;\ x = 0 \Rightarrow [x]_s = 0$;
$N_3)$ there exists a convex non-negative function $\mu(\lambda)$ such that

$$\lim_{\lambda \to \pm 0} \frac{\mu(\lambda)}{|\lambda|} = c_0; \quad \lim_{\lambda \to \pm\infty} \frac{\mu(\lambda)}{|\lambda|} = c_1, \quad c_i \geq 0,\ i = 0,1 \quad \text{and}$$

$$[\lambda x]_s \leq \mu(\lambda)[x]_s,\ \forall x \in S,\ \forall \lambda \in R^1 \left(\text{where } c_0 = c_1 = 1, \text{ or } c_0 = 0,\ c_1 = +\infty\right).$$

And if it is fulfilled the condition:
$N_2)$ $x = 0 \Leftrightarrow [x]_s = 0,\ [x_1]_s \neq [x_2]_s \Rightarrow x_1 \neq x_2,\ \ x_1, x_2 \in S$,
then S is called a pseudonormed (pn)-space, and the functional $[x]_s$ is called k-norm or p-norm respectively.

Note that using a general definition of pn spaces we get the existence of some mappings of g (may be of vector) and Banach spaces B such that using the mappings g, acting from some topological spaces X to another topological space Y, under some conditions on g and a Banach space $B \subseteq Y$, when $R(g) \cap B \neq \varnothing$, kn-spaces may be defined in the form of (see [5,6]);

$$S \equiv \left\{x \in X \middle| g(x) \in B \subseteq Y\right\} \equiv \left\{x \in X \middle| [x]_s \equiv \|g(x)\|_B < +\infty\right\}. \tag{14}$$

Using the integral properties from papers [4-6,9 and known results from book [10] we get the validity of the following statement (everywhere further - $H(\tau)$ is a Heaviside function).

Proposition 1. *Let the numbers* $\mu_\alpha \geq 0, \nu_\alpha \geq 1,\ \mu'_\beta \geq 0,\ \nu'_\beta \geq 1$ *be such that* $\mu_\alpha / \nu_\alpha = \rho$, $\mu_\alpha + \nu_\alpha = \mu'_\beta + \nu'_\beta$ *for* $\beta = \alpha - 2, \alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ *and* $\mu_\alpha > m - 2$ *for* $m = 1,2,3,4$. *Then the space determined in (5) is equivalent to the following*

$$S_{m,\mu,\nu}(\Omega) \equiv \left\{ u(x) \in L_1(\Omega) \middle| [u]_s \equiv \left(\sum_{|\alpha|=m} \int_\Omega |u|^{\mu_\alpha} |D^\alpha u|^{\nu_\alpha} dx \right)^{\frac{1}{\mu_\alpha + \nu_\alpha}} + \right.$$

$$\left. + H(m-2) \left(\sum_{|\alpha|=m-2} \int_\Omega |u|^{\mu'_\alpha} |D^\alpha u|^{\nu'_\alpha} \right)^{\frac{1}{\mu_\alpha + \nu_\alpha}} + \|u\|_X \right\}, \quad X \subseteq L_1(\Omega),\ X \text{ is a Banach space} \tag{15}$$

Remark 1. In exactly the same way in (10) the space of functions with zero boundary values is determined. As it is shown in above papers for some values of μ and ν, independent on m, when $u|_{\partial\Omega} = 0$, p-norm from the definition (15) is equivalent to the p-norm, not containing the last summand. Besides, we are to note that the space $L_p(0,T; S_{m,\bar{\mu},\bar{\nu}}(\Omega))$ and other spaces of such type are determined analogously.

The introduced spaces generally speaking are nonlinear in the sense that in these spaces at the least the additivity condition is not fulfilled (see, for instance [4,5,6]).

Now, since in the introduced spaces, generally speaking, the additivity condition is not fulfilled and it will be investigated an inhomogeneous problem that requires to use the sum of functions from these spaces it is necessary to mention sufficient conditions under which the function $u(x)$, determined in the form of sums of two functions is contained in the corresponding space S, determined in the form of (6)-(12).

Thus, further, everywhere we shall denote by $u(x)$ (or $u(x,t)$) the function determined in the form of: $u(x)=v(x)-w(x)$ on Ω (or $u(x,t)=v(x,t)-w(x,t)$ on Q) satisfying the condition $u(x)=0$ on $\partial\Omega$ (or $u(x,t)=0$ on $\partial\Omega\times(0,T)$). The set of such functions $u(x)$ (or $u(x,t)$) we shall denote by $U(\Omega)$, where $v(x)$ is some function of one of above spaces, and $w(x)$ is a function from the definite Sobolev space. Note that the boundary value of functions from above determined spaces we shall understand as well as in papers [4,5].

Lemma 1. *Let $\mu_0\ge 0$, $p_0\ge\nu_0\ge 1$ be some numbers. Then there exists a number $\mu_1\ge 0$ such, that if $p\ge\max\{\mu_0+\nu_0+\mu_1, p_0+\mu_1\}$ for $p_0>\nu_0$ or $p\ge n$ for $p_0=\nu_0$, then for some $\mu:\mu_0\le\mu\le\mu_0+\mu_1$, satisfying the inequality:*

$$\mu_0\le\mu\le\min\left\{\frac{np(p_0-\nu_0)}{(n-p)p_0};\frac{p-\nu_0}{p}\max\left\{\frac{(\mu_0+\mu_1+\nu_0)n}{n-\nu_0},\frac{n(\mu_1+p_0)}{n-p_0}\right\}\right\} \tag{16}$$

it is valid implication: from the inclusion

$$v(x)\in S_{1,\mu_0,\nu_0}(\Omega)\cap S_{1,\mu_0+\mu_1,\nu_0}(\Omega)\cap W^1_{p_0}(\Omega),\quad w(x)\in W^1_p(\Omega) \text{ and } u(x)=v(x)-w(x)$$

it follows that $u(x)$ is contained in the space $\overset{0}{S}_{1,\mu,\nu_0}(\Omega)\cap\overset{0}{W}{}^1_{p_0}(\Omega)$.

Proof. In Lemma conditions we have: $u(x)\in L_{\bar p}(\Omega)$. By using the Minkowskian inequality, for $p_0>\nu_0$ we receive the validity of the following inequality:

$$\int_\Omega|u|^{\mu_0}|D_iu|^{\nu_0}dx=\int_\Omega|v-w|^{\mu}|D_i(v-w)|^{\nu_0}dx\le C\left\{\int_\Omega|v|^{\mu}|D_iv|^{\nu_0}dx+\int_\Omega|v|^{\mu}|D_iw|^{\nu_0}dx\right\}+$$
$$+C\left\{\int_\Omega|w|^{\mu}|D_iv|^{\nu_0}dx+\int_\Omega|w|^{\mu}|D_iw|^{\nu_0}dx\right\}\le C\left\{\int_\Omega|v|^{\mu}|D_iv|^{\nu_0}dx+\int_\Omega|v|^{\rho}dx\right\}+$$
$$+C\left\{\int_\Omega|D_iv|^{p_0}dx+\int_\Omega|w|^{\rho_1}dx+\int_\Omega|D_iw|^{p}dx\right\}$$

here the numbers are determined by the expressions: $\rho=\frac{p\mu}{p-\nu_0}$, $\rho_1=\frac{p_0\mu}{p_0-\nu_0}$.

From the Lemma conditions it follows that the right hand side of this inequality is bounded, consequently, and the left hand side of the inequality is bounded. The remaining inequalities necessary for the proof are proved in exactly the same way. By the same the lemma is proved.

Corollary 1. *In conditions of Lemma 1, if $p_0=\nu_0$ and $p\ge n$, then the statement of lemma 1 remains valid for μ satisfying the inequality*

$$\mu_0\le\mu\le\frac{n(\mu_0+\mu_1+\nu_0)(p-\nu_0)}{(n-\nu_0)p}.$$

Remark 2. It is easy to see that by choosing $\mu_1 \geq 0$ sufficiently great, we may increase the number μ.

Lemma 2. *Let $\mu_0 \geq 0$, $p_0 \geq \nu_0 \geq 1$, $p_1 \geq 1$ are some numbers. Then there exists $\mu_1 \geq 0$ such that if $p \geq \max\{p_0 + \mu_1, \mu_0 + \nu_0 + \mu_1, p_1 + \mu_1\}$ for $p_0 > \nu_0$ and $p \geq n$ for $p_0 = \nu_0$, then from the inclusion*

$$v(x,t) \in L_{\mu_0+\nu_0}\left(0,T; S_{1,\mu_0,\nu_0}(\Omega)\right) \cap L_{\mu_0+\nu_0+\mu_1}\left(0,T; S_{1,\mu_0+\mu_1,\nu_0}(\Omega)\right) \cap$$

$$\cap L_{p_0+\mu_1}\left(0,T; S_{1,\mu_1,p_0}(\Omega)\right) \cap L_{p_0}\left(0,T; W^1_{p_0}(\Omega)\right) \cap L^\infty\left(0,T; L_{p_1}(\Omega)\right) \cap L^\infty\left(0,T; L_{p_1+\mu_1}(\Omega)\right),$$

$$w(x,t) \in W^1_p(Q), \quad u(x,t) = v(x,t) - w(x,t), \; u|_\Gamma = v|_\Gamma - w|_\Gamma = 0$$

follows that $u(x,t)$ is contained in the class $L_{\mu+\nu_0}\left(0,T; \overset{0}{S}_{1,\mu,\nu_0}(\Omega)\right) \cap$ $\cap L_{p_0}\left(0,T; \overset{0}{W}_{p_0}(\Omega)\right) \cap L^\infty\left(0,T; L_\mu(\Omega)\right)$ for some $\mu : \max\{\mu_0, p_1\} \leq \mu <$ $< \min\{\mu_0 + \mu_1, p_1 + \mu_1\}$, if μ satisfy the condition (15) of lemma 1.

Proof. Since the remaining inclusions are obvious, show the inclusion $u(x,t) \in L_{\mu+\nu_0}\left(0,T; \overset{0}{S}_{1,\mu,\nu_0}(\Omega)\right)$, and for this, it is sufficient to prove the validity of the corresponding inequality exactly the same as at the proof of lemma 1.

Thus, using lemma 1, we have

$$\int_Q |u|^\mu |D_i u|^{\nu_0} dxdt \leq C \int_0^T \left\{ \int_\Omega |v|^\mu |D_i v|^{\nu_0} dx + \int_\Omega \left(|v|^\rho + |D_i v|^{p_0}\right) dx + \int_\Omega \left(|w|^{\rho_1} + |D_i w|^p\right) dx \right\} dt. \quad (17)$$

Hence, since the boundedness of the first summand at the right hand side is obvious, it remains to show the boundedness of other summands. By virtue of lemma 1 we received that for this purpose it is sufficient to estimate the second summand from the second integral of the right hand side. And we use the following obvious multiplicative inequality $\|v\|_{L_\mu(\Omega)} \leq C_1 \|v\|^\theta_{q_0} \|v\|^{1-\theta}_{L_{q_1}}$, where $q_0 > \rho > q_1$, $\theta = \dfrac{p-\nu}{\rho}$, $q_1 > \nu > 0$ are some numbers. Choosing here $q_0 = \widetilde{p}$, $q_1 \leq p_1 + \mu_1$ so that $\theta\rho \leq \overline{p}$, we obtain:

$$L_p(Q) \subset L_{p_0}\left(0,T; W^1_{p_0}(\Omega)\right) \cap L_{\mu_0+\mu_1+\nu_0}\left(0,T; S_{1,\mu_0+\mu_1,\nu_0}(\Omega)\right) \cap L^\infty\left(0,T; S_{p_1+\mu_1}(\Omega)\right).$$

It is easy to see that for each given $\mu_0 \geq 0$ one can choose such a number: $\mu_1 \geq 0$.

Note that in the case when $\widetilde{p} \leq p_1 + \mu_1$ the boundedness of the right hand side of the inequality (17) directly follows from the conditions of lemma.

Thus, in the case $m=1$ all necessary inequalities have been obtained. Now consider the case $m>1$, in anisotropic case, for whose study the anisotropic Sobolev spaces will be used (see [10]).

Lemma 3. *Let the numbers $p \geq 1$, $\mu_i^\ell \geq 0$, $p_0^\ell > \nu_i^\ell \geq 1$ be such that the condition $p \geq \max\left\{\mu_i^\ell + \nu_i^\ell, p_0^i \middle| 0 \leq |\ell| \leq m, 1 \leq i \leq n\right\}$, and $m = 1,2,3,4$ is fulfilled. Then from the inclusion $v \in S_{m,\overline{\mu},\overline{\nu}}(\Omega) \cap W^m_{p_0}(\Omega)$, $w \in W^m_p(\Omega)$ it follows that for $u(x) = v(x) - w(x)$ from* *the class* $U(\Omega)$ *it is valid the inclusion:* $u \in \overset{0}{S}_{m,\bar{\mu},\bar{\nu}}(\Omega) \cap W^m_{\bar{p}_0}(\Omega)$, *if* $\bar{p}_0, \mu, \nu$ *satisfy the following inequalities:* $\frac{p_0^i \mu_i^l}{p_0^i - \nu_i^l} \le p_1^i$, $\frac{p^i \mu_i^l}{p^i - \nu_i^l} \le p_2^i$, *where the vectors* p_1, p_2 *are such that* $\left|\left(\frac{1}{p} - \frac{1}{p_1}\right) : m\right| \le 1$, $\left|\left(\frac{1}{p_0} - \frac{1}{p_2}\right) : m\right| \le 1$.

Proof. Since from $u(x) \in U(\Omega)$ and the lemma conditions it follows directly that $u(x) \in \in L_{\bar{\mu}+\bar{\nu}}(\Omega) \cap W^m_{\bar{p}_0}(\Omega)$ we are to show the inclusion: $u \in \overset{0}{S}_{m,\bar{\mu},\bar{\nu}}(\Omega)$. For this, as in the proof of lemma 1, it is sufficient to prove the validity of some integral inequalities. In assumption that $u(x) \in U(\Omega)$, cite the basic ones,

$$\int_\Omega |u|^{\mu^l} \left|D^l u\right|^{\nu^l} dx \le \left\{ \int_\Omega |v|^{\mu^l} \left|D^l v\right|^{\nu^l} dx + \int_\Omega |v|^{\mu^l} \left|D^l w\right|^{\nu^l} dx + \int_\Omega |w|^{\mu^l} \left|D^l v\right|^{\nu^l} dx \right\} +$$

$$+ \int_\Omega |w|^{\mu^l} \left|D^l w\right|^{\nu^l} dx \le C \left\{ \int_\Omega |v|^{\mu^l} \left|D^l v\right|^{\nu^l} dx + \int_\Omega \left(|v|^{p_2} + \left|D^l v\right|^{p_0^l} \right) dx \right\} + \tag{18}$$

$$+ C \int_\Omega \left(|w|^{p_1} + \left|D^l v\right|^{p} \right) dx + C_1, \quad 0 \le |l| \le m,\, C > 0,\, C_1 \ge 0 - const.$$

Hence, in conditions of lemma we get that the right hand side is bounded, and consequently the left hand side of this inequality is also bounded. This proves the validity of the lemma.

Remark 3. It is clear that such type lemma is valid and in the case when the spaces are isotropic and $p_0 \le \nu_0$. Note also that by choosing $p \ge 1$ sufficiently large we can always achieve the validity of the inequality (18), those contracts the class of functions for which these lemmas are valid, and we are to establish the greatest class of such functions.

We are to note that as it is seen from the proof for the validity of this lemma, the condition that $S_{m,\bar{\mu},\bar{\nu}}(\Omega)$ is the pn - space is not essential.

Lemma 4. *Let the numbers* $m, \mu_i, \nu_i, p^i, p_0^i, p_3^i, p_4^i$ *satisfy the conditions of lemma 3 and the following inequalities:* $\mu_i > m - 1$; $\nu_i \ge 1$; $p^i, p_0^i, p_3^i, p_4^i \ge 1$, $\bar{p}_3 \ge \mu + \nu$, $p \ge \max\{p_0^i, p_3^i, p_4^i\}$; $m = 1,2,3,4$, $i = 1,\dots,n$. *Then for* $u(x,t) \in U(Q)$ *from inclusion* $w \in W_p^{1,m}(Q)$ *and*

$$v(x,t) \in L_{p_3'}\left(0,T; S_{m,\bar{\mu},\bar{\nu}}(\Omega)\right) \cap L_{p_4'}\left(0,T; W^m_{\bar{p}_4}(\Omega)\right) \cap L^\infty\left(0,T; L_{\bar{p}_0}(\Omega)\right),$$

it yields the validity of the inclusion

$$u(x,t) \in L_{p_3'}\left(0,T; \overset{0}{S}_{m,\bar{\mu},\bar{\nu}}(\Omega)\right) \cap L_{p_4'}\left(0,T; W^m_{\bar{p}_4}(\Omega)\right) \cap L^\infty\left(0,T; L_{\bar{p}_0}(\Omega)\right),\ p_3', p_4' > 1.$$

The proof is carried out analogous to the proof of lemma 3, apply this lemma, i.e. for the proof it is necessary to show the validity of corresponding integral inequalities.

Now cite some results on renormalization in pn - spaces. Consider two definition types reduced in papers [4, 5, 6], where in an isotropic case some sufficient condition were obtained for the equivalence of pn - spaces, determined by these two ways. These definitions are the following:

$$S_{m,\mu,\nu}(\Omega)\equiv\left\{u\in L_1(\Omega)\,\middle|\,[u]_S\equiv\sum_{|\alpha|=m}\left(\int_\Omega |u|^{\mu}|D^\alpha u|^{\nu}dx\right)^{\frac{1}{\mu+\nu}}+\right.$$
$$\left.+\sum_{|\beta|\le m-1}\left(\int_\Omega |u|^{\mu'_\beta}|D^\alpha u|^{\nu'_\beta}dx\right)^{\frac{1}{\mu'_\beta+\nu'_\beta}}<\infty\right\},\tag{19}$$

$$\underline{S}_{m,\mu,\nu}(\Omega)\equiv\left\{u\in L_1(\Omega)\,\middle|\,[u]_S\equiv\sum_{|\alpha|=m}\left(\int_\Omega |u|^{\mu_m}|D^\alpha u|^{\nu_m}dx\right)^{\frac{1}{\mu_m+\nu_m}}+\right.$$
$$\left.+H(m-2)\sum_{|\beta|=m-2}\left(\int_\Omega |u|^{\mu_{m-2}}|D^\beta u|^{\nu_{m-2}}dx\right)^{\frac{1}{\mu_{m-2}+\nu_{m-2}}}+\sum_{|\gamma|\le m-2}\left(|u|^{\mu_{|\gamma|}}|D^\gamma u|^{\nu_{|\gamma|}}dx'\right)^{\frac{1}{\mu_{|\gamma|}+\nu_{|\gamma|}}}<\infty\right\}\tag{20}$$

Thus, we have: since the p - norms of these spaces under some correspondences between the degrees are equivalent (see [4, 5, 6]), and consequently these spaces are equivalent, and since we shall consider here only such cases, therefore denote them equally. Note that for the equivalence of spaces determined in (19) and (20) it is sufficient to suppose that the equalities: $\beta\equiv\alpha-1$, $\gamma=\alpha-(m-|\gamma|)$, $\mu_\alpha+\nu_\alpha=\mu_\beta+\nu_\beta=\mu_\gamma+\nu_\gamma$ are fulfilled, and at the second definition $\nu_\beta=2\nu_\alpha$, $\nu_\gamma=(m-|\gamma|+1)\nu_\alpha-1$.

Further, it is shown that the constructed spaces $S_{m,\mu,\nu}(\Omega)$ are contained in spaces $S_{gW_\nu^m(\Omega)}$ with $g_\alpha(u)\equiv|u|^{\mu_\alpha/\nu_\alpha}u$. In the case when the condition $u|_{\partial\Omega}=0$ is fulfilled on the boundary the expression for the p - norm is considerably simplified, since in particular all boundary integrals in definition (20) are annihilated, for instance in the case of the space $\overset{0}{S}_{3,\mu,\nu}(\Omega)$ (see [11]).

Cite some integral inequalities that are applied also in the proof of equivalence of p - norm.

Lemma 5. *Let $\rho_0\ge0,\ \rho_2\ge\rho_1\ge0,\ \rho_2\ge2$ or $\rho_0\ge0,\ \rho_2\ge\rho_1\ge1$. Then for any $u(x)\in C^2(\Omega)$ the inequalities*

$$\int_\Omega |u|^{\rho_0}|D_i u|^{\rho_1+\rho_2}dx\le C\int_\Omega |u|^{\rho_0+\rho_1}|D_i^2u|^{\rho_2}dx+C_1\int_{\partial\Omega}\left[|u|^{\rho_0+\rho_1+\rho_2}+|u|^{\rho_0+1}|D_i u|^{\rho_1+\rho_2-1}\right]dx';$$

$$\int_\Omega |u|^{\rho_0}|D_i u|^{\rho_1+\rho_2}dx\le C\int_\Omega |u|^{\rho_0+\rho_1}|D_iD_ju|^{\rho_2}dx+$$
$$+C_1\int_{\partial\Omega}\left[|u|^{\rho_0+\rho_1+\rho_2}+|u|^{\rho_0+1}|D_i u|^{\rho_1+\rho_2-1}\right]dx';\ i,j=\overline{1,n,...},\ C,C_1>0-const$$

are valid.

For the proof sees [2, 5, 6].

Note that in the last two papers such type inequalities were also proved in the case when integrand expressions both at the left and at the right hand side of inequalities contain the derivatives and of the higher order.

For the completeness of the presentation, and also for the best understanding of further statements cite some of above inequalities from papers [9, 11] in the form of integral inequalities as lemma 5.

Lemma 6. *Let* $\rho_0 \geq \rho_1 + \rho_2 \geq 2$, $\rho_2 \geq 2\rho_1 \geq 0$, $\rho_0 \geq \max\left\{\frac{\rho_1^2}{\rho_2 - 1} + \frac{\rho_1\rho_2}{\rho_2 - \rho_1},\ \rho_1\frac{\rho_1 + \rho_2}{\rho_2 - \rho_1}\right\}$. *Then for any* $u(x) \in C^3(\Omega)$ *the inequalities*

$$\int_\Omega |u|^{\rho_0}\left|D_i^2 u\right|^{\rho_1+\rho_2} dx \leq C\left\{\int_\Omega |u|^{\rho_0+\rho_1}\left|D_i^3 u\right|^{\rho_2} dx + \int_\Omega |u|^{\rho_0-\rho_1-\rho_2} + \left|D_i u\right|^{2(\rho_1+\rho_2)} dx + \right.$$

$$\left. + \int_{\partial\Omega} |u|^{\rho_0}\left(|u|^{\rho_1+\rho_2}\left|D_i u\right|^{\rho_1+\rho_2} + \left|D_i^2 u\right|^{\rho_1+\rho_2}\right) dx'\right\},$$

$$\int_\Omega |u|^{\rho_0}\left|D_i D_j D_k u\right|^{\rho_1+\rho_2} dx \leq C(\varepsilon)\int_\Omega |u|^{\rho_0+\rho_1}\left|D_i^2 D_j D_k u\right|^{\rho_2} dx +$$

$$+ \varepsilon\int_\Omega |u|^{\rho_0+\rho_1-\rho_2}\left|D_i^2 u\right|^{2\rho_1} dx + \left\{\int_{\partial\Omega} |u|^{\rho_0+\rho_1-\rho_2}\left|D_j D_k u\right|^{2\rho_2} dx' + \int_{\partial\Omega} |u|^{\rho_0}\sum_{|\beta|\leq 3}\left|D^\beta u\right|^{\rho_1+\rho_2} dx'\right\}$$

$$D^\beta = D^{\beta_1}D^{\beta_2}D^{\beta_3},\ |\beta| = \sum_{i=1}^{3}\beta_i,\ \varepsilon > 0 \text{ is some number,}$$

are valid.

The proof is carried out by using the integration on parts, the Young inequality, and known inequalities in exactly the same way as at the proof of lemma 5 (i.e. as in papers [2, 5, 6]).

In papers [9, 11] some equivalence results are proved for the case $m = 2$. To show the above-equivalence we use the renormalization in Sobolev spaces (see [1, 10]). As we know [1], it is valid the relation: $\|v\|_{W_p^m(\Omega)} \cong \|v\|_{L_p(\Omega)} + \sum_{|\alpha|=m}\left\|D^\alpha v\right\|_{L_p(\Omega)}$ from which the validity of the following statements yields.

Lemma 7. *Let the above mentioned conditions on the parameters* μ *u* ν *be fulfilled. Then if to define* g *by the expression:* $g(u) \equiv |u|^{1/\nu}u$, *then the spaces determined in (19) and (20) are equivalent to the following:*

$$S_{gW_\nu^3} \equiv \left\{u \in L_1(\Omega)\left|[u]_{S_{3,\mu,\nu}}^{\mu+\nu} \equiv \|u\|_{L_{\mu+\nu}}^{\mu+\nu} + \sum_{|\alpha|=3}\left\|D^\alpha g(u)\right\|_{L_\nu}^{\nu} < \infty\right.\right\}. \tag{21}$$

Proof. It is known from the Sobolev's space theory that the inclusion: $W_p^l(\Omega) \subseteq W_{p_1}^{l_1}(\partial\Omega)$, $p_1 \leq \frac{p(n-1)}{n - p(l - l_1)}$ holds. By using this fact and the inequalities from papers [5,6,9,11] of the inequality type of lemmas 5 and 6 we obtain the inequality among the p-norms to one side, the inequalities from indicated papers and the following type equality

$$\int_\Omega |u|^{\rho_0}\left|D_i^2 u\right|^{\rho_1+\rho_2} dx = -(\rho_1 + \rho_2 - l)\int_\Omega |u|^{\rho_0} D_i u\left|D_i^2 u\right|^{\rho_2+\rho_1-2} D_i^3 u dx -$$

$$-\rho_0\int_\Omega |u|^{\rho_0-2}u(D_iu)^2D_i^2u\left|D_i^2u\right|^{\rho_1+\rho_2-2}dx+\int_{\partial\Omega}|u|^{\rho_0}D_iuD_i^2u\left|D_i^2u\right|^{\rho_1+\rho_2-2}dx'$$

admits to show the inequality and to the other side. Hence the validity of lemma 7 yields.

In the anisotropic case the lemmas of lemma 7 type may be formulated in the form of

Lemma 7'. *Let the numbers $\mu_i\geq 0, \nu_i\geq 1$ be such that $\rho_0=\mu_i/\nu_i\geq 0, i=1,...,n$. Then there exists the one-to-one mapping of the space $S_{1,\bar\mu,\bar\nu}(\Omega)$ into the anisotropic space $W^1_{\bar\nu}(\Omega)$ moreover, if we denote by g the mapping determined by the expression $g(u)\equiv|u|^{\rho_0}u$, then the equality $S_{1,\bar\mu,\bar\nu}(\Omega)=S_{gW^1_{\bar\nu}(\Omega)}$, holds, where $\bar\mu$ and $\bar\nu$ are vectors*

Further, cite some results related with interpolational properties of kn-spaces that are also proved by applying Young inequality and known inequalities from papers [5,6,9,11] of type of inequalities from lemmas 5 and 6.

Theorem 1. *Let $\rho_0\geq 0, p>\gamma_0\geq 1$ be some numbers, and $\mu\geq 0, \nu\geq 1$ are such that, $p\geq\nu\geq\gamma_0, \mu\geq\rho_0$. Then there exists such $\rho\geq 0$ that $\rho_0+\rho>\mu\geq\rho_0$ and the inclusions $S_{1,\rho_0,\gamma_0}(\Omega)\cap S_{1,\rho_0+\rho,\gamma_0}(\Omega)\cap S_{1,\rho,p}(\Omega)\cap W^1_p(\Omega)\subseteq S_{1,\mu,\nu}(\Omega)$,*

$$L_{p_0}\left(0,T;S_{1,\rho_0,\gamma_0}(\Omega)\right)\cap L_{p_{01}}\left(0,T;S_{1,\rho_0+\rho,\gamma_0}(\Omega)\right)\cap L_{p_2}\left(0,T;S_{1,\rho,p}(\Omega)\right)\cap$$
$$\cap L_p\left(0,T;W^1_p(\Omega)\right)\subseteq L_{p_1}\left(0,T;S_{1,\mu,\nu}(\Omega)\right)$$

where $p_0=\rho_0+\gamma_0, p_{01}=\rho_0+\rho+\gamma_0$, $p_1=\mu+\nu$, $p_2=\rho+p$ are valid.

Corollary 2. *Let $\rho_0,\mu\geq 0, \gamma_0\geq 1$ are some numbers. Then, there exists $\rho\geq 0$ such that $\rho_0+\rho>\mu\geq\rho_0$ and the inclusions*

$$S_{1,\rho_0,\gamma_0}(\Omega)\cap S_{1,\rho_0+\rho,\gamma_0}(\Omega)\subseteq S_{1,\mu,\gamma_0}(\Omega)$$
$$L_{p_0}\left(0,T;S_{1,\rho_0,\gamma_0}(\Omega)\right)\cap L_{p_{01}}\left(0,T;S_{1,\rho_0+\rho,\gamma_0}(\Omega)\right)\subseteq L_{p_1}\left(0,T;S_{1,\mu,\gamma_0}(\Omega)\right),$$

where $p_0=\rho_0+\gamma_0, p_{01}=\rho_0+\rho+\gamma_0$, $p_1=\mu+\gamma_0$ are valid.

It is easy to see that these results remain valid and in the anisotropic case, i.e. when $\rho_0,\mu,\rho,p,\nu,\gamma_0$ are vectors and satisfy some conditions as in lemma 7' for brevity we shall not cite this case separately.

Introduce the denotations:

$$\mathbf{P}_{1,\bar\mu,\bar\nu,\rho,\bar\rho,\bar\gamma}(Q)\equiv L_p\left(0,T;S_{1,\bar\mu,\bar\nu}(\Omega)\right)\cap L_{p_1}\left(0,T;S_{1,\bar\mu+\rho,\bar\nu}(\Omega)\right)\cap$$
$$\cap\left\{u(x,t)\,\middle|\,|u|^{\rho_0}u\in W^1_{\bar\gamma}(0,T;B)\right\},\ L_{\bar\gamma}(\Omega)\subseteq B$$

$$\underset{0}{\mathbf{P}}_{1,\bar\mu,\bar\nu,\rho,\bar\rho,\bar\gamma}(Q)\equiv L_p\left(0,T;\overset{0}{S}_{1,\bar\mu,\bar\nu}(\Omega)\right)\cap L_{p_1}\left(0,T;\overset{0}{S}_{1,\bar\mu+\rho,\bar\nu}(\Omega)\right)\cap\left\{u(x,t)\,\middle|\,|u|^{\rho_0}u\in W^1_{\bar\gamma}(0,T;B)\right\},$$

where $\bar\mu,\bar\nu,\bar\rho,\bar\gamma$ are n-dimensional vectors, $p\geq\bar\mu+\bar\nu$, $p_0=\bar\rho+\bar\gamma$, $p_1=p+\rho$, and $\rho,\rho_0\geq 0$ are some numbers $\rho_0={}^{\rho_i}\!/_{\gamma_i}$, $i=\overline{1,n}$.

Theorem 2. *Let $\mu_i,\rho_i,\rho_0\geq 0$, $\nu_i,\gamma_i\geq 1$, $\mu_i\leq\rho_i\leq\mu_i+\rho_0$, $i=\overline{1,n}$ are some numbers. Then there exist numbers $\rho,\eta_i\geq 0$, $\lambda_i\geq 1$ such that*

$$\rho_i\geq\rho\gamma_i,\ \ \rho\lambda_i+\lambda_i\leq\rho_i+\gamma_i,\ \mu_i\leq(\rho+1)\eta_i+\rho\nu_i\leq\mu_i+\rho,\ p_i=\eta_i+\nu_i$$

and it is valid the implication (for $\mathbf{B}\equiv L_{\bar\gamma}(\Omega)$)

$$u(x,t)\in \mathbf{P}_{1,\overline{\mu},\overline{\nu},\rho_0,\overline{p},\overline{\gamma}}(Q)\Rightarrow |u|^{p}u\in L_{\overline{p}}\left(0,T;S_{1,\overline{\eta},\overline{\nu}}(\Omega)\right)\cap W^{1}_{\overline{\lambda}}\left(0,T;L_{\overline{\lambda}}(\Omega)\right). \qquad (22)$$

Moreover, if there additionally exist numbers $\eta_0\ge 0,\sigma_i:\nu_i\ge\sigma_i>1$ *such that* $\frac{\eta_i}{\nu_i}\le\eta_0$, $\sigma_i(\eta_0+1)\le\eta_i+\nu_i$, $k_i=\frac{\mu_i+\nu_i}{(\eta_0+1)}$; $\left|\sum_{i=1}^{n}\left(\frac{1}{\sigma_i}-\frac{1}{k_i}\right)\right|<1$, *then it is compact the imbedding:* $\mathbf{P}_{1,\overline{\mu},\overline{\nu},\rho_0,\overline{p},\overline{\gamma}}(Q)\subset L_{p_1}(Q)$, $p_1^i=p_i(\rho+1)$, $i=\overline{1,n}$.

Proof. The right hand side of the statement follows from the following arguments. We have from the anisotropic variable of theorem 1 that
$L_p\left(0,T;S_{1,\overline{\mu},\overline{\nu}}(\Omega)\right)\cap L_{p_1}\left(0,T;S_{1,\overline{\mu}+\rho,\overline{\nu}}(\Omega)\right)\subseteq L_{p_0}\left(0,T;S_{1,\overline{\mu}^0,\overline{\nu}}(\Omega)\right)$, where $p_i^0=\mu_i^0+\nu_i$, μ_i^0: $\mu_i\le\mu_i^0\le\mu_i+\rho_0$ are some numbers. We have also

$$L_p\left(0,T;S_{1,\overline{\mu},\overline{\nu}}(\Omega)\right)\cap L_{p_1}\left(0,T;S_{1,\overline{\mu}+\rho,\overline{\nu}}(\Omega)\right)\subseteq L_{p_2}\left(0,T;S_{1,\overline{\rho},\overline{\gamma}}(\Omega)\right), p_2=\overline{\rho}+\overline{\gamma},\ \gamma_i\le\nu_i.$$

Now the validity of the inclusion (22) follows from the conditions of the theorem by virtue of the known inequalities of papers [5,6].

Thus, to obtain the last statement it is sufficient to use the imbedding compactness theorem from papers [6,9,10,12] only by applying the imbedding compactness theorem for anisotropy Sobolev's spaces.

Now cite some results belonging to spaces of type determined in (13).

Theorem 3. *Let numbers* $\mu_i\ge 0$, $\nu_i\ge 1$, $i=1,2,\ldots,n$ *be such, that there exist numbers* $\rho_0\ge 0$, $q_1>1$, *satisfying the relations* $\mu_i+\nu_i\ge q_i(\rho_0+1),\rho_0\ge \mu_i/\nu_i$.

Then, the imbedding $S_{1,\overline{\mu},\overline{\nu}}(\Omega)\subset L_\mu(\Omega)$ *is compact, where*

$$s<\tilde{q}^{*}=\frac{n\tilde{q}}{n-\tilde{q}},\ \text{if}\ \tilde{q}<n;\ \frac{1}{\tilde{q}}=\frac{1}{n}\sum_{i=1}^{n}\frac{1}{q_i};\ \infty>s\ge 1-\forall,\ \text{if}\ \tilde{q}>n,\ \text{but}\ \mu=s(\rho_0+1).$$

By virtue of known results from [6,7,9] and above reduced theorem from [12], the proof follows from

Lemma 8. *Let numbers* $\overline{\mu}_i\ge 0$, $\overline{\nu}_i\ge 1$ *be such, that* $\frac{\overline{\mu}_i}{\overline{\nu}_i}=\rho_0\ge 0$, $i=\overline{1,n}$. *Then there exists the one-to-one mapping of the space* $S_{1,\overline{\mu},\overline{\nu}}(\Omega)$ *into the anisotropic Sobolev space* $W^1_{\overline{\nu}}(\Omega)$, *moreover, using the space* $W^1_{\overline{\nu}}(\Omega)$ *and the mapping* $g: g(u)\equiv|u|^{\rho_0}u$, *the space* $S_{1,\overline{\mu},\overline{\nu}}(\Omega)$ *may be defined (by general definition of* pn*-spaces from [6,9]) in the form of* $S_{gW^1_{\overline{\nu}}(\Omega)}$, *exactly,*

$$S_{1,\overline{\mu},\overline{\nu}}(\Omega)\equiv\left\{u(x)\in L_1(\Omega)\,\middle|\,g(u)\in W^1_{\overline{\nu}}(\Omega)\right\}\equiv S_{gW^1_{\overline{\nu}}(\Omega)}.$$

Proof. In conditions of the lemma it is easy to see that from $u(x)\in S_{1,\overline{\mu},\overline{\nu}}(\Omega)$, it follows $|u|^{\rho_0}u\in W^1_{\overline{\nu}}(\Omega)$ and conversely, from $\mathrm{v}\in W^1_{\overline{\nu}}(\Omega)$ it follows that

$$g^{-1}(v)\equiv|v|^{\frac{-\rho_0}{\rho_0+1}}v\in S_{1,\overline{\mu},\overline{\nu}}(\Omega).$$

Now show the equality of spaces $S_{1,\overline{\mu},\overline{\nu}}(\Omega)$ and $S_{gW^1_{\overline{\nu}}(\Omega)}$, and for this it is sufficient to consider the p-norms determined in these spaces. We have:

$$[u]_{S_{1,\bar\mu,\bar\nu}} \equiv \sum_{i=1}^{n}\left(\int_\Omega |u|^{\bar\mu_i}|D_i u|^{\bar\nu_i}dx\right)^{\frac{1}{\bar\mu_i+\bar\nu_i}} + \|u\|_{L_{\bar\mu+\bar\nu}(\Omega)},$$

and

$$[u]_{S_{gW^1_{\bar\nu}}} \equiv \|g(u)\|_{W^1_{\bar\nu}} \equiv (\rho_0+1)\sum_{i=1}^{n}\left(\int_\Omega |u|^{\bar\mu_i}|D_i u|^{\bar\nu_i}dx\right)^{1/\bar\nu_i} + \|u\|_{L_{\bar\mu+\bar\nu}(\Omega)}$$

and hence the last statement of the lemma follows.

The proof of theorem 3. We have from the imbedding compactness theorem for anisotropic Sobolev space ([10]) and Troisi theorem ([12]) that $W^1_{\tilde\nu}(\Omega)\subset L_\nu(\Omega)$ is compact, if $\nu\geq 1$ satisfy the inequality $\nu_i<\frac{\tilde\nu n}{n-\tilde\nu}$ for $\tilde\nu<n$ where $\frac{1}{\tilde\nu}=\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\nu_i}$ etc. (In particular, this case holds in the case $\nu_i=\nu_0$, $i=\overline{1,n}$). Then by using the homomorphism of the mapping determined by the expression $g:g(u)\equiv|u|^{\rho_0}$ and by virtue of known results from [7,9], that the inclusion $S_{1,\mu,\nu}(\Omega)\subset S_{1,\bar\mu,\bar\nu}(\Omega)$ is valid, we get the validity of the statement of theorem 3. By the same theorem 3 is proved.

The validity of the following statement follows from lemma 8.

Corollary 3. *In conditions of lemma 1, the space* $S_{1,\bar\mu,\bar\nu}(\Omega)$ *is weak complete.*

Analogously, using lemma 1 it is proved the following more general

Theorem 3'. *Let numbers* $\mu_i\geq 0,\ \nu_i\geq 1,\ i=1,2,\dots,n$ *be such that, there exist numbers* $\rho_0\geq 0,\ q_1\geq 1$, *satisfying the relations* $\mu_i+\nu_i\geq q_1(\rho_0+1)$, $\rho_0\geq\frac{\mu_i}{\nu_i}$ *for* $\eta_i=\frac{\mu_i+\nu_i}{\rho_0+1}$ *the inequality* $\left(\sum_{i=1}^{n}\left(\frac{1}{q_i}-\frac{1}{\eta_i}\right)\right)<1$ *is fulfilled.*

Then the imbedding $S_{1,\mu,\nu}(\Omega)\subset L_{\mu+\nu}(\Omega)$ is compact.

Theorem 4. *Let numbers* $\mu_i\geq 0,\ \nu_i\geq\bar\nu_i\geq 1,\ \rho_0\geq 0$ *be such that*

$$\frac{\mu_i}{\nu_i}\leq\rho_0,\ \mu_i+\nu_i\geq\bar\nu_i(\rho_0+1),\ i=\overline{1,n};\ \left|\sum_{i=1}^{n}\left(\frac{1}{\bar\nu_i}-\frac{1_i}{\eta_{l_i}}\right)\right|\leq 1,\ \frac{p_i}{\rho_0+1}=\eta_{l_i},\ p=\mu+\nu,\ i=\overline{1,n}.$$

Then for $p_0=\bar\nu(\rho_0+1)$ *the inclusion*

$$\mathbf{P}_{p,\bar\nu}\left(0,T;\overset{0}{S}_{1,\mu,\nu}(\Omega),W_\nu^{-1}(\Omega)\right)\subset L_{p_0}\left(0,T;L_p(\Omega)\right)\ \textit{is compact.}$$

Proof. It follows from the conditions of the theorem that numbers $\bar\mu_i\geq 0,\ \bar\nu_i\geq 1$ may be chosen so that for some numbers $\rho_0\geq 0,\ \eta_i\geq 1$ be fulfilled the relations $\bar\mu_i=\rho_0\nu_i, p_i=\mu_i+\nu_i=\bar\nu_i\eta_i(\rho_0+1)$. Then if follows from theorem 3 that $S_{1,\bar\mu,\bar\nu}(\Omega)\subset L_{\bar\mu+\bar\nu}(\Omega)$ is compact, and since $S_{1,\mu,\nu}(\Omega)\subset S_{1,\bar\mu,\bar\nu}(\Omega)$ we received that the imbedding $S_{1,\mu,\nu}(\Omega)\subset L_{\mu+\nu}(\Omega)$ is also compact.

Further, by definition of the space $\mathbf{P}_{\rho_0,\bar\nu_0}\left(0,T;S_{1,\mu,\nu}(\Omega),W_\nu^{-1}(\Omega)\right)$ we have:

$$\mathbf{P}_{p_0,\bar\nu_0}\left(0,T;S_{1,\mu,\nu}(\Omega),W_\nu^{-1}(\Omega)\right)\equiv L_{p_0}\left(0,T;S_{1,\mu,\nu}(\Omega)\right)\cap W^1_{\bar\nu_0}\left(0,T;W_\nu^{-1}(\Omega)\right).$$

Then, using the compactness result (see [1,5,9]) we get the validity of the statement of theorem 4, i.e. the imbedding

$$\mathbf{P}_{p_0,\bar{v}_0}\left(0,T;S_{1,\mu,\nu}(\Omega),W_\nu^{-1}(\Omega)\right)\subset L_{p_0}\left(0,T;L_p(\Omega)\right) \text{ is compact.}$$

Lemma 9. *Let numbers* $\mu_i \ge 0,\ \nu_i \ge 1$ *be such that* $\frac{\mu_i}{\nu_i}\le \rho_0,\ p_i \equiv \equiv \mu_i+\nu_i=\bar{\mu}_i(\rho_0+1)$. *Then for* $p_0=\min\left\{p_i \middle| i=\overline{1,n}\right\},\ p_0^* < \dfrac{n(\rho_0+1)}{\left(\sum\limits_{i=1}^{n}\frac{1}{\bar{\nu}_i}-1\right)}$ *is valid the inclusion*

$$L_{p_0}\left(0,T;\overset{0}{S}_{1,\mu,\nu}(\Omega)\right)\cap L^\infty\left(0,T;L_2(\Omega)\right)\subset L_{p_0}\left(0,T;L_{p_0^*}(\Omega)\right).$$

Proof. By virtue of lemma 8 in conditions of lemma 9 it is valid the inclusion $\overset{0}{S}_{1,\mu,\nu}(\Omega)\subseteq \overset{0}{S}_{1,\bar{\mu},\bar{\nu}}(\Omega)$, $1\le \bar{\nu}_i \le \nu_i$, $\bar{\mu}_i=\rho_0\bar{\nu}_i$. Then for $g(u)\equiv |u|^{\rho_0}u$ we have: $u\in \overset{0}{S}_{1,\mu,\nu}(\Omega)\Rightarrow g(u)\in \overset{0}{W}{}_{\bar{\nu}}^{1}(\Omega)$.

Now, using the imbedding theorem for anisotropic Sobolev spaces ([10]) we get that $|u|^{\rho_0}u\in L_{\tilde{\beta}^*}(\Omega)$, $\beta_i^*\ge\tilde{\beta}_i^*\ge 1$, for $\left|\sum\limits_{i=1}^{n}\left(\frac{1}{\bar{\nu}_i}-\frac{1}{\tilde{\beta}_i^*}\right)\right|=1$. Hence, in particular, by choosing $\beta_0^*=\min\left|\beta_i^*\middle| i=\overline{1,n}\right|$ we get that $u\in L^\infty(0,T;L_2(\Omega))\cap L_{p_0}\left(0,T;L_{\beta_0^*(\rho_0+1)}(\Omega)\right)$.
Consequently, $u\in L_{p_0}\left(0,T;L_{p_0^*}(\Omega)\right)$, Q.E.D.

The validity of the next statement directly follows from lemma 9.

Corollary 4. *Let the conditions of lemma 9 be fulfilled and* $\nu_0=\min\left\{\nu_i\middle| i=\overline{1,n}\right\}$. *Then for* $\nu_0\ge\dfrac{1}{\rho_0+1}\max\left\{2,3-\dfrac{3}{n}\right\}$ *it is valid the inclusion*

$$L_{p_0}\left(0,T;\overset{0}{S}_{1,\mu,\nu}(\Omega)\right)\cap L^\infty(0,T;L_2(\Omega))\subset L_{\tilde{p}_0}(Q),$$

$$p=\nu_0(\rho_0+1),\rho_0\ge 0,\tilde{p}_0=p_0\left(3+\frac{2}{n-\nu_0}\right).$$

The analogous theorem of theorem 4 type for the anisotropic case has the following form

Theorem 4'. *Let numbers* $\mu_i\ge 0,\ \nu_i\ge 1,\ \rho_0\ge 0$ *be such that*

$$\frac{\mu_i}{\nu_i}\le\rho_0,\mu_i+\nu_i\ge\bar{\nu}_i(\rho_0+1),\ \frac{1}{\tilde{\nu}}=\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\bar{\nu}_i},\ \frac{p_i}{\rho_0+1}<\frac{n\tilde{\nu}}{n-\tilde{\nu}},\ p_i\ge 1, i=\overline{1,n}.$$

In particular, it holds $p_i=\mu_i+\nu_i$, $i=\overline{1,n}$.

Then for $p_0=\bar{\nu}(\rho_0+1)$ *the inclusion*

$$\mathbf{P}_{p_0,\bar{\nu}}\left(0,T;\overset{0}{S}_{1,\mu,\nu}(\Omega),W_{\bar{\nu}}^{-1}(\Omega)\right)\subset L_{p_0}\left(0,T;L_p(\Omega)\right) \text{ is compact.}$$

It follows from theorem 4', theorem from [12], and lemma 9, that it holds

Corollary 5. *Let numbers* $\mu_i \geq 0,\ \nu_i > 1$ *be such that*

$$\frac{\mu_i}{\nu_i} \leq \rho_0,\ p_i \equiv \mu_i + \nu_i = \bar{\nu}_i(\rho_0 + 1),\ \rho_0 \geq 0.$$

Then for

$$p_0 = \min\{p_i \mid i = \overline{1,n}\},\ \tilde{p}^* \leq \frac{n(\rho_0+1)\tilde{\beta}}{n-\tilde{\beta}},\ \frac{1}{\tilde{\beta}} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\bar{\nu}_i},\ \bar{\nu}_i > 1$$

it is valid the inclusion

$$L_{p_0}\left(0,T;\overset{0}{S}_{1,\mu,\nu}(\Omega)\right) \cap L^\infty(0,T;L_2(\Omega)) \subset L_{\tilde{p}^*}\left(0,T;L_{\tilde{p}^*}(\Omega)\right).$$

Remark 4. Such type theorems are also valid for the spaces $S_{m,\bar{\mu},\bar{\nu}}(\Omega)$, $P_{m,\bar{\mu},l\bar{\nu},\rho_0,\bar{\rho},\bar{\gamma}}(Q)$ for $m = 2,3,4$, and the remaining parameters satisfy the conditions, analogous to the conditions of theorem 2 and above mentioned conditions for the equivalence of definitions of pn-spaces with corresponding alternations, depending on m. We shall not cite these theorems here.

2. Solvability theorem of the problem (1)-(3).

Consider the problem (1)-(3) and assume that $\Omega \subset \Re^n (n \geq 1)$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$, $Q \equiv \Omega \times (0,T), T > 0$ and $\rho \geq 0$ are some numbers. Assume that $B_i(x,t,\xi) = 0,\ i = \overline{1,n}$. Let the following conditions be fulfilled:

1) the functions $A_i(x,t,\xi,\zeta),\ i = \overline{1,n}$ are Karatheodory functions and there exist functions $a_i(x,t,\xi),\ b_i(x,t,\xi,\zeta)$ such that the representations

$$A_i(x,t,\xi,\zeta) \equiv a_i(x,t,\xi)\zeta + b_i(x,t,\xi,\zeta),\ \forall(x,t,\xi,\zeta) \in Q \times R^{n+1}$$

2) there exist constants $a_0, b_0, B_0, A_0, c_0, c_1 \geq 0, \mu_i, \sigma_i \geq 0$ such that

$$a_0|\xi_i|^{\mu_i} \leq a_i(x,t,\xi) \leq A_0|\xi_i|^{\mu_i},\ a_0 > 0,\ i = \overline{1,n},$$

$$\sum_{i=1}^{n} b_i(x,t,\xi,\zeta)\zeta_i \geq b_0\left(\sum_{i=1}^{n}|\zeta_i|^{\sigma_i+2} - |\xi_i|^{\sigma_i'+2}\right) - B_0,\ 0 < b_0,\ 0 \leq \sigma_i' < \sigma_i;$$

$$|b_i(x,t,\xi,\zeta)| \leq c_0\left(|\zeta_i|^{\sigma_i+1} + |\xi_i|^{\sigma_i'+1}\right) + c_1,\ 0 < c_0,\ 0 \leq \sigma_i' < \sigma_i,\ i = \overline{1,n}$$

3) the functions $b_i(x,t,\xi,\zeta)$ are such that for any $\zeta, \zeta' \in R^n$ and $(x,t) \in Q,\ \xi \in R^1$ the inequality

$$\sum_{i=1}^{n}[b_{it}(x,t,\xi,\zeta) - b_i(x,t,\xi,\zeta')](\zeta_i - \zeta_i') \geq 0.$$

As is shown in papers [6,8], the summand with $b_i(x,t,\xi,\zeta)$ in conditions 1-3 generates a pseudo monotone operator (more exactly, a variational calculation operator), acting from $L_{p_1}\left(0,T;\overset{0}{W}{}^1_{p_1}(\Omega)\right)$ in $L_{q_1}\left(0,T;W^1_{q_1}(\Omega)\right)$, where $p_1^i = \sigma_i + 2, q_1 = p_1'$ (more exactly for almost all t from $\overset{0}{W}{}^1_{p_1}(\Omega)$ in $W^{-1}_{q_1}(\Omega)$), and the summand with $a_i(x,t,\xi)$ in these conditions generates a weak compact operator from $P_{1,\bar{\mu},q_0}\left(0,T;\overset{0}{S}_{1,\bar{\mu},q_0}(\Omega),W_{q_0}^{-1}(\Omega)\right)$, $p_0^i=\mu_i+2$, $q_0=p_0'$ in $L_{q_0}\left(0,T;W_{q_0}^{-1}(\Omega)\right)$.

4) the function (x,t,ξ) is a Karatheodory function, and there exists a constant $\omega\geq 0$ and functions $f_0\in L^\infty(Q), f_1\in L_q(Q), \hat{p}=\min\{p,p_0,p_1\}$, $q\geq\hat{p}$, $\omega\leq\hat{p}-1$ such that the inequalities $|f(x,t,\xi)|\leq f_0(x,t)|\xi|^\omega+f_1(x,t)$, or $f(x,t,\xi)\xi\geq -f_0(x,t)|\xi|^{\omega+1}-$ $-f_1(x,t)$ are valid.

It is clear that in condition 3 the function $f(x,t,\xi)$ generates a subordinate operator (if the problem (1)-(3) is not degenerated to the Cauchy problem for the ordinary equation with parameter x that is not possible under posed conditions).

Introduce the following denotation (here $\rho\geq 0$ is some number):

$$\mathbf{P}_0(Q)\equiv\mathbf{P}_{1,\bar{\mu},q_0,\rho,\bar{\sigma},2}(Q)\cap\left\{u(x,t)\middle| u|_\Gamma=0,\ \Gamma\equiv\Omega\cup\partial\Omega\times[0,T)\right\}.$$

Definition 2. *The function* $\tilde{u}(x,t)\equiv u(x,t)+w(x,t), u(x,t)\in\mathbf{P}_0(Q)$, $w(x,t)\in W_{\bar{p}}^1(Q)$ *is called the solution of the problem (1)-(3), if it satisfies the equation (1) in the sense of the space* $L_q\left(0,T;W_q^{-1}(\Omega)\right)$, *i.e. for any*

$$v(x,t)\in L_p\left(0,T;\overset{0}{W}{}_p^1(\Omega)\right),\ p\geq\max\left\{p_0^i,p_1^i,\rho_0+2\middle| i=\overline{1,n}\right\}$$

it is fulfilled the equality:

$$\int_Q\frac{\partial|\tilde{u}|^{\rho_0}\tilde{u}}{\partial t}vdxdt+\sum_{i=1}^n\int_Q A_i(x,t,\tilde{u},D\tilde{u})D_ivdxdt+\int_Q f(x,t,\tilde{u})vdxdt=0 \qquad (23)$$

here $w(x,0)=u_0(x)$, $w|_{\Gamma_0}=\varphi(x',t)$, $\forall(x',t)\in\Gamma_0\equiv[0,T)\times\partial\Omega$.

Thus, if we assume that $(u_0(x),\varphi(x',t))\in U(Q)$, i.e. the known function $w(x,t)$ satisfies the corresponding conditions, then we can write the problem (1)-(3) in the form

$$\frac{\partial|u+w|^{\rho_0}(u+w)}{\partial t}-\sum_{i=1}^n D_i\left[A_i(x,t,u+w,D(u+w))\right]+f(x,t,u+w)=0, \qquad (1')$$

$u(x,0)=0$, $x\in\Omega$, $u|_\Gamma=0$, and $w(x,t)$ satisfies the conditions (2)-(3), (2')

So, the problem (1')-(2') is homogeneous as the problems considered in papers [5-7], and we can expect that to it we shall apply the method used in indicated problems. Really, the problem will be studied by the indicated method, but now only by applying here the obtained results on imbedding.

Namely, for the problem (1)-(3) it is valid the following

Theorem 5. *Let conditions 1-4 be fulfilled and $\rho\geq 0$ is some number, satisfying the conditions of theorem 2 with parameters from conditions 1-4.*

Then for any pair $\left(u_0(x),\varphi(x',t)\right)\in U(Q)$ *the problem (1)-(3) is solvable in the space* $P_{1\bar{\mu},q_0,\rho,\bar{\sigma},2}(Q)\cap L_{p_1}\left(0,T;W_{p_1}^1(\Omega)\right)$.

At first we shall study the following auxiliary problem with parameter $\varepsilon > 0$, i.e. to study the considered problem we shall use the known elliptic regularization method ([8]):

$$-\varepsilon\frac{\partial^2|\widetilde{u}_\varepsilon|^{\rho_0}\widetilde{u}_\varepsilon}{\partial t^2}+\frac{\partial|\widetilde{u}_\varepsilon|^{\rho_0}\widetilde{u}_\varepsilon}{\partial t^2}-\sum_{i=1}^{n}D_i\left[A_i\left(x,t,\widetilde{u}_\varepsilon,D\widetilde{u}_\varepsilon\right)\right]+f\left(x,t,\widetilde{u}_\varepsilon\right)=0, \qquad (24)$$

$$u_\varepsilon(x,0)=0,\ \frac{\partial u_\varepsilon}{\partial t}(x,T)=0,\ x\in\Omega,\ u_\varepsilon|_\Gamma=0,\ \Gamma_0\equiv\partial\Omega\times[0,T],\ \widetilde{u}_\varepsilon=u_\varepsilon+w_\varepsilon, \qquad (25)$$

where $(x,t)\in Q\equiv\Omega\times(0,T)$, $\left\{w_\varepsilon(x,t)\middle|\forall\varepsilon\in(0,\varepsilon_0],\ \frac{\partial w_\varepsilon}{\partial t}|_{t=T}=0,\ \varepsilon_0>0\right\}\subset W_p^2(Q)$,

$$w_\varepsilon(x,t)\Rightarrow w(x,t)\ \text{in}\ W_p^1(Q)\ \text{for}\ \varepsilon\to 0,$$

and $w(x,t)$ is the function determined in theorem 5.

Definition 3. *The function* $\widetilde{u}_\varepsilon(x,t)\equiv u_\varepsilon(x,t)+w(x,t)$, $u_\varepsilon(x,t)\in \underset{0}{P}(Q)$, $w(x,t)\in W_{\bar p}^1(Q)$ *(with* $B\equiv W_{\bar q}^{-1}(\Omega)$*) is called the solution of the problem (24)-(25), if it satisfies the equation (24) in the sense of the space* $W_q^{-1}(Q)$*, i.e. for any*

$$v(x,t)\in L_p\left(0,T;\overset{0}{W}{}^1_p(\Omega)\right)\cap W_p^1\left(0,T;L_p(\Omega)\right)\cap\left\{v\middle|\ v(x,t)|_\Gamma=0,\ \Gamma\equiv\Omega\cup\partial\Omega\times[0,T]\right\}$$

it is fulfilled the equality:

$$\varepsilon\int_Q\frac{\partial|\widetilde{u}_\varepsilon|^{\rho_0}\widetilde{u}_\varepsilon}{\partial t}\frac{\partial v}{\partial t}dxdt+\int_Q\frac{\partial|\widetilde{u}_\varepsilon|^{\rho_0}\widetilde{u}_\varepsilon}{\partial t}\frac{\partial v}{\partial t}vdxdt+\sum_{i=1}^{n}\int_Q A_i\left(x,t,\widetilde{u}_\varepsilon,D\widetilde{u}_\varepsilon\right)D_ivdxdt++$$
$$+\int_Q f\left(x,t,\widetilde{u}_\varepsilon\right)vdxdt=0 \qquad (26)$$

here $w(x,0)=u_0(x)$, $w|_\Gamma=\varphi(x',t)$, $\forall(x',t)\in\Gamma_0\equiv[0,T)\times\partial\Omega$.

For the problem (24)-(25) it is valid the following

Lemma 10. *In conditions of theorem 5 the problem (24)-(25) is solvable in the sense of definition 3.*

Proof. We shall use the Galerkin method, and the approximate solutions of $u_{\varepsilon m}(x,t)$ we shall seek in the form of the solution of the equation

$$\left[\left|\widetilde{u}_{\varepsilon m}(x,t)\right|^\rho+N\right]u_{\varepsilon m}(x,t)=\sum_{k=1}^{m}c_{mk}v_k(x,t);\widetilde{u}_{\varepsilon m}(x,t)\equiv u_{\varepsilon m}(x,t)+w_{m\varepsilon}(x,t)$$

here $c_{mk},k=\overline{1,m}$ are unknown coefficients subjected to definition, $\left\{v_k(x,t)\right\}_{k=1}^{\infty}$ is a complete system of function in $W_{\bar p}^1(Q)\cap L^\infty(Q)$, and $N,\rho\geq 0$ are some numbers.

Further, acting in standard way and denoting the left hand side of the equation (24) by $\Psi(\mathrm{u})$, taking into account conditions 1-4, we get the following sequence of inequalities:

$$0=\left\langle\Psi(u_\varepsilon),\left(|\widetilde{u}_\varepsilon|^\rho+N\right)u_\varepsilon\right\rangle\equiv$$

$$\equiv\int_0^t\int_\Omega\left[\varepsilon\frac{\partial|\widetilde{u}_\varepsilon|^{\rho_0}\widetilde{u}_\varepsilon}{\partial\tau}\frac{\partial}{\partial\tau}\left(|\widetilde{u}_\varepsilon|^\rho+N\right)u_\varepsilon+\frac{\partial|\widetilde{u}_\varepsilon|^{\rho_0}\widetilde{u}_\varepsilon}{\partial\tau}\left(|\widetilde{u}_\varepsilon|^\rho+N\right)u_\varepsilon\right]dxdt+$$

$$+\int_0^t\int_\Omega\sum_{i=1}^n\left[A_i\left(x,\tau,\tilde{u}_\varepsilon,D\tilde{u}_\varepsilon\right)D_i\left[\left(|\tilde{u}_\varepsilon|^\rho+N\right)u_\varepsilon\right]dxd\tau+\int_0^t\int_\Omega f\left(x,\tau,\tilde{u}_\varepsilon\right)\left(|\tilde{u}_\varepsilon|^\rho+N\right)u_\varepsilon dxd\tau=$$

$$=\varepsilon\left(\rho_0+1\right)\int_0^t\int_\Omega\left(\left(\rho+1\right)|\tilde{u}_\varepsilon|^{\rho_0+\rho}+N|\tilde{u}_\varepsilon|^{\rho_0}\right)\left(\frac{\partial\tilde{u}_\varepsilon}{\partial\tau}\right)^2dxd\tau+$$

$$+\int_\Omega|\tilde{u}_\varepsilon|^{\rho_0+2}\left(\frac{\rho_0+1}{\rho+\rho_0+2}|\tilde{u}_\varepsilon|^\rho+\frac{\rho_0+1}{\rho_0+2}N\right)dx(t)-$$

$$-\varepsilon\left(\rho_0-1\right)\int_0^t\int_\Omega|\tilde{u}_\varepsilon|^{\rho_0}\left(\frac{\partial\tilde{u}_\varepsilon}{\partial\tau}\right)\left[\left(|\tilde{u}_\varepsilon|^{\rho_0}+N\right)\frac{\partial w_\varepsilon}{\partial\tau}+\rho|\tilde{u}_\varepsilon|^{\rho-2}\tilde{u}_\varepsilon\frac{\partial\tilde{u}_\varepsilon}{\partial\tau}w\right]dxd\tau-$$

$$-\int_0^t\int_\Omega|\tilde{u}_\varepsilon|^{\rho_0+2}\left(\frac{\partial\tilde{u}_\varepsilon}{\partial\tau}\right)\left(|\tilde{u}_\varepsilon|^\rho+N\right)wdxd+\int_0^t\int_\Omega f\left(x,\tau,\tilde{u}_\varepsilon\right)\left(|\tilde{u}_\varepsilon|^\rho+N\right)\left(\tilde{u}_\varepsilon-w\right)dxd+$$

$$+\int_0^t\int_\Omega\sum_{i=1}^n D_i\tilde{u}_\varepsilon\left[A_i\left(x,\tau,\tilde{u}_\varepsilon,D\tilde{u}_\varepsilon\right)\right]\left[\left(\rho+1\right)|\tilde{u}_\varepsilon|^\rho+N\right]dxd\tau-$$

$$-\int_0^t\int_\Omega\sum_{i=1}^n\left(|\tilde{u}_\varepsilon|^\rho D_iw+\rho|\tilde{u}_\varepsilon|^{\rho-2}\tilde{u}_\varepsilon D_i\tilde{u}_\varepsilon w\right)\left[A_i\left(x,\tau,\tilde{u}_\varepsilon,D\tilde{u}_\varepsilon\right)\right]dxd\tau\geq$$

$$\geq\varepsilon\left(\rho_0+1\right)\left(\rho+1\right)\int_0^t\int_\Omega|\tilde{u}_\varepsilon|^{\rho_0+\rho}\left(\frac{\partial\tilde{u}_\varepsilon}{\partial\tau}\right)^2dxd\tau+\frac{\rho_0+1}{\rho_0+\rho+2}\int_\Omega|\tilde{u}_\varepsilon|^{\rho_0+\rho+2}dx\Big|_t+$$

$$+\varepsilon\left(\rho_0+1\right)N\int_0^t\int_\Omega|\tilde{u}_\varepsilon|^{\rho_0}\left(\frac{\partial\tilde{u}_\varepsilon}{\partial\tau}\right)^2dxd\tau+\frac{\rho_0+1}{\rho_0+2}\int_\Omega|\tilde{u}_\varepsilon|^{\rho_0+2}dx\Big|_t-$$

$$-\varepsilon\left(\rho_0+1\right)\int_0^t\int_\Omega|\tilde{u}_\varepsilon|^{\rho_0}\left[\rho|\tilde{u}_\varepsilon|^{\rho-1}|w|\left|\frac{\partial u_\varepsilon}{\partial\tau}\right|+\left(|\tilde{u}_\varepsilon|^\rho+N\right)\left(\left|\frac{\partial w}{\partial\tau}\right|+|w|\right)\right]\left|\frac{\partial u_\varepsilon}{\partial\tau}\right|ddt+$$

$$+\int_0^t\int_\Omega\sum_{i=1}^n\left[A_i\left(x,\tau,\tilde{u}_\varepsilon,D\tilde{u}_\varepsilon\right)\right]\left[\left(\rho+1\right)|\tilde{u}_\varepsilon|^\rho+N\right]D_i\tilde{u}_\varepsilon dxd\tau-$$

$$-\int_0^t\int_\Omega\sum_{i=1}^n\left[A_i\left(x,\tau,\tilde{u}_\varepsilon,D\tilde{u}_\varepsilon\right)\right]\left[\left(|\tilde{u}_\varepsilon|^\rho+N\right)D_iw|+\rho|\tilde{u}_\eta|^{\rho-1}|w||D_i\tilde{u}_\varepsilon|\right]dxd\tau+$$

$$+\int_0^t\int_\Omega f\left(x,\tau,\tilde{u}_\varepsilon\right)\left(|\tilde{u}_\varepsilon|^\pi+N\right)\left(\tilde{u}_\varepsilon-w\right)dxd\tau\geq$$

$$\geq\varepsilon\left(\rho_0+1\right)\left(\rho+1\right)\int_0^\tau\int_\Omega|\tilde{u}_\varepsilon|^{\rho_0-\rho}\left(\frac{\partial\tilde{u}_\varepsilon}{\partial\tau}\right)^2dxd\tau+\varepsilon N\left(\rho_0+1\right)\int_0^\tau\int_\Omega|\tilde{u}_\varepsilon|^{\rho_0}\left(\frac{\partial\tilde{u}_\varepsilon}{\partial\tau}\right)^2dxd\tau-$$

$$-\varepsilon\left(\rho_0+1\right)\rho\|w\|_{L^\infty}\int_0^t\int_\Omega\left(\delta_0|\tilde{u}_\varepsilon|^{\rho_0+\rho}+C\left(\delta_0\right)|\tilde{u}_\varepsilon|^{\rho_0}\right)\left(\frac{\partial\tilde{u}_\varepsilon}{\partial\tau}\right)^2dxd\tau-$$

$$-\varepsilon\left(\rho_0+1\right)\int_0^t\int_\Omega\left(|\tilde{u}_\varepsilon|^{\rho_0+\rho}+N|\tilde{u}_\varepsilon|^{\rho_0}\right)\left[\delta_1\left|\frac{\partial\tilde{u}_\varepsilon}{\partial\tau}\right|^2+C\left(\delta_1\right)\left|\frac{\partial w}{\partial\tau}\right|^2\right]dxd\tau+$$

$$+\frac{\rho_0+1}{\rho_0+\rho+1}\int_\Omega|\tilde{u}_\varepsilon|^{\rho_0+\rho+2}dx+N\frac{\rho_0+1}{\rho_0+2}\int_\Omega|\tilde{u}_\varepsilon|^{\rho_0+2}dx-N\int_\Omega|\tilde{u}_\varepsilon|^{\rho_0+1}|w|dx-$$

$$-\int_0^t\int_\Omega\left(\frac{\rho_0+1}{\rho_0+\rho+1}|\widetilde{u}_\varepsilon|^{\rho_0+\rho+1}+N|\widetilde{u}_\varepsilon|^{\rho_0+1}\right)\left|\frac{\partial w}{\partial\tau}\right|dxd\tau-\frac{\rho_0+1}{\rho_0+\rho+1}\int_\Omega|\widetilde{u}_\varepsilon|^{\rho_0+\rho+1}|w|dx+$$

$$+\int_0^t\int_\Omega\sum_{i=1}^n\left[b_0\left(|D_i\widetilde{u}_\varepsilon|^{p_i}-|\widetilde{u}_\varepsilon|^{p_i'}\right)-B_0+a_0|\widetilde{u}_\varepsilon|^{p_0-2}|D_i\widetilde{u}_\varepsilon|^2\right]\left[(\rho+1)|\widetilde{u}_\varepsilon|^\rho+N\right]dxd\tau-$$

$$-\int_0^t\int_\Omega\sum_{i=1}^n\left[c_0\left(|D_i\widetilde{u}_\varepsilon|^{p_i-1}+|\widetilde{u}_\varepsilon|^{p_i'-1}\right)+c_1+A_0|\widetilde{u}_\varepsilon|^{p_0-2}|D_i\widetilde{u}_\varepsilon|\right]\left[|\widetilde{u}_\varepsilon|^\rho+N\right]|D_iw|dxd\tau-$$

$$-\int_0^t\int_\Omega\sum_{i=1}^n\rho\left[c_0\left(|D_i\widetilde{u}_\varepsilon|^{p_i-1}+|\widetilde{u}_\varepsilon|^{p_i'-1}\right)+c_1+A_0|\widetilde{u}_\varepsilon|^{p_0-2}|D_i\widetilde{u}_\varepsilon|\right]|\widetilde{u}_\varepsilon|^{\rho-1}|D_i\widetilde{u}_\varepsilon||w|dxd\tau-$$

$$-\int_0^t\int_\Omega\sum_{i=1}^n\left[f_0\left(f_0(x,\tau)|\widetilde{u}_\varepsilon|^\omega+f_1(x,\tau)\right)\right]\left[|\widetilde{u}_\varepsilon|^\rho+N\right]\left(|\widetilde{u}_\varepsilon|+|w|\right)dxd\tau.$$

Now using conditions 1-4, the conditions on the function $w(x,t)$ and choosing the number $N>0$ sufficiently large, form the last inequality we get:

$$0=-\varepsilon k_1\int_0^t\int_\Omega|\widetilde{u}_\varepsilon|^{\rho_0+\rho}\left(\frac{\partial\widetilde{u}_\varepsilon}{\partial\tau}\right)dxd\tau+\varepsilon N_1\int_0^t\int_\Omega|\widetilde{u}_\varepsilon|^{\rho_0}\left(\frac{\partial\widetilde{u}_\varepsilon}{\partial\tau}\right)^2dxd\tau+k_2\int_\Omega|\widetilde{u}_\varepsilon|^{\rho_0+\rho+2}dx+$$

$$+N_2\int_\Omega|\widetilde{u}_\varepsilon|^{\rho_0+2}dx-\varepsilon\delta_2(\delta_1)\int_0^t\int_\Omega\left(|\widetilde{u}_\varepsilon|^{\rho_0+\rho-2}-N|\widetilde{u}_\varepsilon|^{\rho_0+2}\right)dxd\tau-r_1\|w\|_{L\rho_0+2}^{\rho_0+2}(t)-$$

$$-r_2\|w\|_{L\rho_0+\rho+2}^{\rho_0+\rho+2}(t)-\varepsilon C(\delta_2)\|w\|_{W^1_{\rho_0+2}}^{\rho_0+2}-\varepsilon C(\delta_2)\|w\|_{W^1_{\rho_0+\rho+2}}^{\rho_0+\rho+2}-$$

$$-\delta_3\int_0^t\int_\Omega\left(|\widetilde{u}_\varepsilon|^{\rho_0+\rho+2}+N|\widetilde{u}_\varepsilon|^{\rho_0+2}\right)dxd\tau-C(\delta_3)\|w\|_{W^1_{\rho_0+2}}^{\rho_0+2}-C(\delta_3)\|w\|_{W^1_{\rho_0+\rho+2}}^{\rho_0+\rho+2}+$$

$$+\int_0^t\int_\Omega\sum_{i=1}^n\left[b_0\left(|D_i\widetilde{u}_\varepsilon|^{p_i}-|\widetilde{u}_\varepsilon|^{p_i'}\right)-B_0+a_0|\widetilde{u}_\varepsilon|^{p_0-2}|D_i\widetilde{u}_\varepsilon|^2\right]\left[(\rho+1)|\widetilde{u}_\varepsilon|^\rho+N\right]dxd\tau-$$

$$-\int_0^t\int_\Omega\sum_{i=1}^n\left[c_0\left(\left(|\widetilde{u}_\varepsilon|^\rho+N\right)|D_i\widetilde{u}_\varepsilon|^{p_i-1}+|\widetilde{u}_\varepsilon|^{p_i'+\rho-1}+N|\widetilde{u}_\varepsilon|^{p_i'-1}\right)+B_0\right]|D_iw|dxd\tau-$$

$$-\int_0^t\int_\Omega\sum_{i=1}^n\left[A_0\left(|\widetilde{u}_\varepsilon|^{P_0+\rho-2}+N|\widetilde{u}_\varepsilon|^{p_0-2}\right)|D_i\widetilde{u}_\varepsilon|+c_1\left(|\widetilde{u}_\varepsilon|^\rho+N\right)\right]|D_iw|dxd\tau-$$

$$-\int_0^t\int_\Omega\sum_{i=1}^n\rho\left[c_0\left(|\widetilde{u}_\varepsilon|^{P'-1}+|D_i\widetilde{u}_\varepsilon|^{p_i-2}\right)+c_1+A_0|\widetilde{u}_\varepsilon|^{p_0-2}|D_i\widetilde{u}_\varepsilon|\right]|\widetilde{u}_\varepsilon|^{\rho-1}|D_i\widetilde{u}_\varepsilon||w|dxd\tau-$$

$$-\int_0^t\int_\Omega\sum_{i=1}^n\left[f_0(x,\tau)|\widetilde{u}_\varepsilon|^\omega+f_1(x,\tau)\right]\left(|\widetilde{u}_\varepsilon|^\rho+N\right)\left(|\widetilde{u}_\varepsilon|+|w|\right)|w|dxd\tau$$

Taking into account that the constants satisfy the conditions from 2-4, and continuing the estimate, we get

$$0=\left\rangle\Psi(u_\varepsilon),\left(|\widetilde{u}_\varepsilon|^\rho+N\right)u_\varepsilon\right\langle\geq\varepsilon k_1\int_0^t\int_\Omega|\widetilde{u}_\varepsilon|^{\rho_0+\rho}\left(\frac{\partial\widetilde{u}_\varepsilon}{\partial\tau}\right)dxd\tau+\varepsilon N\int_0^t\int_\Omega|\widetilde{u}_\varepsilon|^{\rho_0}\left(\frac{\partial\widetilde{u}_\varepsilon}{\partial\tau}\right)dxd\tau-$$

$$-\left(\varepsilon\delta_2C(\delta_1)+\delta_3\right)\int_0^t\int_\Omega\left(|\widetilde{u}_\varepsilon|^{\rho_0+\rho+2}+N|\widetilde{u}_\varepsilon|^{\rho_0+2}\right)dxd\tau+\int_\Omega\left(k_2|\widetilde{u}_\varepsilon|^{\rho_0+\rho+2}+N_2|\widetilde{u}_\varepsilon|^{\rho_0+2}\right)dx-$$

$$-\widetilde{C}_0\left(\rho_0,\rho,N,\delta_1,\delta_2,\delta_3,\|w\|_{W^1_p\cap L^\infty},mes\Omega\right)-\int_0^t\int_\Omega\sum_{i=1}^n\left(b_0|\widetilde{u}_\varepsilon|^{p_i'}+B_0\right)\left[(\rho+1)|\widetilde{u}_\varepsilon|^\rho+N\right]dxd\tau+$$

$$+\int_0^t\int_\Omega\sum_{i=1}^n\left(b_0|D_i\tilde u_\varepsilon|^{p_1}+a_0|\tilde u_\varepsilon|^{\rho_0-2}|D_i\tilde u_\varepsilon|^2\right)\left[(\rho+1)|\tilde u_\varepsilon|^\rho+N\right]dxd\tau-$$

$$-\int_0^t\int_\Omega\sum_{i=1}^n\Big\{c_0\left(\delta_4|\tilde u_\varepsilon|^\rho|D_i\tilde u_\varepsilon|^{p_1}+\delta_5N|D_i\tilde u_\varepsilon|^{p_1}\right)+C(\delta_4,\delta_5)|D_iw|^{p_1}\Big]+$$

$$+\rho c_0|\tilde u_\varepsilon|^{-1\rho}|D_i\tilde u_\varepsilon|^{p_1}|w|\Big\}dxd\tau-$$

$$-\int_0^t\int_\Omega\sum_{i=1}^n\Big[\delta_6\left(|\tilde u_\varepsilon|^{\rho_0+\rho-2}+N|\tilde u_\varepsilon|^{\rho_0-2}\right)|D_i\tilde u_\varepsilon|^2+C(\delta_6)|D_iw|^{p_0}+$$

$$+c_0\rho|\tilde u_\varepsilon|^{p_1+\rho-2}|D_i\tilde u_\varepsilon||w|\Big]dxd\tau-$$

$$-\int_0^t\int_\Omega\sum_{i=1}^n\Big[\rho A_0|\tilde u_\varepsilon|^{\rho_0+\rho-3}|D_i\tilde u_\varepsilon|^2|w|+\left(f_0(x,\tau)|\tilde u_\varepsilon|^\varpi+f_1(x,\tau)\right)\left(|\tilde u_\varepsilon|^\rho+N\right)\left(|\tilde u_\varepsilon|+|w|\right)dxd\tau\geq$$

$$\geq\varepsilon k_1\int_0^t\int_\Omega|\tilde u_\varepsilon|^{\rho_0+\rho}\left(\frac{\partial\tilde u_\varepsilon}{\partial\tau}\right)dxd\tau+\varepsilon N_1\int_0^t\int_\Omega|\tilde u_\varepsilon|^{\rho_0}\left(\frac{\partial\tilde u_\varepsilon}{\partial\tau}\right)dxd\tau+k_2\int_\Omega|\tilde u_\varepsilon|^{\rho_0+\rho+2}dx+$$

$$+N_2\int_\Omega|\tilde u_\varepsilon|^{\rho_0+2}dx-\left(\varepsilon\delta_2C(\delta_1)+\delta_3\right)\int_0^t\int_\Omega\left(|\tilde u_\varepsilon|^{\rho_0+\rho+2}+N|\tilde u_\varepsilon|^{\rho_0+2}\right)dxd\tau-$$

$$-\tilde C_0\left(\rho_0,\rho,N,\delta_1,\delta_2,\delta_3,\|w\|_{W_p^1\cap L^\infty},mesQ\right)+$$

$$+\int_0^t\int_\Omega\sum_{i=1}^n\left(\tilde b_0|D_i\tilde u_\varepsilon|^{p_1}+\tilde a_0|\tilde u_\varepsilon|^{\rho_0-2}|D_i\tilde u_\varepsilon|^2\right)\left[(\rho+1)|\tilde u_\varepsilon|^\rho+N\right]dxd\tau-$$

$$-\tilde C_1\left(\rho_0,\rho,N,p_1,p_0,a_0,b_0,B_0,\|w\|_{W_p^1\cap L^\infty},mesQ\right)-\int_0^t\int_\Omega\rho c_0|\tilde u_\varepsilon|^{-1l}|D_i\tilde u_\varepsilon|^{p_1}|w|dxd\tau-$$

$$-\int_0^t\int_\Omega\sum_{i=1}^n\rho A_0|\tilde u_\varepsilon|^{p_0+\rho-3}|D_i\tilde u_\varepsilon|^2|w|dxd\tau-C_2\left(p_0,p_1,\rho_0,\rho,a_0,b_0,N,\|f_0\|,\|f_1\|,\|w\|,mesQ\right)\geq$$

$$\geq\varepsilon k_1\int_0^t\int_\Omega|\tilde u_\varepsilon|^{\rho_0+\rho}\left(\frac{\partial\tilde u_\varepsilon}{\partial\tau}\right)^2dxd\tau+\varepsilon N_1\int_0^t\int_\Omega|\tilde u_\varepsilon|^{\rho_0}\left(\frac{\partial\tilde u_\varepsilon}{\partial\tau}\right)^2dxd\tau+k_2\int_\Omega|\tilde u_\varepsilon|^{\rho_0+\rho+2}dx+$$

$$+N_2\int_\Omega|\tilde u_\varepsilon|^{\rho_0+2}dx-\left(\varepsilon\delta_2C(\delta_1)+\delta_3\right)\int_0^t\int_\Omega\left(|\tilde u_\varepsilon|^{\rho_0+\rho+2}+N|\tilde u_\varepsilon|^{\rho_0+2}\right)dxd\tau-$$

$$-\tilde C_0\left(\rho_0,\rho,N,\delta_1,\delta_2,\delta_3,\|w\|_{W_p^1\cap L^\infty},mesQ\right)+$$

$$+\int_0^t\int_\Omega\sum_{i=1}^n\left(\tilde b_0|D_i\tilde u_\varepsilon|^{p_1}+\tilde a_0|\tilde u_\varepsilon|^{\rho_0-2}|D_i\tilde u_\varepsilon|^2\right)\left[(\rho+1)|\tilde u_\varepsilon|^\rho+N\right]dxd\tau-$$

$$-\int_0^t\int_\Omega\sum_{i=1}^n\left[\left(\delta_8|\tilde u_\varepsilon|^\rho+C(\delta_8)|w|^\rho\right)|D_i\tilde u_\varepsilon|^{p_1}+\left(\delta_9|\tilde u_\varepsilon|^{p_0+\rho-2}+C(\delta_9)|w|^{p_0+\rho-2}\right)|D_i\tilde u_\varepsilon|^2\right]dxd\tau-$$

$$-\tilde C_3\left(\rho_0,\rho,N,p_1,p_0,a_0,b_0,B_0,N,\|f_0\|,\|f_1\|,\|w\|_{W_p^1\cap L^\infty},mesQ\right)\geq$$

$$\geq\varepsilon\int_0^t\int_\Omega\left(k_1|\tilde u_\varepsilon|^{\rho_0+\rho}+N_1|\tilde u_\varepsilon|^{\rho_0}\right)\left(\frac{\partial\tilde u_\varepsilon}{\partial\tau}\right)^2dxd\tau+\int_\Omega\left(k_2|\tilde u_\varepsilon|^{\rho_0+\rho+2}+N_2|\tilde u_\varepsilon|^{\rho_0+2}\right)dx-$$

$$-\left(\varepsilon\delta_2 C(\delta_1)+\delta_3\right)\int_0^t\int_\Omega\left(|\widetilde{u}_\varepsilon|^{\rho_0+\rho+2}+N|\widetilde{u}_\varepsilon|^{\rho_0+2}\right)dxd\tau+$$

$$+\int_0^t\int_\Omega\sum_{i=1}^n\left(\widetilde{b}_0|D_i\widetilde{u}_\varepsilon|^{p_i}+\widetilde{a}_0|\widetilde{u}_\varepsilon|^{p_0-2}|D_i\widetilde{u}_\varepsilon|^2\right)\left[(\rho+1)|\widetilde{u}_\varepsilon|^\rho+N_0\right]dxd\tau-$$

$$-\widetilde{C}\left(\rho_0,p_1,p,\rho_0,\rho,a_0,b_0,B_0,N,\|f_0\|,\|f_1\|,\|w\|_{W_p^1\cap L^\infty},mesQ\right).$$

Here we have used the properties of given functions, and namely, that

$$w\in W_p^1(Q)\cap L^\infty(Q),p\ge\max\{p_0,p_1,\rho_0+2\}+\rho,$$

$$f_0\in L_{q_1}(Q),f_1\in L_{q_2}(Q),q_i\ge\frac{p-\rho}{p-\rho-1}$$

and the number $N>0$ is chosen sufficiently large.

Thus, we get that the following final inequality hold:

$$0=\left\langle\Psi(u_\varepsilon),\left(|\widetilde{u}_\varepsilon|^\rho+N\right)\right\rangle\ge\varepsilon\int_0^t\int_\Omega\left(k_1|\widetilde{u}_\varepsilon|^{\rho_0+\rho}+N_1|\widetilde{u}_\varepsilon|\rho_0\right)\left(\frac{\partial\widetilde{u}_\varepsilon}{\partial\tau}\right)^2dxd\tau+$$

$$+\int_\Omega k_2|\widetilde{u}_\varepsilon|^{\rho_0+\rho+2}dx+N_2\int_\Omega|\widetilde{u}_\varepsilon|^{\rho_0+2}dx+$$

$$+\int_0^t\int_\Omega\sum_{i=1}^n\left(\overline{b}_0|D_i\widetilde{u}_\varepsilon|^{p_i}+\overline{a}_0|\widetilde{u}_\varepsilon|^{p_0-2}|D_i\widetilde{u}_\varepsilon|^2\right)\left[(\rho+1)|\widetilde{u}_\varepsilon|^\rho+N\right]dxd\tau-\qquad(27)$$

$$-\left(\varepsilon\delta_2 C(\delta_1)+\delta_3\right)\int_0^t\int_\Omega\left(|\widetilde{u}_\varepsilon|^{\rho_0+\rho+2}+N|\widetilde{u}_\varepsilon|^{\rho_0+2}\right)dxd\tau-$$

$$-\widetilde{C}_0\left(p_0,p_1,p,\rho,\rho_0,a_0,b_0,c_0,A_0,B_0,N,\delta_1,\ldots,\delta_9,\|f_0\|,\|f_1\|,\|w\|,mesQ\right).$$

As it is easy to see, hence it follows the validity of the following inequality

$$k_2\int_0^t\int_\Omega|\widetilde{u}_\varepsilon|^{\rho_0+\rho+2}dx+N_2\int_\Omega|\widetilde{u}_\varepsilon|^{\rho_0+2}dx\le$$

$$\le\left(\varepsilon\delta_2 C(\delta_1)+\delta_3\right)\int_0^t\int_\Omega\left(|\widetilde{u}_\varepsilon|^{\rho_0+\rho+2}+N|\widetilde{u}_\varepsilon|^{\rho_0+2}\right)dxd\tau+\qquad(28)$$

$$+C\left(p_0,p_1,p,\rho,\rho_0,a_0,b_0,c_0,A_0,B_0,N,\delta_1,\ldots,\delta_9,\|f_0\|,\|f_1\|,\|w\|,mesQ\right).$$

Applying the Cronwall lemma to the inequality (28) we get the estimate for the left hand side of the inequality (28), i.e. it is bounded and estimated by the constant that is independent on ε, when it converges to zero. Then, by using this estimate in the inequality (27) we get from this inequality the validity of the following a priori estimates:

$$ess\sup_{t\in(0,T)}\left\{\int_\Omega|\widetilde{u}_\varepsilon|^{\rho_0+\rho+2}dx+\int_\Omega|\widetilde{u}_\varepsilon|^{\rho_0+2}dx\right\}\le C<+\infty,$$

$$\varepsilon\int_0^T\int_\Omega\left(|\widetilde{u}_\varepsilon|^{\rho_0-\rho}+|\widetilde{u}_\varepsilon|^{\rho_0}\right)\left(\frac{\partial\widetilde{u}_\varepsilon}{\partial\tau}\right)dxd\tau\le C<+\infty,\qquad(29)$$

$$\int_0^T\int_\Omega\left(|\widetilde{u}_\varepsilon|^\rho+1\right)|D_i\widetilde{u}_\varepsilon|^{p_i}dxdt\le C<+\infty;\quad\int_0^T\int_\Omega\left(|\widetilde{u}_\varepsilon|^\rho+1\right)|\widetilde{u}_\varepsilon|^{p_0-2}|D_i\widetilde{u}_\varepsilon|^2dxdt\le C<+\infty;$$

where $C>0-const$. depending on the norm of functions $f_0(x,t)$, $f_1(x,t)$, $w(x,t)$ in corresponding spaces, parameters of the problem and independent on $u_\varepsilon(x,t)$, $\varepsilon>0$ for $\varepsilon\downarrow 0$.

Thus, the obtained a priori estimates (29) are uniform in ε for $\varepsilon\downarrow 0$. And consequently, the same estimates are valid and for Galerkin's approximate solutions $u_{\varepsilon m}(x,t)$ of the problem (25)-(26).

Then, it follows from (29) by virtue of above mentioned statements of p.1 that the approximate solution of the problem (25)-(26) runs the bounded subset in the space $\underset{0}{\mathbf{P}}(Q)$, when $m\uparrow\infty$, $\varepsilon\downarrow 0$. In the other words, we have:

$\left\{u_{\varepsilon m}(x,t)\middle| m\uparrow\infty \,\&\, \varepsilon\downarrow 0\right\}$ generates a bounded subset in the space $\underset{0}{\mathbf{P}}(Q)$.

Then, arguing as in papers [6,7,9] we get the solvability of the problem (25)-(26) in the sense of definition (here the method suggested in papers [10], [7] is used). We are to note that, as we noted in previous paragraph in the proof of imbedding theorem for vector- valued functions, here it is used that there exist some number $\mu=\mu(\rho_0\rho, p_0, p_1)>0$, such that it is valid the following relation:

$$|u_{\varepsilon m}|^{\mu} u_{\varepsilon m} \in \underset{0\,1,\nu,\eta,\theta}{\mathbf{P}}(Q), \text{ in particular,}$$

$$|u_{\varepsilon m}|^{\rho_0} u_{\varepsilon m} \in \underset{0\,1,\nu,\eta,\theta}{\mathbf{P}}(Q) \text{ for some numbers: } \nu\geq 0, \eta,\theta\geq 1.$$

Further, $\rho\geq 0$ choose so that one of the following conditions be fulfilled:

1) $\rho_0\geq p_0-2,\ \rho_0\geq p_1-2 \Rightarrow \rho: p_0-2\leq\rho_0\leq p_0+\rho-2$, or $p_1-2\leq\rho_0\leq p_1+\rho-2$;

2) $0\leq\rho_0<p_0-2, \Rightarrow \rho: \exists\overline{\rho}_0\geq 0,\ p_0-2\leq\overline{\rho}(\rho_0+1)+2\rho_0\leq p_0+\rho-2$;

3) $0\leq\rho_0<p_1-2, \Rightarrow \rho\geq 0$ -is arbitrary.

Under these conditions we get from a priori estimates (29), that the sequence of approximate solutions runs an unbounded set in the following space:

$$L_p\Big(\Omega; \underset{0\,1,\rho_0,2}{S}(0,T)\Big)\cap L_{p+\rho}\Big(\Omega; \underset{0\,1,\rho_0,2}{S}(0,T)\Big)\cap L_{p_0}\Big(0,T; \overset{0}{S}_{1,(p_0-2),2}(\Omega)\Big)\cap$$

$$L_{p_1}\Big(0,T; \overset{0}{W}{}^{1}_{p_1}(\Omega)\Big)\cap L_{p+\rho}\Big(0,T; S_{1,(p_0+\rho-2),2}(\Omega)\Big)\cap L_{p_1+\rho}\Big(0,T; S_{1,(\rho),p_1}(\Omega)\Big);$$

Hence, using the corresponding condition (1), (2) or (3), interpolation properties of pn-spaces and the above-mentioned imbedding theorem we choose the numbers $\nu\geq 0$, $\eta,\theta\geq 1$. And in exactly the same way as in papers [4]-[7], [9] the solvability of the problem (24)-(25) is shown, and then it is obtained a priori estimate of the form: $\dfrac{|\widetilde{u}_{\varepsilon m}|^{\rho_0}\partial(\widetilde{u}_{\varepsilon m})}{\partial t}\in L_q\big(0,T; W_q^{-1}(\Omega)\big)$ and runs there the bounded set. Consequently we get approximate solutions generate the bounded set in the obtained domain uniform in $m\uparrow\infty$, $\varepsilon\downarrow 0$.

Then, using the method of papers [6,7] the proof of the main solvability theorem is completed (for brevity, we don't cite the full proof, it will be published at the next paper).

Remark. We are to note that this method and obtained here imbedding theorems for anisotropic spaces admit by the same way to study the problem with such type equation in the case, when it has different nonlinearities in different directions. Besides, using this method, the problem with (4) and (5) type equations is studied.

These results will be published in the next paper.

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Soltanov K.N.
Institute of Mathematics and Mechanics of AS Azerbaijan,
9, F. Agayev str., 370141, Baku, Azerbaijan.
Tel.: 38-72-50.

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