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ON SOME MAXIMAL FUNCTIONS, MEASURING SMOOTHNESS, AND METRIC CHARACTERISTICS

Abstract

In present paper some local metric characteristics of locally summably functions and maximal functions measuring smoothness are introduced and relationships between them are investigated.

Let $x=(x_1,x_2,...,x_n)\in R^n$, $\nu=(\nu_1,\nu_2,...,\nu_n)$, $x^\nu=x_1^{\nu_1}\cdot x_2^{\nu_2}...x_n^{\nu_n}$, $|\nu|=\nu_1+\nu_2+...+\nu_n$, ν_i $(i=1,2,...,n)$ be non-negative integers. We shall consider that the power functions $\{x^\nu\}$, $|\nu|\le k$, where $k\in\mathbf{N}\bigcup\{0\}$, have such an arrangement order: x^ν precedes x^μ if either $|\nu|<|\mu|$, or $|\nu|=|\mu|$, but the first different from zero differences $\nu_i-\mu_i$ is negative. Such an order we call partially lexicographic. Apply the orthogonalization process with respect to the scalar product

$$(f,g):=|B(0,1)|^{-1}\int_{B(0,1)}f(t)g(t)dt$$

to the system of power functions $\{x^\nu\}$, $|\nu|\le k$, where by $|B(a,r)|$ we denote the volume of the ball $B(a,r):=\{x\in\mathbf{R}^n:|x-a|\le r\}$. The result of the orthogonalization process we denote by $\{\varphi_\nu\}$, $|\nu|\le k$. The system $\{\varphi_\nu\}$, $|\nu|\le k$ is orthogonal and normed.

By $L^p_{loc}(\mathbf{R}^n)$ $(1\le p<\infty)$ we denote a class of all locally summable in $p-th$ power functions and by $L^\infty_{loc}(R^n)$ - a class of all locally bounded functions defined on in $\mathbf{R}^n$.

Let $f\in L^1_{loc}(\mathbf{R}^n)$. Put (see [1], [2]):

$$P_{k,B(a,r)}f(x):=\sum_{|\nu|\le k}\left(\frac{1}{|B(a,r)|}\int_{B(a,r)}f(t)\varphi_\nu\left(\frac{t-a}{r}\right)dt\right)\varphi_\nu\left(\frac{x-a}{r}\right).$$

$P_{k,B(a,r)}f$ is polynomial of degree at most k. The totality of all polynomials in $\mathbf{R}^n$ of degree not higher than k denote by $\mathbf{P}_k$. Thus, $P_{k,B(a,r)}f\in\mathbf{P}_k$.

For the functions $f\in L^p_{loc}(\mathbf{R}^n)$ $(1\le p\le\infty)$ we denote

$$\Omega_k(f,B(a,r))_p:=\left(\frac{1}{|B(a,r)|}\int_{B(a,r)}\left|f(t)-P_{k-1,B(a,r)}f(t)\right|^p dt\right)^{\frac{1}{p}}\quad(1\le p<\infty),$$

$$\Omega_k(f,B(a,r))_\infty:=ess\sup\left\{\left|f(t)-P_{k-1,B(a,r)}f(t)\right|:t\in B(a,r)\right\},$$

$$O_k(f,B(a,r))_p:=|B(a,r)|^{\frac{1}{p}}\cdot\Omega_k(f,B(a,r))_p\quad(1\le p\le\infty).$$

$\Omega_k(f,B(a,r))_p$ $(1\le p\le\infty)$ is the mean oscillation of the k-th order of the function f in the ball $B(a,r)$ in the metrics L^p (see [3]). Introduce some local metric characteristics of the function $f\in L^p_{loc}(\mathbf{R}^n)$. Denote

$$m_f^k(x;\delta)_p:=\sup\left\{\Omega_k(f,B(x,r))_p : r\le\delta\right\}\quad \left(x\in\mathbf{R}^n,\ \delta>0\right),$$

$$\mu_f^k(x;\delta)_p:=\sup\left\{O_k(f,B(x,r))_p : r\le\delta\right\}\quad \left(x\in\mathbf{R}^n,\ \delta>0\right).$$

Let $x\in R^n, k\in N$ and for any ν, with condition $|\nu|\le k-1$, there exits limit

$$\lim_{r\to 0}D^\nu P_{k-1,B(x,r)}f(x)=:D_\nu f(x).$$

Denote

$$P_{k-1,x}f(t):=\sum_{|\nu|\le k-1}D_\nu f(x)\frac{(t-x)^\nu}{\nu!},$$

$$n_f^k(x;\delta)_p:=\sup\left\{|B(x,r)|^{-\frac{1}{p}}\left\|f-P_{k-1,x}f\right\|_{L^p(B(x,r))} : r\le\delta\right\}\quad \left(x\in\mathbf{R}^n,\ \delta>0\right),$$

where $\nu!=\nu_1!\nu_2!\ldots\nu_n!$ and

$$D^\nu g:=\frac{\partial^{|\nu|}g}{\partial x_1^{\nu_1}\partial x_2^{\nu_2}\ldots\partial x_n^{\nu_n}}.$$

Let Φ be a class of all positive monotonically increasing in $(0,+\infty)$ functions, and Φ_k $(k\in\mathbf{N})$ be a class of all functions $\varphi\in\Phi$ such that $\frac{\varphi(t)}{t^k}$ almost decreases*.

Now let $f\in L^p_{loc}(\mathbf{R}^n)$, $1\le p\le\infty$, $\varphi\in\Phi_k$, $k\in\mathbf{N}$. Denote

$$f^\#_{k,\varphi,p}(x):=\sup_{r>0}\frac{1}{\varphi(r)}\left(\frac{1}{|B(x,r)|}\int_{B(x,r)}\left|f(t)-P_{k-1,B(x,r)}f(t)\right|^p dt\right)^{\frac{1}{p}},\quad x\in\mathbf{R}^n,$$

$$N_{k,\varphi,p}f(x):=\sup_{r>0}\frac{1}{\varphi(r)}\left(\frac{1}{|B(x,r)|}\int_{B(x,r)}\left|f(t)-P_{k-1,x}f(t)\right|^p dt\right)^{\frac{1}{p}},\quad x\in\mathbf{R}^n,$$

with corresponding modification in the case $p=\infty$. Maximal functions $f^\#_{k,\varphi,p}(x)$ and $N_{k,\varphi,p}f(x)$ in case $\varphi(t)=t^\alpha$ $(t>0;\alpha>0)$ have been considered in papers [1] and [4] respectively.

In [3] it has been proved

Proposition 1. *Let* $f\in L^p_{loc}(\mathbf{R}^n)$, $1\le p\le\infty$, $k\in\mathbf{N}$, $\varphi\in\Phi_k$. *Then it is valid the*

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* The positive function $h(t)$, $t\in(0,+\infty)$ is called almost decreasing, if

$$\exists c>0\quad \forall t_1,t_2\in(0,+\infty):\ \left(t_1<t_2\Rightarrow h(t_1)\ge c\cdot h(t_2)\right)$$

equality $f^{\#}_{k,\varphi,p}(x)=\sup\limits_{r>0}\dfrac{m_f^k(x;r)_p}{\varphi(r)},\quad x\in\mathbf{R}^n$.

The following statements are proved similarly.

Proposition 2. *Let* $f\in L^p_{loc}(\mathbf{R}^n)$, $1\le p\le\infty$, $k\in\mathbf{N}$, $\varphi\in\Phi_k$. *Then it holds the equality*

$$f^{\#}_{k,\varphi,p}(x)=\gamma_n^{-\frac{1}{p}}\cdot\sup_{r>0}\frac{\mu_f^k(x;r)_p}{r^{n/p}\cdot\varphi(r)},\, x\in\mathbf{R}^n,$$

where $\gamma_n=\dfrac{\pi^{n/2}}{\Gamma\left(\dfrac{n}{2}+1\right)}$ *is the volume of the unit ball in* $\mathbf{R}^n$.

Proposition 3. *Let* $f\in L^p_{loc}(\mathbf{R}^n)$, $1\le p\le\infty$, $k\in\mathbf{N}$. *Then the relation*

$$m_f^k(x;\delta)_p=\gamma_n^{-\frac{1}{p}}\cdot\sup_{0<r\le\delta}\frac{\mu_f^k(x;r)_p}{r^{n/p}},\quad x\in\mathbf{R}^n,\delta>0$$

is valid.

By means of lemma 1.1 from [5] we can get

$$m_f^k(x;\delta)_p\le c\cdot n_f^k(x,\delta)_p,\quad \delta>0,\ \ x\in\mathbf{R}^n, \tag{1}$$

where a constant $c>0$ doesn't depend on f,δ,x and p.

Theorem 1. *Let* $f\in L^p_{loc}(\mathbf{R}^n)$, $1\le p\le\infty$, $k\in\mathbf{N}$ *and*

$$\int_0^1 t^{-k}m_f^k(x;t)_p\,dt<+\infty. \tag{2}$$

Then it is valid the inequality

$$n_f^k(x;\delta)_p\le c\left(m_f^k(x;\delta)_p+\delta^{k-1}\int_0^\delta\frac{m_f^k(x;t)_p}{t^k}dt\right), \tag{3}$$

where $c>0$ *doesn't depend on* f,δ *and* x.

Proof. If the condition (2) is fulfilled, the condition

$$\int_0^1 t^{-k}\cdot m_f^k(x;t)_1\,dt<+\infty \tag{4}$$

is also fulfilled.

It is known [2] that if $0<\eta<\xi<+\infty$, it is valid the inequality

$$\left|P_{k-1,B(x,\xi)}f(t)-P_{k-1,B(x,\eta)}f(t)\right|\le$$
$$\le c\left(\left(1+\left(\frac{|t-x|}{\xi}\right)^{k-1}\right)m_f^k(x;\xi)_1+\int_\eta^\xi\left(1+\left(\frac{|t-x|}{y}\right)^{k-1}\right)\frac{m_f^k(x;y)_1}{y}dy\right), \tag{5}$$

where $t\in\mathbf{R}^n$, and a constant $c>0$ doesn't depend on η,ξ,f,t and x. By virtue of the Markov's inequality (see [1]), by considering (5) we have

$$\left\|D^{\nu}\left(P_{k-1,B(x,\xi)}f-P_{k-1,B(x,\eta)}f\right)\right\|_{L^{\infty}(B(x,\xi))}\leq$$
$$\leq c\cdot\xi^{-|\nu|}\left\|P_{k-1,B(x,\xi)}f-P_{k-1,B(x,\eta)}f\right\|_{L^{\infty}(B(x,\xi))}\leq$$
$$\leq c_1\cdot\xi^{-|\nu|}\left(m_f^k(x;\xi)_1+\xi^{k-1}\int_{\eta}^{\xi}y^{-k}\cdot m_f^k(x;y)_1\,dy\right). \qquad (6)$$

It is easy to show that if the condition (4) is fulfilled, then $m_f^k(x;\xi)_1=0\left(\xi^{k-1}\right)$, $\xi\to 0$. Considering this, by means of the inequality (6) we establish the existence of the limit

$$\lim_{r\to 0}D^{\nu}P_{k-1,B(x,r)}f(x)=:D_{\nu}f(x),\ |\nu|\leq k-1. \qquad (7)$$

Further, we get from (6), by considering (7), that for $|\nu|\leq k-1$

$$\left|D^{\nu}P_{k-1,B(x,r)}f(x)-D_{\nu}f(x)\right|\leq c\cdot r^{-|\nu|}\left(m_f^k(x;r)_p+r^{k-1}\int_0^r t^{-k}\cdot m_f^k(x;t)_p\,dt\right), \qquad (8)$$

where c doesn't depend on p,f,r and x.

Denote

$$P_{k-1,x}f(t):=\sum_{|\nu|\leq k-1}D_{\nu}f(x)\frac{(t-x)^{\nu}}{\nu!}.$$

Then for $1\leq p<\infty$ we have

$$\left(\frac{1}{|B(x,r)|}\int_{B(x,r)}\left|f(t)-P_{k-1,x}f(t)\right|^p dt\right)^{\frac{1}{p}}\leq$$
$$\leq m_f^k(x;r)_p+\left\|P_{k-1,B(x,r)}f-P_{k-1,x}f\right\|_{L^{\infty}(B(x,r))}. \qquad (9)$$

If $t\in B(x,r)$, by means of the inequality (8) we get

$$\left|P_{k-1,B(x,r)}f(t)-P_{k-1,x}f(t)\right|\leq\sum_{|\nu|\leq k-1}\left|D^{\nu}P_{k-1,B(x,r)}f(x)-D_{\nu}f(x)\right|\frac{|t-x|^{|\nu|}}{\nu!}\leq$$
$$\leq c\left(m_f^k(x;r)_p+r^{k-1}\int_0^r y^{-k}\cdot m_f^k(x;y)_p\,dy\right),$$

where $c>0$ doesn't depend on r,x and f.

Taking this into account by virtue of (9) we get the required inequality (3) in the case $1\leq p<\infty$. The arguments for the case $p=\infty$ are analogous. The theorem is proved.

Proposition 4 is proved similar to proposition 1.

Proposition 4. *Let* $f\in L_{loc}^p\left(\mathbf{R}^n\right)$, $1\leq p\leq\infty$, $k\in\mathbf{N}$, $\varphi\in\Phi_k$. *Then it is valid the equality*

$$N_{k,\varphi,p}f(x)=\sup_{r>0}\frac{n_f^k(x;r)_p}{\varphi(r)},\ x\in\mathbf{R}^n.$$

Theorem 2. *Let* $f\in L_{loc}^p\left(\mathbf{R}^n\right)$, $1\leq p\leq\infty$, $k\in\mathbf{N}$, $\varphi\in\Phi_k$ *and*

$$\delta^{k-1}\int_0^{\delta}\frac{\varphi(t)}{t^k}dt=O(\varphi(\delta))\quad(\delta>0). \qquad (10)$$

Then there exist numbers $c_1 > 0,\ c_2 > 0$ *such that*

$$c_1 \cdot f^{\#}_{k,\varphi,p}(x) \le N_{k,\varphi,p} f(x) \le c_2 \cdot f^{\#}_{k,\varphi,p}(x),\ x \in \mathbf{R}^n, \qquad (11)$$

where constants c_1 *and* c_2 *don't depend on* f *and* x.

Proof. By virtue of propositions 1 and 4, and the inequality (1) we have

$$f^{\#}_{k,\varphi,p}(x) = \sup_{r>0} \frac{m_f^k(x;r)_p}{\varphi(r)} \le c \cdot \sup_{r>0} \frac{n_f^k(x;r)_p}{\varphi(r)} = c \cdot N_{k,\varphi,p} f(x),$$

or

$$f^{\#}_{k,\varphi,p}(x) \le c \cdot N_{k,\varphi,p} f(x), \quad x \in \mathbf{R}^n, \qquad (12)$$

where a constant $c > 0$ doesn't depend on f, x, p and φ. On the other hand, by applying the inequality (3) we get

$$N_{k,\varphi,p} f(x) = \sup_{r>0} \frac{n_f^k(x;r)_p}{\varphi(r)} \le c \cdot \sup_{r>0} \frac{1}{\varphi(r)} \left\{ m_f^k(x;r)_p + \right.$$

$$\left. + r^{k-1} \int_0^r \frac{m_f^k(x;t)_p}{t^k} dt \right\} \le c \left\{ f^{\#}_{k,\varphi,p}(x) + \sup_{r>0} \frac{r^{k-1}}{\varphi(r)} \int_0^r \left(\frac{m_f^k(x;t)_p}{\varphi(t)} \right) \cdot \frac{\varphi(t)}{t^k} dt \right\} \le \qquad (13)$$

$$\le c \cdot f^{\#}_{k,\varphi,p}(x) \left\{ 1 + \sup_{r>0} \frac{r^{k-1}}{\varphi(r)} \int_0^r \frac{\varphi(t)}{t^k} dt \right\},\ x \in \mathbf{R}^n,$$

where a constant $c > 0$ doesn't depend on f, x and φ. By virtue of the condition (10) the required proof is obtained from (12) and (13). The theorem is proved.

Note that the relation (11) in the case $\varphi(\delta) = \delta^\alpha$ $(\delta > 0)$, $k-1 < \alpha \le k$, is proved in [1].

It also holds the following

Lemma. *Let* $f \in L^1_{loc}(\mathbf{R}^n)$, $B(x,r) \subset B(x_0,R)$, $0 < r \le R < +\infty$. *Then it is valid the relation*

$$\forall t \in B(x,r): \ \left| P_{k,B(x,r)} f(t) - P_{k,B(x_0,R)} f(t) \right| \le c \cdot \int_r^{2R} \frac{m_f^{k+1}(x;2t)}{t} dt,$$

where a constant $c > 0$ *depends only on* n *and the system* $\{\varphi_v\}$.

We introduce for $f \in L^1_{loc}(\mathbf{R}^n)$ the denotation

$$\Delta_h^1(f;x) := f(x+h) - f(x), \quad \Delta_h^k(f;x) := \Delta_h^1\left(\Delta_h^{k-1}(f;x)\right).$$

By means of the lemma it is proved

Theorem 3. *Let* $f \in L^1_{loc}(\mathbf{R}^n)$, $k \in \mathbf{N}$, $h \in \mathbf{R}^n$. *Then for almost all* $x \in \mathbf{R}^n$ *it is valid the inequality*

$$\left|\Delta_h^k(f;x)\right| \le c \cdot \sum_{i=0}^k \int_0^{2(k+1)|h|} \frac{m_f^k(x+ih;2t)_1}{t} dt,$$

where a constant $c > 0$ *doesn't depend on* f, x *and* h.

By means of proposition 1 we get from this theorem

Corollary. *Let* $f \in L^1_{loc}(\mathbf{R}^n)$, $k \in \mathbf{N}$, $\varphi \in \Phi_k$, $h \in \mathbf{R}^n$. *Then for almost all* $x \in \mathbf{R}^n$ *it is valid the inequality*

$$\left|\Delta_h^k(f;x)\right| \le c \cdot \left\{\sum_{i=0}^{k} f_{k,\varphi,1}^{\#}(x+ih)\right\} \cdot \int_0^{|h|} \frac{\varphi(y)}{y} dy, \tag{14}$$

where a constant $c>0$ doesn't depend on f, x and h.

Now introduce some denotations:

$$C_p^{k,\varphi} := \left\{ f \in L^p(\mathbf{R}^n) : f_{k,\varphi,1}^{\#} \in L^p(\mathbf{R}^n) \right\}, \quad \left(k \in \mathbf{N}, 1 \le p \le \infty, \varphi \in \Phi_k\right);$$

$$\|f\|_{C_p^{k,\varphi}} := \|f\|_{L^p(\mathbf{R}^n)} + \left\|f_{k,\varphi,1}^{\#}\right\|_{L^p(\mathbf{R}^n)};$$

$$\omega_f^k(\delta)_p := \sup_{|h|\le\delta} \left\|\Delta_h^k(f;\cdot)\right\|_{L^p(\mathbf{R}^n)}, \quad \left(f \in L^p(\mathbf{R}^n),\ 1 \le p \le \infty,\ k \in \mathbf{N}\right);$$

$$H_p^{k,\varphi} := \left\{ f \in L^p(\mathbf{R}^n) : \omega_f^k(\delta)_p = O(\varphi(\delta)), \delta > 0 \right\} \quad \left(k \in \mathbf{N}, 1 \le p \le \infty, \varphi \in \Phi_k\right).$$

$$\|f\|_{H_p^{k,\varphi}} := \|f\|_{L^p(\mathbf{R}^n)} + \sup_{\delta>0} \frac{\omega_f^k(\delta)_p}{\varphi(\delta)}.$$

We get from the inequality (14)

$$\left\|\Delta_h^k(f;\cdot)\right\|_{L^p(\mathbf{R}^n)} \le c \cdot \left\|f_{k,\varphi,1}^{\#}\right\|_{L^p(\mathbf{R}^n)} \cdot \int_0^{|h|} \frac{\varphi(y)}{y} dy,$$

and hence

$$\omega_f^k(\delta)_p \le c \cdot \left\|f_{k,\varphi,1}^{\#}\right\|_{L^p} \cdot \int_0^{\delta} \frac{\varphi(y)}{y} dy, \delta > 0. \tag{15}$$

We get from this inequality that, if $f \in C_p^{k,\varphi}$, then $\omega_f^k(\delta)_p = O(\psi(\delta))(\delta > 0)$, where $\psi(\delta) := \int_0^{\delta} y^{-1} \cdot \varphi(y) dy$. Besides, by means of the inequality (15) we have

$$\exists c > 0\ \forall f \in C_p^{k,\varphi} : \|f\|_{H_p^{k,\psi}} \le c \|f\|_{C_p^{k,\varphi}}.$$

Thus, it is proved

Theorem 4. *Let* $k \in \mathbf{N}$, $1 \le p \le \infty$, $\varphi \in \Phi_k$, $\psi(\delta) := \int_0^{\delta} t^{-1} \cdot \varphi(t) dt$. *Then it holds the imbedding*

$$C_p^{k,\varphi} \subset H_p^{k,\psi}.$$

In the case $\varphi(\delta) = \delta^{\alpha}$ $(\delta > 0)$, $k-1 \le \alpha < k$ this result exists in [1].

Remark 1. It follows from the results of [6] that, if $\psi(\delta) = O(\varphi(\delta))$ $(\delta > 0)$, then

$$C_\infty^{k,\varphi} = H_\infty^{k,\varphi}.$$

(The case $\varphi(\delta) = \delta^{\alpha}$, $k-1 \le \alpha < k$, see also [1]).

2. In [1] it is shown that if $\varphi(\delta) = \delta^{\alpha}$, $k-1 \le \alpha < k$, $1 \le p < \infty$, then $C_p^{k,\varphi} \ne H_p^{k,\varphi}$.

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