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DEFINITION OF THE UNKNOWN COEFFICIENT OF A PARABOLIC EQUATION WITH NON-LOCAL BOUNDARY AND COMPLEMENTARY CONDITIONS

Abstract

The inverse non-selfadjoint boundary value problem is investigated for linear parabolic equations.

Consider on the domain $Q=(0,1)\times(0,T)$ the problem

$$\begin{cases} u_t(t,x)-u_{xx}(t,x)+a(t)u(t,x)=F(t,x), & (1)\\ u(0,x)=\varphi(x), \quad 0\le x\le 1, & (2)\\ u(t,0)=0,\ u_x(t,0)=u_x(t,1), \quad 0\le t\le T, & (3)\\ u_x(t,0)+\beta u(t,1)=g(t), \quad 0\le t\le T, & (4)\end{cases}$$

where $0<T<+\infty$; F,φ,g are given functions, β is a given number, and $u(t,x), a(t)$ are desired functions ([1], [2], [3], [4], [5]).

Definition. *Under the classic solution of the problem (1)-(4) we understand the pair $\{u(t,x), a(t)\}$ of functions $u(t,x), a(t)$ possessing the properties:*

a) $u(t,x), u_t(t,x), u_{xx}(t,x)\in C(\overline{Q})$;

b) $a(t)\in C([0,T])$;

c) all conditions of (1)-(4) are satisfied in a classic sense.

To investigate the problem (1)-(4) we must study the following spectral problem

$$\begin{cases} X''(x)+\lambda X(x)=0, \quad 0\le x\le 1, & (5)\\ X(0)=0, \quad X'(0)=X'(1). & (6)\end{cases}$$

It is clear that the problem (5), (6) is non-selfadjoint. Corresponding conjugate problem has the form

$$\begin{cases} Y''(x)+\lambda Y(x)=0, & (7)\\ Y(0)=Y(1), \quad Y'(1)=0. & (8)\end{cases}$$

It is obvious that the problem (5), (6) has eigen values

$$\lambda_k=(2\pi k)^2, \quad k=0,1,2,\ldots$$

and eigen-functions

$$\overline{X}_0(x)=x, \quad \overline{X}_k(x)=\sin 2\pi kx, \quad k=1,2,\ldots. \qquad (9)$$

A system of eigen functions $\overline{X}_k(x), (k=0,1,2,\ldots)$ doesn't form a basis in $L_2(0,1)$ ([2]). Complement the eigen functions $\overline{X}_k(x), (k=0,1,2,\ldots)$ with respect to the whole system by the adjoined functions

$$\overline{\overline{X}}_k(x)=x\cos 2\pi kx, \quad k=1,2,\ldots \qquad (10)$$

of the boundary value problem (5), (6).

Over denote the functions systems (9) and (10) as follows

$$X_0(x)=x, \quad X_{2k-1}(x)=x\cos 2\pi kx, \quad X_{2k}(x)=\sin 2\pi kx, \quad k=1,2,\ldots. \qquad (11)$$

Further, we find eigen and adjoint functions of the problem (7), (8) and over denote as follows:

$$Y_0(x)=2,\quad Y_{2k-1}(x)=4\cos 2\pi kx,\quad Y_{2k}(x)=4(1-x)\sin 2\pi kx,\quad k=1,2,\ldots. \tag{12}$$

It is easy to calculate that sequences (11) and (12) form a boirthogonal system of functions on the interval $(0,1)$, i.e.

$$(X_i,Y_j)\equiv\int_0^1 X_i(x)Y_j(x)dx=\delta_{ij}. \tag{13}$$

Here δ_{ij} is a Kronecker's symbol.

Thus, the system (11) forms a basis in $L_2(0,1)$ and the system (12) forms a biorhogonal system of functions in $L_2(0,1)$, and it is obvious that for each classic solution $\{u(t,x),a(t)\}$ of the problem (1)-(4) its first component $u(t,x)$ has the form

$$u(t,x)=\sum_{k=0}^{\infty}u_k(t)X_k(x), \tag{14}$$

where

$$u_k(t)=\int_0^1 u(t,x)Y_k(x)dx,\quad k=0,1,\ldots. \tag{15}$$

By multiplying both sides of the equality (1) by $Y_k(x)$ $(k=0,1,2,\ldots)$ and by integrating with respect to x from zero to the unit we obtain the following equation system with respect to $u_k(t)\,(k=0,1,2,\ldots)$:

$$u_0'(t)+a(t)u_0(t)=F_0(t), \tag{16}$$

$$u_{2k-1}'(t)+\left[(2\pi k)^2+a(t)\right]u_{2k-1}(t)=F_{2k-1}(t), \tag{17}$$

$$u_{2k}(t)+\left[(2\pi k)^2+a(t)\right]u_{2k}(t)+4\pi k u_{2k-1}(t)=F_{2k}(t),\ k=1,2,\ldots, \tag{18}$$

where

$$F_k(t)=\int_0^1 F(t,x)Y_k(x)dx,\quad k=0,1,2,\ldots. \tag{19}$$

By solving the equation system (16), (17) and (18) we get

$$u_0(t)=A_0e^{-\int_0^t a(s)ds}+\int_0^t F_0(\tau)e^{-\int_\tau^t a(s)ds}d\tau, \tag{20}$$

$$u_{2k-1}(t)=A_{2k-1}e^{-(2\pi k)^2t-\int_0^t a(s)ds}+\int_0^t F_{2k-1}(\tau)e^{-(2\pi k)^2(t-\tau)-\int_\tau^t a(s)ds}d\tau, \tag{21}$$

$$\begin{gathered}u_{2k}(t)=\left[A_{2k}-4\pi kA_{2k-1}t\right]e^{-(2\pi k)^2t-\int_0^t a(s)ds}+\\+\int_0^t\left[F_{2k}(\tau)-4\pi kF_{2k-1}(\tau)(t-\tau)\right]e^{-(2\pi k)^2(t-\tau)-\int_\tau^t a(s)ds}d\tau,\quad k=1,2,\ldots\end{gathered}, \tag{22}$$

where A_k $(n=0,1,2,\ldots)$ are arbitrary constants. By using the representation (11) and initial condition (2) we get

$$u(0,x)=\sum_{k=0}^{\infty}u_k(0)X_k(x)=\sum_{k=0}^{\infty}\varphi_kX_k(x)=\varphi(x), \tag{23}$$

where $\varphi_k = \int_0^1 \varphi(x) Y_k(x) dx$.

It follows from (23) that

$$u_k(0) = \varphi_k, \quad k = 0,1,2,\ldots. \tag{24}$$

Using the initial condition (24), appearing in (20)-(22) determine the constants A_k $(k = 0,1,2,\ldots)$ in the form

$$A_k = \varphi_k, \quad k = 0,1,2,\ldots. \tag{25}$$

It follows from (14), (20)-(22) and (25) that

$$u(t,x) = \left[\varphi_0 e^{-\int_0^t a(s)ds} + \int_0^t F_0(\tau) e^{-\int_\tau^t a(s)ds} d\tau \right] X_0(x) +$$
$$+ \sum_{k=1}^{\infty} \left[\varphi_{2k-1} e^{-(2\pi k)^2 t - \int_0^t a(s)ds} + \int_0^t F_{2k-1}(\tau) e^{-(2\pi k)^2 (t-\tau) - \int_\tau^t a(s)ds} d\tau \right] \times$$
$$\times X_{2k-1}(x) + \sum_{k=1}^{\infty} \left\{ \left[\varphi_{2k} - 4\pi k \varphi_{2k-1} t \right] e^{-(2\pi k)^2 t - \int_0^t a(s)ds} + \right.$$
$$\left. + \int_0^t \left[F_{2k}(\tau) - 4\pi k (t-\tau) F_{2k-1}(\tau) \right] e^{-(2\pi k)^2 (t-\tau) - \int_\tau^t a(s)ds} d\tau \right\} X_{2k}(x). \tag{26}$$

To determine the second component $a(t)$ of the solution, by using (4) we find

$$u_{xt}(t,0) + \beta u_t(t,1) = g'(t).$$

Hence we shall have

$$g'(t) + a(t) g(t) = (\beta + 1) F_0(t) + \sum_{k=1}^{\infty} \left[2\pi k F_{2k}(t) + (\beta + 1)(2\pi k)^2 F_{2k-1}(t) \right] -$$
$$- \sum_{k=1}^{\infty} \left[(2\pi k)^3 \varphi_{2k} + (\beta + 3)(2\pi k)^2 \varphi_{2k-1} - 2(2\pi k)^4 t \varphi_{2k-1} \right] \cdot e^{-(2\pi k)^2 t - \int_0^t a(s)ds} -$$
$$- \sum_{k=1}^{\infty} \int_0^t \left[(2\pi k)^3 F_{2k}(\tau) + (\beta + 3)(2\pi k)^2 F_{2k-1} - 2(2\pi k)^4 (t-\tau) F_{2k-1}(\tau) \right] \times$$
$$\times e^{-(2\pi k)^2 (t-\tau) - \int_\tau^t a(s)ds} d\tau. \tag{27}$$

It is valid the following

Theorem. *Let*

1) $\varphi(x) \in C^V([0,1])$, $\varphi(0) = \varphi''(0) = \varphi^{IV}(0) = 0$, $\varphi'(0) = \varphi'(1)$, $\varphi'''(0) = \varphi'''(1)$;
2) $g(t) \in C'([0,T])$, $g(t) \neq 0 \ \ \forall t \in [0,T]$;
3) $F(x,t) \in C(\overline{Q})$ *and at each fixed* $t \in [0T]$
   $F(\cdot,x) \in C^V([0,1])$, $F(\cdot,0) = F_{xx}(\cdot,0) = F_{xxxx}(\cdot,0) = 0$, $F_x(\cdot,0) = F_x(\cdot,1)$,
   $F_{xxx}(\cdot,0) = F_{xxx}(\cdot,1)$.

Then under sufficiently small values of T the problem (1)-(4) has a classic solution.

Proof. Write (27) in the form of the operator equation

$$a(t)=P[a(t)]. \tag{28}$$

It is obvious that for any $a(t)$ from the space $C([0,T])$, $P[a(t)]\in C([0,T])$, i.e.

$$P:C([0,T])\to C([0,T]).$$

Now prove the compressibility of the operator P.

Let $a(t),b(t)\in C([0,T])$.

Then

$$|P[a(t)]-P[b(t)]|\le\frac{1}{|g(t)|}\sum_{k=1}^{\infty}\left\{(2\pi k)^3|\varphi_{2k}|+\left((\beta+3)(2\pi k)^2+2T(2\pi k)^4\right)|\varphi_{2k-1}|\right\}\times$$

$$\times\left|e^{-\int_0^t a(s)ds}-e^{-\int_0^t b(s)ds}\right|+\frac{1}{|g(t)|}\int_0^T\left\{\left[(2\pi k)^3|F_{2k}(\tau)|+\left((\beta+3)(2\pi k)^2+4T(2\pi k)^4\right)\times\right.\right.$$

$$\left.\left.\times|F_{2k-1}(\tau)|\right]\cdot\left|e^{-\int_\tau^t a(s)ds}-e^{-\int_\tau^t b(s)ds}\right|\right\}d\tau.$$

By $c_i\ \left(i=\overline{1,s}\right)$ denote positive constants independent on t.

By virtue of conditions 1)-3)

$$\sum_{k=1}^{\infty}\left\{(2\pi k)^3|\varphi_{2k}|+\left((\beta+3)(2\pi k)^2+2T(2\pi k)^4\right)|\varphi_{2k-1}|\right\}\le c_1,$$

$$\int_0^T\left\{\sum_{k=1}^{\infty}\left[(2\pi k)^3|F_{2k}(\tau)|+\left((\beta+3)(2\pi k)^2+4T(2\pi k)^4\right)|F_{2k-1}(\tau)|\right]\right\}d\tau\le c_2,$$

$$\max_{0\le t\le T}\left|\frac{1}{g(t)}\right|=c_3,$$

$$\left|e^{-\int_0^t a(s)ds}-e^{-\int_0^t b(s)ds}\right|\le c_4\cdot T\max_{0\le t\le T}|a(t)-b(t)|,$$

$$\left|e^{-\int_\tau^t a(s)ds}-e^{-\int_\tau^t b(s)ds}\right|\le c_5\cdot T\max_{0\le t\le T}|a(t)-b(t)|.$$

Then

$$\max_{0\le t\le T}|P[a(t)]-P[b(t)]|\le c_3(c_1c_4+c_2c_5)T\max_{0\le t\le T}|a(t)-b(t)|.$$

Choosing T sufficiently small, we obtain that

$$c_3(c_1c_4+c_2c_5)T=q<1.$$

Consequently, $P[a(t)]$ is a compressible operator. Then by a fixed-point theorem, the equation (28) has a unique solution $a(t)$ from $C([0,T])$.

Substituting the found solution (27) in (26) we find the first component $u(t,x)$ of the solution $\{u(t,x),a(t)\}$. It is easy to prove that the pair $\{u(t,x),a(t)\}$ satisfies (1)-(4).

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