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THE BOUNDARY VALUE PROBLEM IN THE CLASS OF GENERALIZED ANALYTIC FUNCTIONS-JUMP PROBLEM

Abstract

In the class of generalized analytic functions the solution of classical Reimann problem - jump problem is found in the form of generalized integral of Cauchy-Stiltjes.

Consider the class of generated analytic functions $u_{p,2}(A,B,G)$ in the I.N.Vecua sense, i.e. the class of regular solutions of equations (see [1], p.158)

$$\partial_{\bar{z}}F(z)+A(z)F(z)+B(z)\overline{F}(z)=0, \tag{1}$$

where $A(z), B(z)\in L_p(G),\ p>2, \partial_{\bar{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial}\right)$.

Let G be a finite domain (simple connected) bounded by Jordan rectifiable curve Γ.

Let $w(z)\in u_{p,2}(A,B,C)$. Denote by $w^+(t)=\lim\limits_{\substack{z\to t\\ z\in G}} w(z)\left(w^-(t)=\lim\limits_{\substack{z\to t\\ z\overline{\in} G}} w(z)\right)$ by non-tangent paths, where $t\in\Gamma$ is fix point.

The statement of problem is:

To find generalized analytic function $w(z)$ from the class $u_{p,2}(A,B,E/\Gamma)$ equal to zero at infinity and satisfying to boundary condition:

$$w^+(t)-w^-(t)=\mu'(t) \text{ almost everywhere on } \Gamma, \tag{2}$$

where $\mu(t)$ is given function with bounded variation on Γ and E is finite complex plane.

In the class of analytic functions such problem is considered for the first time. Similar problem in the class of analytic functions is the component of general boundary value Riemann problem, which is also called jump problem.

In the class of analytic function mentioned problem was investigated by F.D.Gahov and his scholars (see [3]), A.A.Babayev and his scholars (see [4]) and etc.

In present paper the solution of stated problem (2) will be found in the form of generalized integral of Cauchy-Stiltjes.

Now we will solve this problem.

Definition 1. *The following expression*

$$F(z)=\frac{1}{2\pi i}\int_\Gamma \Omega_1(z,t)d\mu(t)-\int_\Gamma \Omega_2(z,t)d\overline{\mu}(t) \tag{3}$$

we will call integral of Cauchy-Stiltjes type, where $\Omega_1(z,t)$ and $\Omega_2(z,t)$ are normalized kernels of class $u_{p,2}(A,B,G)$, $\mu(t)$ is function with bounded variation on Γ.

Property. Integral (3) is generalized analytic function out of Γ, i.e. $F(z)\in u_{p,2}(A,B,E\setminus\Gamma)$ and $F(\infty)=0$.

Definition 2. *Let point $t_0\in\Gamma$ and s_0 is corresponding to t_0 value on the arc $(0\le s\le l,\ l$ is the length of $\Gamma)$. Denote by Γ_ε the arc, the ends of which are points*

$t(s_0-\varepsilon), t(s_0+\varepsilon)$ and which doesn't contain t_0. The finite limit of the following expression (if it is exist)

$$\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{\Gamma_\varepsilon}\Omega_1(t_0,t)d\mu(t)-\int_{\Gamma_\varepsilon}\Omega_2(t_0,t)d\overline{\mu}(t)= \\ =\frac{1}{2\pi i}\int_{\Gamma}\Omega_1(t_0,t)d\mu(t)-\int_{\Gamma}\Omega_2(t_0,t)d\overline{\mu}(t). \tag{4}$$

If in particular $d\mu(t)=\mu'(t)dt$, then integral of Cauchy-Stiltjes type is called integral of Cauchy-Lebesgue type.

Taking into account that at the point t_0 there exists tangent to Γ, we will draw the normal to Γ at this point and we will take point z on the straight line $\overline{z}t_0$ to angle of $\psi_0, 0\le\psi_0<\frac{\pi}{2}$ to normal. Point z can be inside and outside of G. Let the distance zt_0 is equal to ε.

Theorem 1. *If at the point $t_0=t(s_0)$ the curve Γ have tangent, function $\mu(s)$ and its variation $v(s)$ have finite derivatives, and $\left|\frac{dt}{ds}\right|_{s=s_0}=1$, then expression*

$$\frac{1}{2\pi i}\left[\int_{\Gamma}\Omega_1(z,t)d\mu(t)-\int_{\Gamma}\Omega_2(z,t)d\overline{\mu}(t)-\int_{\Gamma_\varepsilon}\Omega_1(t_0,t)d\mu(t)-\int_{\Gamma_\varepsilon}\Omega_2(t_0,t)d\overline{\mu}(t)\right] \tag{5}$$

tends to limit $\pm\frac{1}{2}\mu'(t_0)$ ($\pm\frac{1}{2}\mu'(t_0)$, if $z\to t_0$ outside of G, $-\frac{1}{2}\mu'(t_0)$, if $z\to t_0$ outside of G). Tendency to limit is uniform with respect to $\psi_0, |\psi_0|\le\frac{\pi}{2}\theta, \theta<1$.

So as $\mu'(s)$ and $v'(s)$ exist almost everywhere on Γ and curve Γ is rectifiable and have almost at all points the tangent, moreover, $\left|\frac{dt}{ds}\right|=1$, then the presence in (5) of limit $\pm\frac{1}{2}\mu'(t_0)$ takes place almost everywhere on Γ.

Show the short proof.

by virtue formula (see [1], p.179)

$$\Omega_1(z,t)=\frac{1}{t-z}+O\left(|z-t|^{\alpha}\right),\quad \Omega_2(z,t)=O\left(|z-t|^{\alpha}\right),\quad \alpha=\frac{2}{p};\ p>2; \\ \Omega_1(z,t)\frac{1}{t-z}+M_1(z,t),\quad \Omega_2(z,t)=M_2(z,t), \tag{6}$$

where $M_1(z,t)=\frac{m_1(z,t)}{|z-t|^{\alpha}}$, $M_2(z,t)=\frac{m_2(z,t)}{|z-t|^{\alpha}}$ are continuous functions on z and t everywhere, except that points for which $z=t$, and m_1 and m_2 uniformly bounded by z and t.

Taking into account (6) into (5), we obtain

$$\frac{1}{2\pi i}\left[\int_\Gamma \Omega_1(z,t)d\mu(t)-\int_\Gamma \Omega_2(z,t)d\overline{\mu}(t)-\int_{\Gamma_\varepsilon} \Omega_1(t_0,t)d\mu(t)-\int_{\Gamma_\varepsilon} \Omega_2(t_0,t)d\overline{\mu}(t)\right]=$$

$$=\frac{1}{2\pi i}\left[\int_\Gamma \frac{d\mu(z)}{t-z}-\int_{\Gamma_\varepsilon}\frac{d\mu(z)}{t-t_0}\right]+\frac{1}{2\pi i}\left[\int_\Gamma M_1(z,t)d\mu(t)-\int_{\Gamma_\varepsilon} M_1(t_0,t)d\mu(t)\right]+ \qquad (7)$$

$$+\frac{1}{2\pi i}\left[\int_{\Gamma_\varepsilon} M_2(t_0,t)d\overline{\mu}(t)-\int_\Gamma M_2(z,t)d\overline{\mu}(t)\right]=J_1+J_2+J_3.$$

Each of three expressions in square brackets investigates separately.
Consider the first difference:

$$J_1=\frac{1}{2\pi i}\left[\int_\Gamma \frac{d\mu(z)}{t-z}-\int_{\Gamma_\varepsilon}\frac{d\mu(z)}{t-t_0}\right]. \qquad (8)$$

By virtue of the main lemma of I.I.Privalov for ordinary integrals of Cauchy-Stiltjes type, J_1 tends to $\pm\frac{1}{2}\mu'(t_0)$ for $\varepsilon\to 0$ ($+\frac{1}{2}\mu'(t_0)$ if $z\to t_0$, $z\in G$; $-\frac{1}{2}\mu'(t)$ if $z\to t_0$, $z\overline{\in} G$). For that the tendency to limit is uniformly with respect to ψ_0, $\left(|\psi_0|\le\frac{\pi}{2}\theta,<1\right)$ (see [2], p.184).

Consequently, for complete proof of theorem it is enough to verify that second and third expressions in square brackets in (7), (which denoted by J_2 and J_3) correspondingly, tends to zero for $\varepsilon\to 0$ uniformly with respect to ψ_0.

We have

$$J_2=\frac{1}{2\pi i}\left[\int_\Gamma \frac{m_1(z,t)d\mu(z)}{|z-t|^\alpha}-\int_{\Gamma_\varepsilon}\frac{m_2(z,t)d\mu(z)}{|t-t_0|^\alpha}\right]=$$

$$=\frac{1}{2\pi i}\left[\int_{\Gamma_\varepsilon}[M_1(z,t)-M_1(t_0,t)]d\mu(z)-\int_{t(s_0-\varepsilon)}^{t(s_0+\varepsilon)} M_1(z,t)d\mu(z)\right], \qquad (9)$$

where by $M_1(z,t)$ we denote the ratio $M_1(z,t)=\frac{m_1(z,t)}{|z-t|^\alpha}$. Using the property of $M_1(z,t)$ and $\mu(t)$ we can prove that $J_2\to 0$ for $\varepsilon\to 0$. We use the following relations

$$|M_1(z,t)-M_1(t_0-t)|\xrightarrow{\varepsilon\to 0}0 \text{ and } \int_{t(s_0-\varepsilon)}^{t(s_0+\varepsilon)} M_1(z,t)d\mu(t)\xrightarrow{\varepsilon\to 0}0.$$

Theorem 2. *Let* $z=t_0+i\varepsilon e^{i(\varphi_0+\psi_0)}$, $z^*=t_0-i\varepsilon e^{i(\varphi_0-\psi_0)}$. *The difference of values of integral of Cauchy-Stiltjes type at the points* z *and* z^*, *placed inside and outside of* Γ

$$J=\frac{1}{2\pi i}\left[\int_\Gamma \Omega_1(z,t)d\mu(t)-\Omega_2(z,t)d\overline{\mu}(t)-\right.$$

$$\left.-\int_\Gamma \Omega_1(z^*,t)d\mu(t)-\Omega_2(z^*,t)d\overline{\mu}(t)\right]\to\mu'(t_0) \qquad (10)$$

when $\varepsilon\to 0$ *for all* t_0 *of the line* Γ *except maybe the points of set of measure zero.*

Proof. Transform J by the following way:

$$J=\frac{1}{2\pi i}\left[\int_{\Gamma}\Omega_1(z,t)d\mu(t)-\Omega_2(z,t)d\overline{\mu}(t)-\int_{\Gamma}\Omega_1(t_0,t)d\mu(t)-\Omega_2(t_0,t)d\overline{\mu}(t)\right]+$$

$$+\frac{1}{2\pi i}\left[\int_{\Gamma_*}\Omega_1(t_0,t)d\mu(t)-\Omega_2(t_0,t)d\overline{\mu}(t)-\int_{\Gamma}\Omega_1(z^*,t)d\mu(t)-\Omega_2(z^*,t)d\overline{\mu}(t)\right]=J_1+J_2\ .$$

According to the theorem 1 $J_1\to\frac{1}{2}\mu'(t_0)$, $J_2\to-\left(-\frac{1}{2}\mu'(t_0)\right)=\frac{1}{2}\mu'(t_0)$.

Remark 1. Denotation $F^+(t)$ and $F^-(t)$ means that $F^+(t)-F^-(t)\to\mu'(t)$.

Theorem 3. *Let $\mu(t)$ be given function with bounded variation on Γ. The solution of the jump problem is integral of Cauchy-Stiltjes type with measure $\mu(t)$:*

$$F(z)=\frac{1}{2\pi i}\int_{\Gamma}\Omega_1(z,t)d\mu(t)-\Omega_2(z,t)d\overline{\mu}(t).$$

Really, consider integral of Cauchy-Stiltjes type with measure $\mu(t)$:

$$F(z)=\frac{1}{2\pi i}\int_{\Gamma}\Omega_1(z,t)d\mu(t)-\Omega_2(z,t)d\overline{\mu}(t).$$

By virtue of property $F(z)\in u_{p,2}(A,B,E\setminus\Gamma)$ and $F(\infty)=0$. If point $t_0\in\Gamma$ and at this point there exists tangent to Γ, then by theorem 2 $F^+(t)-F^-(t)=\mu'(t)$. Also from theorem 1 it follows that the last relation provides almost at all points of Γ.

Remark 2. At the present statement of the problem, generally speaking, the solution is not unique. But if Γ is smooth curve (then at these points exist tangent) and $\mu(t)$ is absolutely continuous, then the solution of given problem is unique.

References

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