MIRZOEV S.S., GUSEINOV M.M.

ON SOLVABILITY OF OPERATOR-DIFFERENTIAL EQUATIONS IN THE SPACES OF SMOOTH VECTOR-FUNCTIONS

Abstract

In present paper the theorem on existence of smooth solutions of one class of operator-differential equation on the whole axis in abstract Hilbert spaces, was obtained, the connection between conditions of solvability and norms of operators of mediate derivatives at these spaces was established and exact values of these norms was found.

Let H be separable Hilbert space, A be positively defined self-adjoint operator, and H_γ is scale of Hilbert spaces, generated by operator A $(\gamma \geq 0)$, i.e. $H_\gamma = D(A^\gamma)$, $(x,y)_\gamma = (A^\gamma x, A^\gamma y)$, $x, y \in D(A^\gamma)$.

We define Hilbert space [1] as follows:

$$L_2(R:H) = \left\{ f : \int_{-\infty}^{\infty} \|f\|_H^2 \, dt = \|f\|_{L_2}^2 < \infty \right\}$$

and

$$W_2^p(R:H) = \left\{ u : u^{(p)} \in L_2(R:H),\ A^p u \in L_2(R:H) \right\},$$

and norm in this space defines by the following way

$$\|u\|_{W_2^p(R:H)} = \left(\left\|u^{(p)}\right\|_{L_2}^2 + \left\|A^p u\right\|_{L_2}^2 \right)^{1/2}.$$

Further, we will denote by $\mathcal{L}(X,Y)$ the space of bounded operators, which act from space X into space Y.

Consider in space H operator-differential equation

$$P(d/dt)u(t) = P_0(d/dt : A)u(t) + P_1(d/dt)u(t) = f(t), \quad t \in R = (-\infty, \infty), \tag{1}$$

where

$$P_0(d/dt : A)u = \prod_{l=1}^{\aleph} \left(d/dt - \omega_l A\right)^{r_l} u, \tag{2}$$

$$P_1(d/dt)u = \sum_{j=0}^{n} A_{n-j} u^{(j)} \tag{3}$$

and $r_1 + r_2 + \ldots + r_\aleph = n$, ω_l $(l = 1, \ldots, \aleph)$ are numbers, for which $\operatorname{Re}\omega_l < 0$, $l = 1, .., m$, $r_1 + r_2 + \ldots + r_m = k$, where $k \leq n$ and $\operatorname{Re}\omega_l > 0$, $l \leq m+1, \ldots, \aleph$ and $k + r_{m+1} + \ldots + r_\aleph = n$, operators $A_j \in \mathcal{L}(H_j, H) \cap \mathcal{L}(H_{j+s}, H_s)$, where $s > 0$ is integer fix number, $f(t) \in W_2^s(R:H)$, $u(t) \in W_2^{n+s}(R,H)$, and derivatives $\dfrac{d^j u}{dt^j}$ $(j = 1, .., n)$ understoods in the sense of distribution theory [1].

At first, consider equation

$$P_0(d/dt : A)u(t) = f(t) \tag{4}$$

where $f \in W_2^s(R:H)$, $u \in W_2^{n+s}(R:H)$.

Takes place following

Lemma 1. *Operator* $\mathcal{P}_0$, *generated by the left-hand side of equation (4) isomorphicaly maps space* $W_0^n(R:H)$ *onto* $W_2^s(R:H)$.

Proof. After Fourier transformation from equation (4) we obtain

$$\hat{u}(\lambda) = P_0^{-1}(i\lambda : A)\hat{f}(\lambda), \ \lambda \in R,$$

where $\hat{u}(\lambda)$ and $\hat{f}(\lambda)$ are Fourier transformations of vector-functions $u(t)$ and $f(t)$, correspondingly. By Plancharelle theorem

$$\|u\|_{W_2^{n+s}}^2 = \left\|\lambda^{n+s}\hat{u}(\lambda)\right\|_{L_2}^2 + \left\|A^{n+s}\hat{u}(\lambda)\right\|_{L_2}^2 \Leftarrow \left\|\lambda^{n+s}P_0^{-1}(i\lambda : A)\hat{f}(\lambda)\right\|_{L_2}^2 +$$

$$+\left\|\lambda^{n+s}P_0^{-1}(i\lambda : A)\right\|_{L_2}^2 \le \sup_{\lambda \in R}\left\|\lambda^n P_0^{-1}(i\lambda : A)\right\|_{H\to H}^2 \cdot \left\|\lambda^s \hat{f}(\lambda)\right\|_{L_2}^2 +$$

$$+\sup_{\lambda \in R}\left\|A^n P_0^{-1}(i\lambda : A)\right\|_{H\to H}^2 \cdot \left\|A^s f(\lambda)\right\|_{L_2}^2 \le$$

$$\le const\left(\left\|\lambda^s\hat{f}(\lambda)\right\|_{L_2}^2 + \left\|A^s\hat{f}(\lambda)\right\|_{L_2}^2\right) = const\|f\|_{W_2^s}^2.$$

Further, from theorem on mediate derivatives we conclude, that $\left\|P_0(d/dt : A)u\right\|_{W_2^s} \le const\|u\|_{W_2^{n+s}}$, therefore by Banach theorem on inverse operator, $\mathcal{P}_0$ makes isomorphism between spaces $W_2^{n+s}(R:H)$.

From theorem on mediate derivatives it follows, that for $u \in W_2^{n+s}(R:H)$

$$\left\|A^{n-j}u^{(j)}\right\|_{W_2^s}^2 = \left\|A^{n+s-j}u^{(j)}\right\|_{L_2}^2 + \left\|A^{n-j}u^{(j+s)}\right\|_{L_2}^2 \le const\|u\|_{W_2^{n+s}}^2,$$

therefore from Lemma 1 follows, that numbers

$$N_{j,s}(R:H) = \sup_{0\ne u\in W_2^{n+s}(R:H)}\left\|A^{n-j}u^{(j)}\right\|_{W_2^s} \cdot \left\|P_0(d/dt : A)\right\|_{W_2^s}^{-1}, \ j = 0,...,n$$

are finite.

Theorem 1. *Let* A *be positively defined self-adjoint operator,* $A_j \in \mathcal{L}(H_j, H)\mathcal{L}(H_{j+s}, H_s), j = 0,\dots,n$ *and inequality holds*

$$\alpha = \sum_{j=0}^{n}\max\left\{\left\|A_{n-j}\right\|_{H_{n-j}\to H}, \left\|A_{n-j}\right\|_{H_{n-j+s}\to H_s}\right\} \cdot N_{j,s}(R) < 1.$$

Then operator $\mathcal{P}$, *generated by the left-hand side of equation (1) isomorphically maps space* $W_2^{n+s}(R:H)$ *on* $W_2^s(R:H)$.

Proof. So as for $u \in W_2^{n+s}(R:H)$

$$\left\|P_1(d/dt)u\right\|_{W_2^s}^2 = \left\|\sum_{j=0}^{n}A_{n-j}u^{(j+s)}\right\|_{L_2}^2 + \left\|\sum_{j=0}^{n}A^s A_{n-j}u^{(j)}\right\|_{L_2}^2 \le$$

$$\le \sum_{j=0}^{n}\left\|A_{n-j}A^{-(n-j)}\right\|_{H\to H}^2 \cdot \left\|A^{n-j}u^{(s+j)}\right\|_{L_2}^2 +$$

$$+\sum_{j=0}^{n}\left\|A^{s}A_{n-j}A^{-(n+s-j)}\right\|^{2}_{H\to H}\cdot\left\|A^{n+s-j}u^{(j)}\right\|^{2}_{L_2}=$$

$$=\sum_{j=0}^{n}\max\left(\left\|A_{n-j}\right\|^{2}_{H_{n-j}\to H},\left\|A_{n-j}\right\|^{2}_{H_{n+s-j}\to H_s}\right)\times$$

$$\times\left(\left\|A^{n-j}u^{(s+j)}\right\|^{2}_{L_2}+\left\|A^{n+s-j}u^{(j)}\right\|^{2}_{L_2}\right)\le const\|u\|^{2}_{W_2^{n+s}},$$

then operator $\mathcal{P}_1$, generated by differential expression $P_1(d/dt)\,u$ is bounded operator from space $W_2^{n+s}(R:H)$ into space $W_2^{s}(R:H)$.

Rewrite equation (1) in the form

$$\mathcal{P}u\equiv\mathcal{P}_0u+\mathcal{P}_1u=f\,, \tag{5}$$

where $u\in W_2^{n+s}(R:H)$, $f\in W_2^{s}(R:H)$. By Lemma 1 there exists bounded inverse operator $\mathcal{P}_0^{-1}: W_2^{s}(R:H)\to W_2^{n+s}(R:H)$. After substitution $u=\mathcal{P}_0^{-1}v$ from equation (5) we obtain equation:

$$v+\mathcal{P}_1\mathcal{P}_0^{-1}v=f$$

in the space $W_2^{s}(R:H)$. Show, that $\left\|\mathcal{P}_1\mathcal{P}_0^{-1}\right\|_{W_2^s\to W_2^s}<1$. Really, for any $v\in W_2^{s}(R:H)$

$$\left\|\mathcal{P}_1\mathcal{P}_0^{-1}v\right\|_{W_2^s}=\left\|\mathcal{P}_1u\right\|_{W_2^s}\le\sum_{j=0}^{u}\left\|A_{n-j}u^{(j)}\right\|_{W_2^s}\le$$

$$\le\sum_{j=0}^{u}\left(\left\|A_{n-j}u^{(j+s)}\right\|^{2}_{L_2}+\left\|A^{s}A_{n-j}u^{(j)}\right\|^{1/2}_{L_2}\right)\le$$

$$\le\sum_{j=0}^{n}\max\left(\left\|A_{n-j}\right\|_{H_{n-j}\to H},\left\|A_{n-j}\right\|_{H_{n+s-j}\to H_s}\right)\cdot\left(\left\|A^{n+s-j}u^{(j)}\right\|^{2}_{L_2}+\left\|A^{n-j}u^{(j+s)}\right\|^{2}_{L_2}\right)^{1/2}=$$

$$=\sum_{j=0}^{n}\max\left(\left\|A_{n-j}\right\|_{H_{n-j}\to H},\left\|A_{n-j}\right\|_{H_{n+s-j}\to H_s}\right)\left\|A^{n-j}u^{(j)}\right\|_{W_2^s}\le$$

$$\le\sum_{j=0}^{n}\max\left(\left\|A_{n-j}\right\|_{H_{n-j}\to H},\left\|A_{n-j}\right\|_{H_{n+s-j}\to H_s}\right)N_{j,s}(R)\left\|\mathcal{P}_0u\right\|_{W_2^s}=$$

$$=\alpha\left\|\mathcal{P}_0u\right\|_{W_2^s}=\alpha\|v\|_{W_2^s}.$$

So as according to the statement of theorem $\alpha<1$, then $\left\|\mathcal{P}_1\mathcal{P}_0^{-1}\right\|_{W_2^s\to W_2^s}\le\alpha<1$. Consequently, operator $E+\mathcal{P}_1\mathcal{P}_0^{-1}$ we transform in space $W_2^{s}(R:H)$ and we can find u:

$$u=\mathcal{P}_0^{-1}\left(E+\mathcal{P}_1\mathcal{P}_0^{-1}\right)^{-1}f\,.$$

Theorem is proved.

Form this theorem it is seen, that for obtaining the solvability condition of equation (1), we should find exact values of norm of mediate derivatives $N_{j,s}(R)\ (j=0,\dots,n)$.

Takes place following

Theorem 2. *The number* $N_{j,s}(R)=b_{n,j}$, *where*

$$b_{n,j}=\sup_{\xi\in R}\left|\xi^{j}P_0^{-1}(i\xi;1)\right| \tag{6}$$

and

$$P_0(i\xi;1)=\prod_{j=1}^{N}(i\xi-\omega_j)^{r_j},\ \xi\in R.$$

Proof. For $\beta\in\left[0,b_{n,j}^{-2}\right)$ consider polynomial operator beam [see (6)]

$$Q_{j,s}(\lambda:\beta:A)=\left[(i\lambda)^{2s}E+A^{2s}\right]P_j(\lambda;\beta;A)\equiv$$
$$\equiv\left[(i\lambda)^{2s}E+A^{2s}\right]\left[P_0(\lambda:A)P_0^{*}(-\bar{\lambda}:A)-\beta(i\lambda)^{2j}A^{2j}\right]$$

From spectral expansion of operator A it follows, that for $\beta\in\left[0,b_{nj}^{-2}\right)$ the operator beam $Q_{j,s}(\lambda:\beta:A)$ represented in the form

$$Q_{j,s}(\lambda:\beta:A)=F_{j,s}(\lambda:\beta:A)F_{j,s}^{*}(-\bar{\lambda}:\beta:A)$$

where

$$F_{j,s}(\lambda:\beta:A)=\prod_{l=1}^{n+s}\left(\lambda E-\alpha_{l,j}(\beta)A\right),$$

moreover, $\operatorname{Re}\alpha_{l,j}(\beta)<0,\ l=1,\ldots,n+s,\ j=0,\ldots,n$.

Using Fourier transformation and Plancharelle theorem for all $\beta\in\left[0,b_{n,j}^{-2}\right)$ and $u\in W_2^{n+s}(R:H)$ we can verify equality

$$\left\|P_0(d/dt:A)u\right\|_{W_2^s}^2-\beta\left\|A^{n-j}u^{(j)}\right\|_{W_2^s}^2=\left\|F_{j,s}(d/dt:\beta:A)u\right\|_{L_2}^2. \tag{7}$$

From here we find, that for all $\beta\in\left[0,b_{n,j}^{-2}\right)$ and $u\in W_2^{n+s}(R:H)$

$$\left\|P_0(d/dt:A)u\right\|_{W_2^s}^2\geq\beta\left\|A^{n-j}u^{(j)}\right\|_{W_2^s}^2.$$

Passing to the limit for $\beta\to b_{n,j}^{-2}$ in last inequality we obtain, that

$$N_{j,s}(R)\leq b_{n,j}.$$

In order to prove equality $N_{j,s}(R)=b_{n,j}$ we define continuous functional

$$\mathcal{E}(u)=\left\|P_0(d/dt:A)u\right\|_{W_2^s}^2-\beta\left\|A^{n-j}u^{(j)}\right\|_{W_2^s}^2$$

in space $W_2^{n+s}R:H$ and using spectral expansion of operator A, for any $\varepsilon>0$ we find vector-function $u_\varepsilon(t)=g_\varepsilon(t)\varphi_\varepsilon$, where $\varphi_\varepsilon\in H_{2(n+s)}$, and $g_\varepsilon(t)$ is numeric function from space $W_2^{n+s}(R)$, such that

$$\mathcal{E}(u_\varepsilon)=\int_{-\infty}^{\infty}\left(F_j\left(i\xi;b_{nj}^{-2}+\varepsilon,A\right)\varphi_\varepsilon,\varphi_\varepsilon\right)\left|g_\varepsilon(\xi)\right|^2d\xi<0.$$

Consequently, $N_{j,s}(R)=b_{n,j},j=0,\ldots,n$.

Form theorems 1 and 2 we obtain

Theorem 3. *Let A be positively defined self-adjoint operator, $A_j\in\mathcal{L}(H_j,H)\cap$ $\cap\mathcal{L}(H_{j+s},H_s)$, $j=0,\ldots,n$ and following inequality holds*

$$\alpha = \sum_{j=0}^{n} \max\left(\left\| A_{n-j} \right\|_{H_{n-j} \to H}, \left\| A_{n-j} \right\|_{H_{n-j+s} \to H_s} \right) b_{n,j} < 1.$$

Then operator $\mathcal{P}$ makes isomorphism between spaces $W_2^{n+s}(R;H)$ and $W_2^s(R;H)$. Here numbers b_{nj} are defined from equality (6).

It must be noted, that solvability of boundary value problems in spaces of smooth functions, was investigated, for example, in paper [2-5].

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Mirzoev S.S., Guseynov M.M.

Baku State University named after M.Rasul-zadeh,
23, Z.I. Khalilov str., 370148, Baku, Azerbaijan.

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