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Uniqueness classes of generalized solutions for degenerate parabolic equations in unbounded noncylindrical domais

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Abstract. *We will prove an analogue of the inequality Saint-Venant's well known in mechanic. In paper is to study of solutions of the initial boundary problems for degenerate non-linear parabolic equations in some classes of unbounded domains. Uniqueness classes of generalized solutions are found.*

Keywords. degenerate · higher-order parabolic equations · nonlinear · uniqueness classes.

Mathematics Subject Classification (2010): 35J25 · 35B40

1 Introduction

In paper is to study of solutions of the boundary problems for degenerate nonlinear parabolic equations in unbounded domains $G \subset R^{n+1}_{x,t}$, $n \geq 1$, where $G \subset \{(t,x) : x \in R^n,$ $\infty > T > t - y(|x|)\}$, $y(s) > -T$–any continuously monotone increasing function. We get apriori estimates that analogies of Saint-Venant's principle. Correspondingly results is obtained in works [1]-[6], [8]. On basics this estimates the finding uniqueness classes of generalized solutions correspondingly to classes of Tixonov in case $y(s) > -T$. If $y(s)$ sufficiently small increasing we is proved uniqueness of solutions. This classes of functions in case $y(s) = const$ passing to Taklind classes.

Let $G \subset \{(x,t) : T > t > -y(|x|)\}$ is unbounded domain with piece-wise smooth boundary $\partial G = \Gamma_T \bigcup \Gamma$, $\Gamma_T \subset \{(x,t) : t = T\}$, $\Gamma-$ parabolic part of boundary G, $\Gamma_1 = \{(x,t) \in \Gamma : \ v(x,t) = 1\}$, $v(x,t)$-unique normal vector to ∂G.

$$\frac{\partial u}{\partial t} - \sum_{|\alpha| \leq 2m} (-1)^{|\alpha|} D^{\alpha} A_{\alpha}(x,t,u,Du,\ldots,D_x^m u) = 0, \tag{1.1}$$

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$$\forall (x,t) \in G,$$

$$D_x^{\alpha} \left|_{\partial G \setminus \Gamma_1} = 0, \ |\alpha| \le m-1 \right. \tag{1.2}$$

$$u|_{\Gamma_1} = 0 \tag{1.3}$$

Assume that the coefficients $A_{\alpha}(x,t,\xi)$ are measurable with respect to $(x,t) \in G$, continuously with respect to $\xi \in R^M$, where M is the number of different multi-indices of length no more than m and satisfying the conditions

$$\sum_{|\alpha| \le m} A_{\alpha}(x,t,\xi)\, \xi_{\alpha}^m > \omega(x) |\xi^m|^P - C_1 \omega(x) \sum_{i=1}^{m-1} |\xi_i|^p - f_1(x,t) \tag{1.4}$$

$$|A_{\alpha}(x,t,\xi)| \le C_2 \omega(x) \sum_{i=0}^{m} |\xi_i|^p + f_2(x,t) \tag{1.5}$$

where $\xi = \left(\xi^0, ..., \xi^m\right), \xi^i = \left(\xi_{\alpha}^i\right), \ |\alpha| = i, \quad p > 1,$

$$f_1(x,t) \in L_p(0, T, L_{p,loc}(\Omega_t)),$$

$$f_2(x,t) \in L_{1,loc}(G),$$

$$\Omega_{\tau} = G \bigcap \{(x,t) : t = \tau\}.$$

We define $G(\tau) = G \bigcap \{(x,t) : |x| < \tau\},$

$$G_h^s(\tau) = G(\tau) \bigcap \{(x,t) : s > t > \tau\},$$

$$G(\tau_1, \tau_2) = G(\tau_2) \setminus G(\tau_1), \ S(\tau) = \partial G(\tau) \setminus \partial G,$$

$$\delta_t(\tau) = \Omega_t \bigcap S(\tau).$$

If $Q \subset R_{x,t}^{n+1}$ $-$be bounded domain $V_t = Q \bigcap \{t = const\}$, then for $S_t \subset \partial V_t$ we define space $\overset{0}{W}{}_2^{m}(V_t, S_t)$, where $W_2^m(V_t)$ consists smooth functions in V_t which vanishing in neighborhood $\partial V_t \setminus S_t$. The space $L_2\left(s, h; \overset{0}{W}{}_2^{m}(V_t, S_t)\right)$ - space of functions $v(x,t)$which have finite norm

$$\|v\|^2 = \int_s^h \|v(\cdot,t)\|_{W_2^m(V_t)}^2 dt < \infty$$

The space $L_p\left(0, T, W_{q,\omega}^m(\Omega_t)\right)$ defined as $\left\{u(x,t) : \int_0^T \left(\|u\|_{W_{q,\omega}^m(\Omega_t)}\right)^p dt < \infty\right\}.$

$W_{q,\omega}^m(\Omega_t)$ is a closure in Ω_t the functions from $C^m\left(\overline{\Omega}\right)$ with respect to the norm

$$\|u\|_{W_{q,\omega}^m(\Omega_t)} = \left(\int\limits_{\Omega_t} \omega(x) \sum_{|\alpha| \le m} |D_x^{\alpha} u|^q \, dxdt\right)^{1/q}.$$

Assume that $\omega(x), \ x \in G$ is a measurable non negative function satisfying the conditions:

$$\omega(x) \in L_{1,loc} \ \text{ and } \ \omega \in A_{\sigma} \ \ (see[7]) \tag{1.6}$$

The function $u(x,t) \in L_p\left(-y(\tau), T; \ \overset{0}{W}{}_{p,\omega}^{m}(\Omega_t(\tau), \partial\Omega_t(\tau) \setminus \Gamma)\right)$, $\frac{\partial u}{\partial t} \in L_p(G(\tau))$, $u/\Gamma_1 = 0$ and for any $s, h : T > s > h > -\infty$ is fulfilled of the integral identity

$$\int\limits_{G_h^s(\tau)} \left[\frac{\partial u}{\partial t}\varphi + \sum_{|\alpha|\le m} A_\alpha\left(x,t,u,D_u^m\right) D^\alpha \varphi\right] dxdt = 0 \tag{1.7}$$

for any $\varphi(x,t) \in L_2\left(-y(\tau), T;\ \overset{0}{W}{}^{m}_{p,\omega}\left(\Omega_t\left(\tau\right), \partial\Omega_t \backslash \Gamma\right)\right)$, then is said $u(x,t)$ be a generalized solution of the problem (1.1)–(1.3).

2 Main results

Let we choose sequence $\{\tau_i\}\,,\ i = 0, 1, 2, ..$ which satisfying of condition

$$c_1\tau_{i-1} \le \Delta_i \equiv \tau_i - \tau_{i-1} \le c_2\tau_{i-1},\ 0 < c_1 \le c_2 < \infty \tag{2.1}$$

and also monotone increasing function $\mu_0\left(\tau\right) \equiv h\left(\tau\right)^{m/(2m-1)} \cdot \left(y\left(\tau\right)\right)^{-m/(2m-1)}$, where for any $i > i_0,\ i < \infty$ can be take number $E \equiv E\left(\tau_i\right)$, that hold inequality

$$\int\limits_{(1+c_2)\tau_i}^{\tau_E} \left(\mu_0\left(s\right)\right)^{-(2m-1)/m} ds \ge T + y\left(\tau_E\right) \tag{2.2}$$

Theorem 2.1 *Let* $u\left(x,t\right)$ *is generalized solution of problem* (1.1)-(1.3)*. If there exists a constant* $a^* > 0$ *which dependent at known parameters, such that for solution* $u\left(x,t\right)$ *hold estimate*

$$\int\limits_{G(\tau_i)} u^p \omega dxdt \le \exp\left(a\tau_i \mu_0^{1/m}\left(\tau_i\right)\right) \qquad i > i_1$$

with $a < a^*$*, then* $u\left(x,t\right) \equiv 0$.

Proof. We choose cut off function $\xi\left(s\right) : \xi\left(s\right) = 1$ at $s < 0,\ \ \xi\left(s\right) = 0$ at $s > 1$ and define $c_3 = \max\limits_{s,j\le m}\left|\xi^{(j)}\left(s\right)\right|$. The function $\xi\left(s\right) \in C^m$. We use of weighted Nirenberg-Gariardo interpolation inequality

$$\int\limits_{\Omega_t(\tau_1,\tau_2)} \omega\left|\nabla_x^j u\right|^p dx \le c_4 \left(\int\limits_{\Omega_t(\tau_1,\tau_2)} \omega\left|\nabla_x^m u\right|^p dx\right)^{j/m} \times \tag{2.3}$$

$$\times \left(\int\limits_{\Omega_t(\tau_1,\tau_2)} \omega u^p dx\right)^{(m-j)/m} + c_5\left(\tau_2 - \tau_1\right)^{-2j} \int\limits_{\Omega_t(\tau_1,\tau_2)} \omega u^p dx,$$

$$\forall \tau_2 > \tau_1 \ge 0,\ j \le m, u\left(x\right) \in W_{p,\omega}^m\left(\Omega_t\left(\tau_1,\tau_2\right)\right).$$

For any $s, h : T > s > h > -\infty, \tau > \tau_0$ we have apriori estimate

$$\int\limits_{\Omega_t(\tau-\delta)} \omega\left(x\right) u^p dx + c_6 \int\limits_{G_\tau^s(\tau-\delta)} \omega\left(x\right)\left|\nabla_x^m u\right|^p dxdt \le \frac{c_7}{\delta^{2m}} \int\limits_{G_\tau^s(\tau)} \omega u^p dxdt +$$

$$+ \int\limits_{\Omega_s(\tau)} \omega(x)\, u^p dx\,. \tag{2.4}$$

This estimate we can be show if substitute to integral identity (1.7) $\varphi(x,t) = u(x,t)\,\xi\left(\frac{|x|-\tau+\delta}{2\delta}\right)$. There exists $\mu^* > 0$ such that for any $\mu > \mu^*,\ \tau > \tau_0$ $v > 0,\ s \le T$ have apriori estimate

$$\int\limits_{\Omega_s(\tau)} \omega(x)\, u^p dx \le c_8 \exp\left(-c_9 \mu^{1/m} + 2\mu^2 v\right) \int\limits_{G_s^\tau(s)} \omega(x)\, u^p dxdt +$$

$$+ (1 + c_{10}) \exp\left(2\mu^2 v\right)\ .$$

For get this estimate to integral identity (1.7) we substitute function

$$\varphi(x,t) = u(x,t) \exp\left(-2\mu^2 t\right) \xi\left(\frac{|x| - \tau}{\delta}\right),$$

where $\delta = c_{11} \left(\mu^*\right)^{-\frac{1}{m}} \le 1$.

Let define $\eta_\delta = \xi\left(\frac{|x|-\tau}{\delta}\right)$. Then after transformation we get

$$c_{12} \int\limits_{G_{s-v}^s(\tau)} \omega \mu^2 u^p dxdt + c_{13} \int\limits_{G_{s-v}^s} \omega \left|\nabla_x^m u\right|^p dxdt \le$$

$$\le c_{14} \int\limits_{G_{s-v}^s(\tau+\delta)} \omega \mu^2 u^p dxdt + c_{15} \int_{G_{s-v}^s(\tau+\delta)} \omega \left|\nabla_x^m u\right|^p dxdt + \int\limits_{\Omega_{s-v}(\tau+\delta)} \omega u^p dx \tag{2.5}$$

If we define $J(\tau) = \int\limits_{G_{s-v}^s(\tau)} \left(\omega \mu^2 u^p + \omega \left|\nabla_x^m u\right|^p\right) dxdt$ from (2.5) we have $J(\tau) \le \vartheta J(\tau + \delta) + F(\tau + \delta)$, with $\vartheta < 1$. If doing iteration inequality

$$J^i(\tau_0) \le \sum_{i=1}^{E} (M(i))\, J_{t_i}^{t_i - 1}(\tau_i)\,, \tag{2.6}$$

where iteration is $E(\tau_0)$ times. Then we have estimate

$$\int\limits_{\Omega_{t_0}(\tau_0)} \omega u^p dx \le c_{16}\, \tau_i \left(\mu_0(\tau_i)\right)^{1/m} + c_{17} = g(\tau_0)\,.$$

If $g(\tau_0) \to 0$, at $\tau_0 \to \infty$we have

$$\int\limits_{\Omega_{t_0}^{(\tau_0)}} \omega u^p dx = 0,\ \forall \tau_0 > \tau^*,\ t_0 \le T.$$

Thus Theorem 1 is proved.

This Theorem 1 also can be is consider as theorem of Fragmen-Lindelyof for behavior of solutions. Following example is consider.

1. In case $y(\tau) < y_0 < \infty$ we can choose function $\mu_0(\tau)$and condition (2.2) equivalent to Taklind condition
$$\int_{\tau}^{\infty} h^{-1}(s)\,ds = \infty \tag{2.7}$$
2. Let $y(\tau) \to \infty$ at $\tau \to \infty$. Then any function $h(s)$ which satisfying to condition (2.2) also satisfying to condition (2.7). Thus unique classes no wide.
3. Let $y(\tau) = (\ln r)^{\alpha}$, $\alpha > 0$, $\tau > \tau_0 > 1$. Then function $h(\tau)$ can take as $h(\tau) = (2\alpha)^{-1}\tau\ln\tau$.
4. For any $0 < \alpha < 1$ we can choose $y(\tau) = \exp\left((\ln\tau)^{\alpha}\right)$ and $h(\tau) = \alpha^{-1}\tau(\ln\tau)^{1-\alpha}$.

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