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On the solvability of the inverse boundary value problem for the equation of transverse vibrations of an elastic beam

Yashar T. Mehraliyev · Yusif M. Sevdimaliyev ·
Khalid B. Mamedov

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Abstract. *In this paper we study the inverse boundary value problem for the equation of transverse vibrations of the beam with an additional integral condition. The original problem is first reduced to an equivalent problem. For it the question of uniqueness of the solution is studied and the theorem of uniqueness of the solution is proved.*

Keywords. equations of vibrations of a beam · inverse problem · uniqueness of solution.

Mathematics Subject Classification (2010): 35Q74

1 Introduction

In instrument-making, machine-building it is necessary to regulate vibration processes in one-dimensional distributed systems and the relevance of these problems with increasing the speed of mechanisms and increasing the size of the structure increases. For such problems, mathematical models of transverse oscillations are based on the refined theory [6]. Restoration of unknown parameters in the corresponding problem and other practical problems lead to the problems of determining the coefficients or the right side of the differential equation according to some known data of its solution [1,9]. Such problems are called inverse problems of mathematical physics, which in many works [2,4,5,7,8] were studied for partial differential equations. In inverse problems with initial and boundary conditions typical for a particular direct problem, additional information is given. The need for additional information is due to the presence of unknown coefficients or the right side of the equations.

In this paper we investigate the inverse boundary value problem with additional integral conditions for the equation of transverse vibrations of the beam in the case of rigid fixing of the ends.

Y.T. Mehraliyev
Baku State University, Baku, Azerbaijan
E-mail: Yashar_aze@mail.ru

Y.M. Sevdimaliyev
Baku State University, Baku, Azerbaijan
E-mail: yusifsev@mail.ru

Kh.B. Mamedov
Baku State University, Baku, Azerbaijan
E-mail: xalid1955.20.09

Formulation of the problem and its reduction to an equivalent problem

1. Consider the area $D_T = \{(x.t) : 0 \leq x \leq 1, 0 \leq t \leq T\}$. The equation of vibrations of the beam

$$u_{tt}(x,t) + u_{xxxx}(x,t) = a(t)u(x,t) + b(t)u_t(x,t) + f(x,t) \quad (1.1)$$

and we substitute for it the following inverse boundary value problem: find the triple $\{u(x,t), a(t), b(t)\}$ of functions $u(x,t), a(t), b(t)$ satisfying equation (1.1) with initial conditions

$$u(x,0) = \varphi(x), u_t(x,0) = \psi(x)(0 \leq x \leq 1), \quad (1.2)$$

with boundary conditions

$$u(0,t) = u_x(1,t) = u_{xx}(0,t) = u_{xxx}(1,t) = 0(0 \leq t \leq T), \quad (1.3)$$

and additional conditions

$$\int_0^1 u(x,t)dx = h_1(t)(0 \leq t \leq T), \quad (1.4)$$

$$u(1,t) = h_2(t)(0 \leq t \leq T), \quad (1.5)$$

where $f(x,t)$, $\varphi(x)$, $\psi(x)$, $h_i(t)$ $(i = 1,2)$ are given functions.

Definition 1.1 *The classical solution of the inverse boundary value problem* (1.1)-(1.5) *is a triple* $\{u(x,t), a(t), b(t)\}$ *of functions* $u(x,t), a(t), b(t)$ *satisfying the following conditions:*

1 *the function* $u(x,t)$ *and its derivatives* $u_t(x,t)$, $u_{tt}(x,t)$, $u_x(x,t)$, $u_{xx}(x,t)$, $U_{xxx}(x,t)$, $u_{xxxx}(x,t)$ *are continuous in* D_T *:*
2 *functions* $a(t)$ *and* $b(t)$ *are continuous on* $[0;T]$ *:*
3 *equation* (1.1) *and conditions* (1.2)-(1.5) *are satisfied in the usual sense.*

The following theorem is true.

Theorem 1.1 *If* $\varphi(x) \in C[0,1]$, $\psi(x) \in C[0,1]$, $h_i(t) \in C^2[0,T]$ $(i = 1,2)$, $h_1(t) \equiv h_1(t)h_2'(t) - h_2(t)h_1'(t) \neq 0$ $(0 \leq t \leq T)$, $f(x,t) \in C(D_T)$ *and the condition approval is fulfilled*

$$\int_0^1 \varphi(x)dx = h_1(0), \int_0^1 \psi(x)dx = h_1'(0),$$

$$\varphi(1) = h_2(0), \psi(1) = h_2'(0).$$

Then the problem of finding the classical solution of the problem (1.1) - (1.5) is equivalent to the problem of determining functions $u(x,t)$, $a(t)$ and $b(t)$, and having properties 1) and 2) determining the classical solution of the problem (1.1) - (1.5), from (1.1) - (1.3),

$$h_1''(t) - u_{xxx}(0,t) = a(t)h_1(t) + b(t)h_1'(t) + \int_0^1 f(x,t)dx(0 \leq t \leq T) \quad (1.6)$$

$$h_2''(t) + u_{xxxx}(1,t) = a(t)h_2(t) + b(t)h_2'(t) + f(1,t)(0 \leq t \leq T). \quad (1.7)$$

Proof. Suppose that $\{u(x,t), a(t), b(t)\}$ is the solution of the problem (1.1) - (1.5). Considering $h_i(t) \in C[0,t]$ $(i = 1,2)$ from (1.4) and (1.5), respectively, we obtain:

$$\int_0^1 u_t(x,t)\,dx = h_1'(t),\ \int_0^1 u_{tt}(x,t)\,dx = h_1''(t)\ \ (0 \le t \le T) \tag{1.8}$$

$$u_t(1,t) = h_t'(t)\,,\ u_{tt}(1,t) = h_2''(t)\ \ (0 \le t \le T)\,. \tag{1.9}$$

Integrating equation (1.1) by x from 0 to 1, we have:

$$\begin{gathered}\frac{d^2}{dt^2}\int_0^1 u(x,t)dx + u_{xxx}(1,t) - u_{xxx}(0,t) = a(t)\int_0^1 u(x,t)dx + b(t)\times \\ \times\frac{d}{dt}\int_0^1 u(x,t)dx + \int_0^1 f(x,t)dx \quad (0 \le t \le T).\end{gathered} \tag{1.10}$$

Hence given (1.3), (1.4), (1.8) we come to implementation (1.6).

Substituting x=1 in equation (1.1), we find:

$$u_{tt}(1,t) + u_{xxxx}(1,t) = a(t)u(1,t) + b(t)u_t(1,t) + f(1,t)(0 \le t \le T) \tag{1.11}$$

Given (1.5) and (1.9) of (1.10), execution (1.7) follows.

Let $\{u(x,t), a(t), b(t)\}$ is the solution of the problem (1.1)-(1.3), (1.6), (1.7). Then from (1.6) and (1.10) taking into account (1.3) we have:

$$y''(t) = a(t)y(t) + b(t)y'(t)\ (0 \le t \le T), \tag{1.12}$$

where

$$y(t) = \int_0^1 u(x,t)dx - h_1(t)\ (0 \le t \le T). \tag{1.13}$$

Because of

$$\int_0^1 \varphi(x)dx = h_1(0),\ \int_0^1 \psi(x)dx = h_1'(0),$$

we find

$$y(0) = \int_0^1 u(x,0)dx - h_1(0) = \int_0^1 \varphi(x)dx - h_1(0) = 0$$

$$y'(0) = \int_0^1 u_t(x,0) - h_1'(0) = \int_0^1 \psi(x)dx - h_1'(0) = 0. \tag{1.14}$$

From (1.12), given (1.14) it is obvious that $y(t) \equiv 0$ $(0 \le t \le T)$. Hence, by virtue of (1.13), we easily come to fulfillment (1.4).

Next, from (1.7) and (1.11), we obtain:

$$\frac{d^2}{dt^2}(u(1,t)-h_2(t))=a(t)\,(u(1,t)-h_2(t))+b(t)\frac{d}{dt}(u(1,t)-h_2(t))\ \ (0 \le t \le T) \tag{1.15}$$

Because of $\varphi(1) = h_2(0)$, $\psi(1) = h_2'(0)$, we have

$$u(1,0) - h_2(0) = \varphi(1) - h_2(0) = 0$$

$$u_t(1,0) - h_2'(0) = \psi(1) - h_2'(0) = 0. \tag{1.16}$$

From (1.15), (1.16) it is clear that the condition (1.5) is fulfilled. The theorem is proved.

2 Solvability of the inverse boundary value problem

The first component $u(x,t)$ of the solution $\{u(x,t), a(t), b(t)\}$ of the problem (1.1)-(1.3), (1.6), (1.7) we will look in the form:

$$u(x,t)=\sum_{k=1}^{\infty}u_k(t)\sin\lambda_k x\left(\lambda_k=\frac{\pi}{2}(2k-1)\right), \tag{2.1}$$

where

$$u_k(t)=\int_0^1 u(x,t)\sin\lambda_k x dx.$$

Then applying the formal scheme of the Fourier, we have from (1.1) and (1.2) have:

$$u_k''(t)+\lambda^4{}_k u_k(t)=F_k(t,u,a,b)(0\le t\le T; k=1,2...), \tag{2.2}$$

$$u_k(0)=\varphi_k, u_k'(0)=\psi_k(k=1,2...), \tag{2.3}$$

where

$$F_k(t,u,a,b)=a(t)u_k(t)+b(t)u_k'(t)+f_k(t),$$

$$f_k(t)=2\int_0^1 f(x,t)\sin\lambda_k x dx,$$

$$\varphi_k=2\int_0^1\varphi(x)\sin\lambda_k x dx, \psi_k=2\int_0^1\psi(x)\sin\lambda_k x dx(k=1,2...).$$

Having solved the problem (2.2), (2.3), we find:

$$u_k(t)=\varphi_k\cos\lambda_k^2 t+\frac{1}{\lambda_k^2}\psi_k\sin\lambda_k^2 t+$$

$$+\frac{1}{\lambda_k^2}\int_0^t F_k(\tau,u,a,b)\sin\lambda_k^2(t-\tau)d\tau \qquad (k=1,2...). \tag{2.4}$$

After substitution of expressions from (2.4) to (2.1),

$$u(x,t)=\sum_{k=1}^{\infty}\left\{\varphi_k\cos\lambda_k^2 t+\frac{1}{\lambda_k^2}\psi_k\sin\lambda_k^2\, t+\right.$$

$$\left.+\frac{1}{\lambda_k^2}\int_0^t F_k(\tau,u,a,b)\sin\lambda_k^2(t-\tau)d\tau\right\}\sin\lambda_k x. \tag{2.5}$$

Now from (1.6) and (1.7) taking into account (2.1) we have:

$$a(t)=[h(t)]^{-1}\left\{h_2'(t)\left(h_1''(t)-\int_0^1 f(x,t)dx\right)-h_1'(t)\left(h_2''(t)-f(0,t)\right)+\right.$$

$$+\sum_{k=1}^{\infty}\lambda_k^3\left(h_2'(t)-(-1)^{k-1}\lambda_k h_1'(t)\right)u_k(t)\Bigg\} \ , \tag{2.6}$$

$$b(t)=[h(t)]^{-1}\left\{h_1(t)\left(h_2''(t)-f(0,t)\right)-h_2'(t)\left(h_1''(t)-\int_0^1 f(x,t)dx\right)+\right.$$

$$\left.+\sum_{k=1}^{\infty}\lambda_k^3\left((-1)^{k+1}\lambda_k h_1(t)-h_2(t)\right)u_k(t)\right\} \ , \tag{2.7}$$

where

$$h(t)=h_1(t)h_2^{'}-h_2(t)h_1^{'}(t)\neq 0(0\leq t\leq T).$$

Substituting the expression (2.4) in (2.6) and (2.7), respectively, we obtain:

$$a(t)=[h(t)]^{-1}\left\{h_2^{'}(t)(h_1^{''}(t)-\int_0^1 f(x,t)dx)-h_1^{'}(t)(h_2^{''}(t)-f(0,t)+\right.$$

$$+\sum_{k=1}^{\infty}\lambda_k^3\left(h_2^{'}(t)-(-1)^{k+1}\lambda_k h_1^{'}(t)\right)\times$$

$$\left.\times\left[\varphi_k\cos\lambda_k^2 t+\frac{1}{\lambda_k^2}\psi_k\sin\lambda_k^2 t+\frac{1}{\lambda_k^2}\int_0^t F_k(\tau;u,a,b)\sin\lambda_k^2(t-\tau)d\tau\right]\right\}, \tag{2.8}$$

$$b(t)=[h(t)]^{-1}\left\{h_1(t)(h_2^{''}(t)-f(0,t))-h_2(t)\left(h_1^{''}(t)-\int_0^1 f(x,t)dx\right)+\right.$$

$$+\sum_{k=1}^{\infty}\lambda_k^3\left((-1^{k+1}\lambda_k h_1(t)-h_2(t)\right)\times$$

$$\left.\times\left[\varphi_k\cos\lambda_k^2 t+\frac{1}{\lambda_k^2}\psi_k\sin\lambda_k^2 t+\frac{1}{\lambda_k^2}\int_0^t F_k(\tau,u,a,b)\sin\lambda_k^2(t-\tau)d\tau\right]\right\}. \tag{2.9}$$

Thus, the solution of the problem (1.1) – (1.3), (1.6), (1.7), it was reduced to the solution of system (2.5), (2.8), (2.9) relatively unknown functions $u(x,t), a(t), b(t)$.

To study the uniqueness of the solution of the problem (1.1) - (1.3), (1.6), (1.7) the following Lemma plays an important role.

Lemma 2.1 *If* $\{u(x,t),a(t),b(t)\}$ *is any solution to the problem* (1.1)-(1.3)*,* (1.6)*,* (1.7)*, then function* $u_k(t)=2\int_0^1 u(x,t)\sin\lambda_k x dx$ $(k=1,2...)$ *satisfies system* (2.4) *at* $[0;T]$.

Proof. Suppose that $\{u(x,t),a(t),b(t)\}$-is some solution of the problem (1.1)-(1.3), (1.6), (1.7). Multiplying both parts of equation (1.1) by a function $2\sin\lambda_k x$ $(k=1,2...)$, integrating the obtained equality with respect to x from 0 to 1 and using the relations

$$\int_0^1 u_{tt}(x,t)\sin\lambda_k x dx=\frac{d^2}{dx^2}\left(\int_0^1 u(x,t)\sin\lambda_k x dx\right)=u_k''(t),\quad (k=1,2...),$$

$$\int_0^1 u_{xxxx} \sin \lambda_k x dx = \lambda_k^4 \left(\int_0^1 u(x,t) \sin \lambda_k x dx \right) = \lambda_k^4 u_k(t) (k = 1, 2...)$$

we obtain that the equation (2.2) is satisfied.

Similarly, from (1.2) we obtain that the condition (2.3) is satisfied. Thus, $u_k(t)$ $(k = 1, 2...)$ it is the solution of the problem (2.2), (2.3). Hence, it follows directly that the functions $u_k(t)$ $(k = 1, 2...)$ satisfy on $[0, T]$ system (2.4). The Lemma is proved.

Obviously, if $u_k(t) = 2 \int_0^1 u(x,t) \sin \lambda_k x dx$ $(k = 1, 2...)$ is the solution of the system (2.4), the triangle $\{u(x,t), a(t), b(t)\}$ of functions $u(x,t) = \sum\limits_{k=1}^{\infty} u_k(t) \sin \lambda_k x$, $a(t), b(t)$ is the solution of the system (2.5), (2.8), (2.9).

It follows from the Lemma that there is a consequence.

Corollary 2.1 *Let the system* (2.5), (2.8), (2.9) *has the only solution. Then the task* (1.1) – (1.3), (1.6), (1.7) *can not have more than one solution, i.e. if the problem* (1.1) – (1.3), (1.6), (1.7) (1.2) *has a solution, then it is the only one.*

We denote $B_{2,T}^{5,3}$ [9] the set of all functions of the form $u(x,t) = \sum\limits_{k=1}^{\infty} u_k(t) \sin \lambda_k x (\lambda_k) = = \frac{\pi}{2}(2k-1))$ considered in D_T, where $u_k(t) \in C^1 [0; T]$ $(k = 1, 2...)$ and

$$I(u) = \left\{ \sum_{k=1}^{\infty} \left(\lambda_k^5 \| u_k(t) \|_{C[0;T]} \right)^2 \right\}^{\frac{1}{2}} + \left\{ \sum_{k=1}^{\infty} \left(\lambda_k^3 \| u_k'(t) \|_{C[0;T]} \right)^2 \right\}^{\frac{1}{2}} < +\infty.$$

The norm in this set is defined as:

$$\| (u(x,t) \|_{B_{2,T}^{5,3}} = I(u).$$

Through $E_T^{5,3}$ denote the space $B_{2,T}^{5,3} \times C[0,T] \times C[0,T]$ vector of functions $z(x,t) = \{u(x,t), a(t), b(t)\}$ with the norm

$$\| z(x,t) \|_{E_{2,T}^{5,4}} = \| u(x,t) \|_{B_{2,T}^{5,4}} + \| a(t) \|_{C[0,T]} + \| b(t) \|_{C[0,T]}.$$

It is known that $B_{2,T}^{5,3}$ and $E_T^{5,3}$ are Banach spaces.

Consider the operator $E_T^{5,3}$ in space $\varPhi(u,a,b) = \{\varPhi_1(u,a,b), \varPhi_2(u,a,b), \varPhi_3(u,a,b)\}$ where $\varPhi_1(u,a,b) = \tilde{u}(x,t) = \sum\limits_{k=1}^{\infty} \tilde{u}_k(t) \sin \lambda_k x$, $\varPhi_2(u,a,b) = \tilde{a}(t)$, $\varPhi_3(u,a,b) = \tilde{b}(t)$, and $\tilde{u}_k(t)$ $(k = 1, 2...)$, $\tilde{a}(t), \tilde{b}(t)$ are equal respectively right parts of (2.4),(2.8),(2.9). Of (2.4) it is not difficult to see that

$$\tilde{u}_k'(t) = -\lambda_k^2 \varphi_k \sin \lambda_k^2 t + \psi_k \cos \lambda_k^2 t +$$

$$+ \int_0^t F_k(\tau, u, a, b) \cos \lambda_k^2 (t - \tau) dt \ (k = 1, 2...). \tag{2.10}$$

With the help of easy transformations, respectively, we find:

$$\left\{ \sum_{k=1}^{\infty} \left(\lambda_k^5 \| \tilde{u}_k(t) \|_{C[0;T]} \right)^2 \right\}^{\frac{1}{2}} \le \sqrt{5} \left\{ \sum_{k=1}^{\infty} \left(\lambda_k^5 |\varphi_k| \right)^2 \right\}^{\frac{1}{2}} + \sqrt{5} \left\{ \sum_{k=1}^{\infty} \left(\lambda_k^3 |\psi_k| \right)^2 \right\}^{\frac{1}{2}} +$$

$$+\sqrt{5T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^3\left|f_k(\tau)\right|\right)^2d\tau\right)^{\frac{1}{2}}+\sqrt{5T}\left\|a(t)\right\|_{C[0,T]}\left\{\sum_{k=1}^{\infty}\left(\lambda_k^5\left\|u_k(t)\right\|_{C[0;T]}\right)^2\right\}^{\frac{1}{2}}+$$

$$+\sqrt{5T}\left\|b(t)\right\|_{C[0,T]}\left\{\sum_{k=1}^{\infty}\left(\lambda_k^3\left\|u_k'(t)\right\|_{C[0;T]}\right)^2\right\}^{\frac{1}{2}}, \qquad (2.11)$$

$$\left\{\sum_{k=1}^{\infty}\left(\lambda_k^3\left\|u_k'(t)\right\|_{C[0;T]}\right)^2\right\}^{\frac{1}{2}}\leq\sqrt{5}\left\{\sum_{k=1}^{\infty}\left(\lambda_k^5\left|\varphi_k\right|\right)^2\right\}^{\frac{1}{2}}+\sqrt{5}\left\{\sum_{k=1}^{\infty}\left(\lambda_k^3\left|\psi_k\right|\right)^2\right\}^{\frac{1}{2}}+$$

$$+\sqrt{5T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^3\left|f_k(\tau)\right|\right)^2d\tau\right)^{\frac{1}{2}}+\sqrt{5T}\left\|a(t)\right\|_{C[0,T]}\left\{\sum_{k=1}^{\infty}\left(\lambda_k^5\left\|u_k(t)\right\|_{C[0;T]}\right)^2\right\}^{\frac{1}{2}}+$$

$$+\left\|b(t)\right\|_{C[0,T]}\left\{\sum_{k=1}^{\infty}\left(\lambda_k^3\left\|u_k'(t)\right\|_{C[0;T]}\right)^2\right\}^{\frac{1}{2}}, \qquad (2.12)$$

$$\left\|\tilde{a}(t)\right\|_{C[0,T]}\leq\left\|[h(t)]^{-1}\right\|_{C[0,T]}\times$$

$$\times\left\{\left\|h_2'(t)\left(h_1''(t)-\int_0^1 f(x,t)dx\right)-h_1'(t)\left(h_2'(t)-f(0,t)\right)\right\|_{C[0,T]}+\right.$$

$$+\left\|\left|h_2'(t)\right|+\left|h_1'(t)\right|\right\|_{C[0,T]}\left(\sum_{K=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left[\left\{\sum_{k=1}^{\infty}\left(\lambda_k^5\left|\varphi_k\right|\right)^2\right\}^{\frac{1}{2}}+\left\{\sum_{k=1}^{\infty}\left(\lambda_k^3\left|\psi_k\right|\right)^2\right\}^{\frac{1}{2}}+\right.$$

$$+\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^3\left|f_k(\tau)\right|\right)^2d\tau\right)^{\frac{1}{2}}+T\left\|a(t)\right\|_{C[0,T]}\left\{\sum_{k=1}^{\infty}\left(\lambda_k^5\left\|u_k(t)\right\|_{C[0;T]}\right)^2\right\}^{\frac{1}{2}}+$$

$$\left.\left.+T\left\|b(t)\right\|_{C[0,T]}\left\{\sum_{k=1}^{\infty}\left(\lambda_k^3\left\|u_k'(t)\right\|_{C[0;T]}\right)^2\right\}^{\frac{1}{2}}\right]\right\}, \qquad (2.13)$$

$$\left\|\tilde{b}(t)\right\|_{C[0,T]}\leq\left\|[h(t)]^{-1}\right\|_{C[0,T]}\times$$

$$\times\left\{\left\|h_1(t)(h_2''(t)-f(0,t))-h_2(t)\left(h_1''(t)-\int_0^1 f(x,t)dx\right)\right\|_{C[0,T]}+\right.$$

$$+\left\|\left|h_1(t)\right|+\left|h_2(t)\right|\right\|_{C[0,T]}\left(\sum_{K=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left[\left\{\sum_{k=1}^{\infty}\left(\lambda_k^5\left|\varphi_k\right|\right)^2\right\}^{\frac{1}{2}}+\left\{\sum_{k=1}^{\infty}\left(\lambda_k^3\left|\psi_k\right|\right)^2\right\}^{\frac{1}{2}}+\right.$$

$$+\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^3\left|f_k(\tau)\right|\right)^2d\tau\right)^{\frac{1}{2}}+T\left\|a(t)\right\|_{C[0,T]}\left\{\sum_{k=1}^{\infty}\left(\lambda_k^5\left\|u_k(t)\right\|_{C[0;T]}\right)^2\right\}^{\frac{1}{2}}+$$

$$+ T \|b(t)\|_{C[0,T]} \left\{ \sum_{k=1}^{\infty} \left(\lambda_k^3 \left\| u_k'(t) \right\|_{C[0;T]} \right)^2 \right\}^{\frac{1}{2}} \Bigg] \Bigg\}. \tag{2.14}$$

Let the problems (1.1)-(1.3),(1.6),(1.7) satisfy the following conditions:

1.$\varphi(x) \in C^4[0,1], \varphi^{(5)}(x) \in L_2(0,1), \varphi(0) = \varphi_1'(1) = \varphi''(0) = \varphi'''(1) = \varphi^{(4)}(0) = 0;$

2.$\psi(x) \in C^2[0,1]\, \psi'''(x) \in L_2(0,1), \psi(0) = \psi'(1) = \psi''(0) = 0;$
3.$f(x,t), f_x(x,t), f_{xx}(x.t) \in L(D_T), f_{xxx}(x,t) \in L_2(D_T),$

$$f(0,t) = f_x(1,t) = f_{xx}(0,t) = 0 (0 \le t \le T);$$

4.$h_i(t) \in C^2[0,T]\, (i = 1,2), h(t) \equiv h_1(t)h_2'(t) - h_2(t)h_1'(t) \ne 0 (0 \le t \le T)$. Then from (2.10) and (2.12) we have:

$$\|\tilde{u}(x,t)\|_{B_{2,T}^{5,3}} \le A_1(T) + B_1(T) \left(\|a(t)\|_{C[0,T]} + \|b(t)\|_{C[0,T]} \right) \|u(x,t)\|_{B_{2,T}^{5,3}}, \tag{2.15}$$

where

$$A_1(T) = 2\sqrt{5} \left\| \varphi^{(5)}(x) \right\|_{L_2(0,1)} + 2\sqrt{5} \left\| \psi'''(x) \right\|_{L_2(0,1)} + 2\sqrt{5T} \|f_{xxx}(x,t)\|_{L_2(D_T)},$$

$$B_1(T) = 2\sqrt{5}T.$$

Then from (2.13) and (2.14) respectively get:

$$\|\tilde{a}(t)\|_{C[0,T]} \le A_2(T) + B_2(T) \left(\|a(t)\|_{C[0,T]} + \|b(t)\|_{C[0,T]} \right) \|u(x,t)\|_{B_{2,T}^{5,3}}, \tag{2.16}$$

$$\left\| \tilde{b}(t) \right\|_{C[0,T]} \le A_3(T) + B_2(T) \left(\|a(t)\|_{C[0,T]} + \|b(t)\|_{C[0,T]} \right) \|u(x,t)\|_{B_{2,T}^{5,3}}, \tag{2.17}$$

where

$$A_2(T) = \left\| [h(t)]^{-1} \right\|_{C[0,T]} \times$$

$$\times \left\{ \left\| h_2'(t) \left(h_1'(t) - \int_0^1 f(x,t)dx \right) - h_1'(t) \left(h_2'(t) - f(0,t) \right) \right\|_{C[0,T]} + \right.$$

$$\left\| |h_2'(t)| + |h_1'(t)| \right\|_{C[0,T]} \times$$

$$\times \left(\sum_{K=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} \left[\left\| \varphi^{(5)}(x) \right\|_{L_2(0,1)} + \left\| \psi'''(x) \right\|_{L_2(0,1)} + \sqrt{T} \|f_{xxx}(x,t)\|_{L_2(D_T)} \,] \right\},$$

$$B_2(T) = \left\| [h(t)]^{-1} \right\|_{C[0,T]} \left\| |h_1'(t)| + |h_2'(t)| \right\|_{C[0,T]} \times$$

$$\left. \times \left(\sum_{K=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} \left[\left\| \varphi^{(5)}(x) \right\|_{L_2(0,1)} + \left\| \psi'''(x) \right\|_{L_2(0,1)} + \sqrt{T} \|f_{xxx}(x,t)\|_{L_2(D_T)} \right] \right\}$$

$$A_3(T) = \left\| [h(t)]^{-1} \right\|_{C[0,T]} \times$$

$$\times \left\{ \left\| h_1(t) \left(h_2''(t) - f(0,t) \right) - h_2(t) \left(h_1''(t) - \int_0^1 f(x,t)\, dx \right\|_{C[0,T]} + \right. \right.$$

$$+\left\||h_1(t)|+|h_2(t)|\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{1/2}\times$$

$$\times\left[\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}+\left\|\psi'''(x)\right\|_{L_2(0,1)}+\sqrt{T}\left\|f_{xxx}(x,t)\right\|_{L_2(0,1)}\right]\Bigg\},$$

$$B_3(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\||h_1(t)|+|h_2(t)|\right\|_{C[0,T]}\left(\sum_{K=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}T.$$

From inequalities (2.15) - (2.17) we conclude

$$\|\tilde{u}(x,t)\|_{B_{2,T}^{5,3}}+\|\tilde{a}(t)\|_{C[0,T]}+\left\|\tilde{b}(t)\right\|_{C[0,T]}\leq$$

$$\leq A(T)+B(T)\left(\left(\|a(t)\|_{C[0,T]}+\|b(t)\|_{C[0,T]}\right)\|\tilde{u}(x,t)\|_{B_{2,T}^{5,3}}\right), \tag{2.18}$$

where

$$A(T)=A_1(T)+A_2(T)+A_3(T), B(T)=B_1(T)+B_2(T)+B_3(T).$$

The following theorem is proved by means of inequality (2.18).

Theorem 2.1 *Let are executed* (1.1)-(1.4) *and*

$$B(T)\left(A(T)+2\right)^2<1. \tag{2.19}$$

Then the problem (1.1)-(1.3), (1.6), (1.7) *has in the ball* $K=K_R\left(\|z\|_{E_T^{5,3}}\leq A(T)+2\right)$ *of the space* $E_T^{5,3}$ *unique solution.*

Proof. In the space $E_T^{5,3}$, we consider the operator equation

$$z=\Phi z, \tag{2.20}$$

where $z=\{u,a,b\}$, are the components of the operator is defined right-hand sides of equations (2.5), (2.8), (2.9) accordingly.

Consider the operator $\Phi(u,a,b)$ in a ball $K=K_R\left(\|z\|_{E_T^{5,4}}\leq A(T)+2\right)$ of $E_T^{5,3}$. Analogously (2.18) get that for any $z, z_1, z_2\in K_R$ fair estimates

$$\|\Phi z\|_{E_T^{5,3}}\leq A(T)+B(T)\left(\left(\|a(t)\|_{C[0,T]}+\|b(t)\|_{C[0,T]}\right)\|\tilde{u}(x,t)\|_{B_{2,T}^{5,3}}\right), \tag{2.21}$$

$$\|\Phi z_1-\Phi z_2\|_{E_T^{5,3}}\leq B(T)R\left(\|a_1(t)-a_2(t)\|_{C[0,T]}+\right.$$

$$\left.+\left(\|b_1(t)-b_2(t)\|_{C[0,T]}+\|u_1(x,t)-u_2(x,t)\|_{B_{2,T}^{5,3}}\right)\right). \tag{2.22}$$

From estimates (2.21) and (2.22), with consideration (2.19) it is clear that the operator Φ acts in the bowl $K=K_R$ is the compressive. That's why in the bowl$K=K_R$operator Φ has a single fixed point. $\{u,a,b\}$ which is the solution of the equation (2.20), that is $\{u,a,b\}$ has in the bowl $K=K_R$ a single solution of system (2.5), (2.8), (2.9).

Function $u(x,t)$, as an element of space $B_{2,T}^{5,3}$, continuous and has continuous derivatives

$u_x(x,t), u_{xx}(x,t), u_{xxx}(x,t), u_{xxxx}(x,t), u_t(x,t), u_{tx}(x,t), u_{txx}(x,t)$ and D_T.

From (2.2) it is not difficult to see that $u_k''(t) \in C[0,T]$ and

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k \left\|u_k''(t)\right\|_{C[0,T]}\right)^2\right)^{1/2} \leq \sqrt{2}\left(\sum_{k=1}^{\infty}\left({\lambda^5}_k \left\|u_k(t)\right\|_{C[0,T]}\right)^2\right)^{1/2} +$$

$$+\left\|\left\|a(t)u_x(x,t)+b(t)u_{tx}(x,t)+f_x(x,t)\right\|_{C[0,T]}\right\|_{L_2[0,1]}.$$

It follows that $u_{tt}(x,t)$ continuous in D_T. We can show that the equation (1.1) with condition (1.2), (1.3), (1.6), (1.7) satisfied in the usual sense. And this in turn means that $\{u(x,t),a(t),b(t)\}$ is the solution of task (1.1)-(1.3), (1.6), (1.7) and by virtue of the corollary of Lemma it is the only one in the bowl $K = K_R$. The theorem is proved.

With the help of theorem 1, from the last theorem immediately follows the uniqueness of solvability of the original problem (1.1)-(1.5).

Theorem 2.2 *Let all conditions of the theorem 2 be satisfied, as well as the conditions of approval*

$$\int_0^1 \varphi(x)dx = h_1(0), \int_0^1 \psi(x)dx = h_1'(0), \varphi(1) = h_2(0), \psi(1) = h_2'(0).$$

Than the tasks (1.1)-(1.5) *has in the bowl* $K = K_R \left(\|z\|_{E_T^{5,4}} \leq A(T)+2\right)$ *in space* $E_T^{5,3}$ *a single classical solution.*

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