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Building of homogeneous solutions for a transversally isotropic spherical shell

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Abstract. *In this paper, we solve the problem of the equilibrium of a transversally isotropic spherical shell under homogeneous boundary conditions on the face surfaces and a given system of forces symmetric with respect to the axis of rotation on the lateral part of the boundary. The behavior of the solution is studied at a small value of the relative thickness of the shell.*

Keywords. shell · sphere · asymptotics · homogeneous solution

Mathematics Subject Classification (2010): 74K25, 34M30

1 Introduction

General representation of solutions of the equations of anisotropic elasticity theory in spherical coordinates

1.1. Let $V = [R_1, R_2] \times [\theta_1, \theta_2] \times [0, 2\pi]$ be the volume occupied by the spherical layer (Fig.1). The layer is referred to a spherical coordinate system r, θ, φ (Fig. 2), varying within the following ranges $R_1 \leq r \leq R_2, \theta_1 \leq \theta \leq \theta_2, 0 \leq \varphi \leq 2\pi$. The shell is made of a transversely isotropic material. The surface $\theta = const$ is the surface of isotropy. The

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spherical parts of the boundary of the layer will be called the face surfaces $\Gamma_j (j = 1, 2)$ and the remaining part of the boundary will be called the lateral surface. Suppose that from the side of the face surfaces the layer is subject to the load

$$\sigma_r = Q_r(\theta), \ \ \tau_{r\theta} = T_i(\theta) \text{ for } r = R_i \ \ (i = 1, 2). \tag{1.1}$$

The character of the boundary conditions on the lateral surface will not be specified, but we shall consider them as such that the layer is in equilibrium. We give here a complete system of equations describing the spatial stress-strain state of a spherical layer. Equilibrium equations in the axisymmetric case in stresses, in the absence of mass forces in a spherical coordinate system, have the form:

$$\begin{aligned} &\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{2\sigma_r - \sigma_\theta - \sigma_\varphi + \tau_{r\theta} ctg\theta}{r} = 0, \\ &\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{(\sigma_\theta - \sigma_\varphi)\, ctg\theta + 3\tau_{r\theta}}{r} = 0, \end{aligned} \tag{1.2}$$

where $\sigma_r, \sigma_\theta, \sigma_\varphi, \tau_{r\theta}$ are the components of the stress tensor. The relations of the generalized Hooke law have the form:

$$\begin{aligned} \sigma_r &= G_1 \left\lfloor b_{11} e_r + b_{12}(e_\theta + e_\varphi) \right\rfloor, \\ \sigma_\theta &= G_1 \left\lfloor b_{12} e_r + b_{22} e_\theta + b_{23} e_\varphi \right\rfloor, \end{aligned} \tag{1.3}$$

$$\sigma_\varphi = G_1 \left\lfloor b_{12} e_r + b_{23} e_\theta + b_{22} e_\varphi \right\rfloor$$

$$\tau_{r\theta} = G_1 \tau_{r\theta},$$

where

$$e_r = \frac{\partial u_r}{\partial r}, \ \ e_\theta = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}, \ e_\varphi = \frac{u_r}{r} + \frac{ctg\theta u_\theta}{r}, \ \ e_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial r} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \tag{1.4}$$

the components of the strain tensor $u_r = u_r(r, \theta), \ \ u_\theta = u_\theta(r, \theta)$, are the components of the displacement vector b_{ij}, G_1 are material constants.

$$mb_{11} = 2G_0 E_0 (1 - \nu^2) \quad mb_{22} = 2G_0(1 - v_1 \nu_2)$$

$$mb_{12} = 2G_0 \nu_1 (1 + \nu) \quad mb_{23} = 2G_0(\nu + \nu_1 \nu_2)$$

$$m = 1 - \nu - 2\nu_1 \nu_2, \ \ G_0 = G G_1^{-1}, \ \ E_0 = E_1 E^{-1},$$

here $\nu, \nu_1, \nu_2, G, G_1, E, E_1$ are the technical constants of the material.

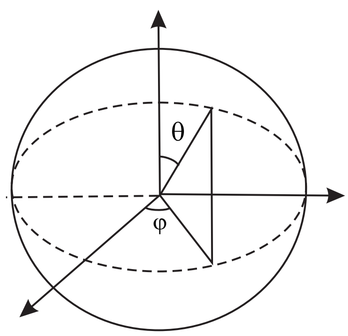

Fig. 1.

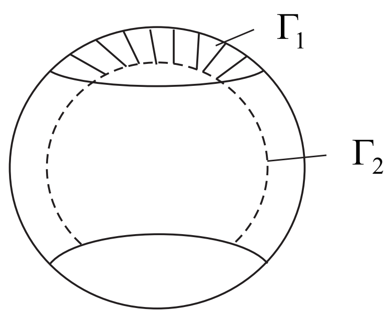

Fig. 2.

Substituting (1.4) and (1.3) into (1.2), after simple computations we obtain [13]:

$$b_{11}\frac{\partial^2 u_r}{\partial r^2}+\frac{2b_{11}}{r}\frac{\partial u_r}{\partial r}+\frac{2}{r^2}\left(b_{12}-b_{22}-b_{23}\right)u_r+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}+$$

$$+\frac{ctg\theta}{r^2}\frac{\partial u_r}{\partial \theta}+\frac{b_{12}+1}{r}\frac{\partial}{\partial r}\left(\frac{\partial u_\theta}{\partial \theta}+ctg\theta\, u_\theta\right)+ \tag{1.5}$$

$$+\frac{b_{12}-b_{22}-b_{23}-1}{r^2}\left(\frac{\partial u_\theta}{\partial \theta}+ctg\theta\, u_\theta\right)=0$$

$$\frac{b_{12}+1}{r}\frac{\partial^2 u_r}{\partial r\partial\theta}+\frac{b_{22}+b_{23}+2}{r^2}\frac{\partial u_r}{\partial \theta}+\frac{\partial^2 u_\theta}{\partial r^2}+\frac{\partial^2 u_r}{\partial r^2}+\frac{2}{r}\frac{\partial u_\theta}{\partial r}+$$

$$+\frac{b_{22}}{r^2}\frac{\partial}{\partial \theta}\left(\frac{\partial u_\theta}{\partial \theta}+ctg\theta\, u_\theta\right)+\frac{2\left(G_0-1\right)}{r^2}u_\theta=0.$$

The partial solutions of equation (1.5) which satisfy boundary conditions (1.1) on the face surfaces of the layer are called the non-homogeneous solutions. The techniques proposed in [10] are used to construct non-homogeneous mono solutions. However, this is not the only technique to relieve the facial surfaces from the load. One of the known methods is as follows: the region V arbitrarily extends to a closed spherical layer $V_0 = [R_1,\, R_2]\times[0,\,\pi]\times[0,\,2\pi]$, and the load $[Q_i(\theta)\,,\; T_i(\theta)\,]$ given on the faces is extended arbitrarily enough to closed spherical surfaces $\Gamma_i^0\,(r=R_1,\,R_2)$. External forces given on Γ_i^0 will be denoted by $Q_i^*,\; T_i^*$. Moreover, $(\theta,\varphi)\subset_i$, $\Gamma_i^*=Q_i,\; T_i^*=T_i$ and besides this, it is necessary that the external forces $\Gamma_i^*,\; T_i^*$ satisfy the equilibrium conditions

$$Q_i^*=\sum_{n=1}^{\infty}\sigma_{ni}P_n\left(\cos\theta\right)T_i^*=\sum_{n=1}^{\infty}\tau_{ni}\frac{dP_n}{d\theta}. \tag{1.6}$$

The coefficients of these series are determined from known formulas of the analysis

$$\sigma_{ni}=\frac{2n+1}{2}\int_0^{\pi}Q_i^*\left(\theta\right)P_n\left(\cos\theta\right)\sin\theta\, d\theta,$$

$$\tau_{ni}=\frac{2n+1}{2n\left(n+1\right)}\int_0^{\pi}T_i^*\left(\theta\right)\frac{dP_n}{d\theta}\sin\theta\, d\theta. \tag{1.7}$$

Then the components of the displacement vector can be found in the form of series:

$$u_r=\sum_{n=1}^{\infty}u_{rn}\left(r\right)P_n\left(\cos\theta\right),$$

$$u_\theta=\sum_{n=1}^{\infty}u_{\theta n}\left(r\right)\frac{dP_n\left(\cos\theta\right)}{d\theta}. \tag{1.8}$$

Here function $P_n\left(\cos\theta\right)$ is the Legendre function of the first kind.

Substituting (1.8),(1.7) into the equilibrium equations (1.5) and the boundary conditions (1.1) with respect to $u_{rn}(r)$, $u_{\theta n}(r)$, we obtain the following systems of ordinary differential equations and the boundary conditions for them:

$$\begin{gathered} b_{11}u''_{rn}(r)+\frac{2b_{11}}{r}u'_{rn}+\frac{2}{r^2}(b_{12}-b_{22}-b_{23})u_{rn}-\frac{n(n+1)}{r^2}u_{rn}- \\ -\frac{b_{12}n(n+1)}{r}u'_{\theta n}-\frac{(b_{12}-b_{22}-b_{23}-1)n(n+1)}{r^2}u_{\theta n}=0, \\ \frac{b_{12}+1}{r}u'_{rn}+\frac{(b_{22}+b_{23}+2)}{r^2}u_{rn}+u''_{\theta n}+\frac{2}{r}u'_{\theta n}- \\ -\frac{b_{22}n(n+1)}{r^2}u_{\theta n}+\frac{2(G_0-1)}{r^2}u_{\theta n}=0 \end{gathered} \tag{1.9}$$

$$\begin{gathered} G_1\left[b_{11}u'_{rn}+\frac{2b_{12}}{r}u_{rn}-\frac{n(n+1)}{r}u_{\theta n}\right]_{r=R_i}=\sigma_{ni}, \\ G_1\left[\frac{1}{r}u_{rn}-u'_{\theta n}-\frac{u_{\theta n}}{r}\right]_{r=R_i}=\tau_{ni}\ (i=1,2). \end{gathered} \tag{1.10}$$

The dashes denote the derivatives with respect to r. To solve the problems obtained here various methods, including numerical ones, can be used, for example, the Godunov orthogonal sweep method. The described method for constructing inhomogeneous solutions is universal and does not depend on various shell parameters, including its thickness. However, as shown in [10], if the relative thickness of the shell is sufficiently small and the load given on the faces is sufficiently smooth, then it is expedient to use the first iterative process of A.L. Goldenweiser's asymptotic method which is less time-consuming and allows to achieve the ultimate goal faster.

2 Construction of homogeneous solutions.

Any solution of the equilibrium equations (1.5) that satisfies the condition of stresses absence on the faces is said to be a homogeneous solution. The asymptotic analysis of homogeneous solutions for an isotropic spherical shell, carried out in [16], made it possible to distinguish three main types, each of which is determined by the type of asymptotic expansions in the small parameter.

We show that for a transversally isotropic enriched shell there are three basic types of homogeneous solutions, that is, any solution of the equilibrium equations (1.5) satisfying homogeneous conditions can be represented in the form

$$\underline{u}=\underline{u}^{(0)}+\underline{u}^{(1)}+\underline{u}^{(2)}. \tag{2.1}$$

We assume that the front parts of the boundary are free of stresses

$$\sigma_r=0,\ \tau_{r\theta}=0\ \text{ as }\ r=R_i\ (i=1,\,2). \tag{2.2}$$

To construct homogeneous solutions in the equations of equilibrium, we make a substitution

$$\xi=\frac{1}{\varepsilon}\ln\frac{r}{r_0},\quad r_0=\sqrt{R_1R_2},\quad \varepsilon=\frac{1}{2}\ln\frac{R_2}{R_1},\quad \xi\in[-1,\,1]\,. \tag{2.3}$$

Substituting (2.3) into (1.5),(1.3) respectively, we obtain

$$\begin{aligned}
& b_{11}\frac{\partial^2 u_r}{\partial \xi^2}+\varepsilon b_{11}\frac{\partial u_r}{\partial \xi}+2\varepsilon^2\left(b_{12}-b_{22}-b_{23}\right)u_r+\varepsilon^2\left(\frac{\partial^2 u_r}{\partial \theta^2}+ctg\frac{\partial u_r}{\partial \theta}\right)+\\
& +\varepsilon\left(b_{12}+1\right)\frac{\partial}{\partial \xi}\left(\frac{\partial u_\theta}{\partial \theta}+ctg\theta u_\theta\right)+\\
& +\varepsilon^2\left(b_{12}-b_{22}-b_{23}-1\right)\left(\frac{\partial u_\theta}{\partial \theta}+ctgu_\theta\right)=0,\\
& \varepsilon\left(b_{12}+1\right)\frac{\partial^2 u_r}{\partial \xi\partial \theta}+\varepsilon^2\left(b_{22}+b_{23}+2\right)\frac{\partial u_r}{\partial \theta}+\frac{\partial^2 u_\theta}{\partial \xi^2}+\varepsilon\frac{\partial u_\theta}{\partial \xi}+\\
& +\varepsilon^2 b_{22}\frac{\partial}{\partial \theta}\left(\frac{\partial u_\theta}{\partial \theta}+ctg\theta u_\theta\right)+2\varepsilon^2\left(G_0-1\right)u_\theta=0.
\end{aligned} \tag{2.4}$$

$$\begin{aligned}
& \sigma_r=G_1\varepsilon^{-1}e^{-\varepsilon\xi}\left[b_{11}\frac{\partial u_r}{\partial \xi}+\varepsilon b_{12}\left(2u_r+\frac{\partial u_\theta}{\partial \theta}+ctg\theta u_\theta\right)\right],\\
& \sigma_\varphi=G_1\varepsilon^{-1}e^{-\varepsilon\xi}\left[b_{12}\frac{\partial u_r}{\partial \xi}+\varepsilon\left(b_{22}+b_{23}\right)u_r+\varepsilon b_{22}\frac{\partial u_\theta}{\partial \theta}+\varepsilon b_{23}ctg\theta u_\theta\right],\\
& \sigma_\theta=G_1\varepsilon^{-1}e^{-\varepsilon\xi}\left[b_{12}\frac{\partial u_r}{\partial \xi}+\varepsilon\left(b_{22}+b_{23}\right)u_r+\varepsilon b_{23}\frac{\partial u_\theta}{\partial \theta}+\varepsilon b_{22}ctg\theta u_\theta\right],\\
& \tau_{r\theta}=G_1\varepsilon^{-1}e^{-\varepsilon\xi}\left[\varepsilon\frac{\partial u_r}{\partial \theta}+\frac{\partial u_\theta}{\partial \xi}-\varepsilon u_\theta\right].
\end{aligned} \tag{2.5}$$

The solution of equations (2.4) will be sought in the form:

$$u_r=a\left(\xi\right)m\left(\theta\right),\ \ u_\theta=b\left(\xi\right)\frac{dm\left(\theta\right)}{d\theta}, \tag{2.6}$$

where $m\left(\theta\right)$ is the solution of the Legendre equation

$$\frac{d^2m\left(\theta\right)}{d\theta^2}+ctg\theta\frac{dm\left(\theta\right)}{d\theta}+\left(z^2-\frac{1}{4}\right)m\left(\theta\right)=0. \tag{2.7}$$

The parameter z is determined from condition (2.2).

Substituting (2.6) into (2.4), taking into account (2.7), after separating the variables with respect to the pair of functions $a\left(z,\xi\right)$, $b\left(z,\xi\right)$, we obtain the following system of ordinary differential equations

$$\begin{aligned}
& L_1\left(a,b\right)=b_{11}a''+\varepsilon b_{11}a'+\varepsilon^2\left[b_{12}-b_{22}-b_{23}-\left(z^2-\frac{1}{4}\right)\right]a-\\
& -\varepsilon\left(b_{12}+1\right)\left(z^2-\frac{1}{4}\right)b'-\varepsilon^2\left(b_{12}-b_{22}-b_{23}-1\right)\left(z^2-\frac{1}{4}\right)b=0,\\
& L_2\left(a,b\right)=\varepsilon\left(b_{12}+1\right)a'+\varepsilon^2\left(b_{22}+b_{23}+2\right)a+b''+\varepsilon b'+\\
& +\varepsilon^2\left[2\left(G_0-1\right)-b_{22}\left(z^2-\frac{1}{4}\right)\right]b=0.
\end{aligned} \tag{2.8}$$

Here the dashes denote the derivatives with respect to ξ.

Substituting (2.6) into (2.2) with allowance for (2.5) and (2.7), we obtain the following homogeneous boundary conditions for the functions $a(z,\xi)$, $b(z,\xi)$:

$$M_1(z)(a,b)|_{\xi=\xi_i} = \left\{ b_{11}a' + \varepsilon b_{12}\left[2a - \left(z^2 - \frac{1}{4}\right)b\right]\right\}_{\xi=\xi_i} = 0,$$
$$M_2(z)(a,b)|_{\xi=\xi_i} = \left[b' + \varepsilon(a-b)\right]_{\xi=\xi_i} = 0, \qquad (i = 1,\ 2). \tag{2.9}$$

Thus, the system of equations (2.8) together with boundary conditions (2.9) generates a spectral problem for a pair of functions $a(z,\xi)$, $b(z,\xi)$.

The solution of system (2.8) will be sought in the form:

$$a(z,\xi) = Ae^{\varepsilon\lambda\xi},\ \ b(z,\xi) = Be^{\varepsilon\lambda\xi}, \tag{2.10}$$

where A, B are constants. Then the solution of (2.8) can be represented in the form

$$a(z,\xi) = e^{-\frac{1}{2}\varepsilon\xi}\left[d_1C_1e^{\varepsilon s_1\xi} + d_1C_2e^{-\varepsilon s_1\xi} + d_2C_3e^{\varepsilon s_2\xi} + d_2C_4e^{-\varepsilon s_2\xi}\right],$$
$$b(z,\xi) = e^{-\frac{1}{2}\varepsilon\xi}\left[D_{11}C_1e^{\varepsilon s_1\xi} + D_{21}C_2e^{-\varepsilon s_1\xi} + D_{12}C_3e^{\varepsilon s_2\xi} + D_{22}C_4e^{-\varepsilon s_2\xi}\right], \tag{2.11}$$

where

$$s_n = \frac{1}{2}\sqrt{1+4\tau_n}, \quad \lambda_n(\lambda_n+1) = s_n^2 - \frac{1}{4},$$

τ_n are the roots of the quadratic equation

$$\tau^2 - 2q_1\tau + q_2 = 0, \tag{2.12}$$

where

$$2q_1 = b_{11}^{-1}\left[\left(b_{11}b_{22} - b_{12}^2 - 2b_{12}\right)\left(z^2 - \frac{1}{4}\right) - 2(b_{12} - b_{22} - b_{23}) - 2b_{11}(G_0 - 1)\right],$$

$$q_2 = b_{11}^{-1}\left(z^2 - \frac{9}{4}\right)\left[b_{22}\left(z^2 - \frac{1}{4}\right) - 2(b_{12} - b_{22} - b_{23})(G_0 - 1)\right],$$

$$d_k = s_k^2 - \frac{1}{4} + 2(G_0 - 1) - b_{22}\left(z^2 - \frac{1}{4}\right),$$

$$D_{1k} = -\left[(b_{12}+1)\left(s_k - \frac{1}{2}\right) + b_{22} + b_{23} + 2\right]$$
$$D_{2k} = \left[(b_{12}+1)\left(s_k + \frac{1}{2}\right) - b_{22} - b_{23} + 2\right], \quad (k = 1,\ 2).$$

Satisfying homogeneous boundary conditions (2.9), we obtain the characteristic equation

$$\Delta(z,\varepsilon) = 4\left[(A_{11}B_{12} - A_{12}B_{11})(A_{22}B_{21} - A_{21}B_{22})\,sh^2(s_2+s_1)\varepsilon + \right.$$
$$\left. + (A_{11}B_{22} - A_{22}B_{11})(A_{21}B_{12} - A_{12}B_{21})\,sh^2(s_2 - s_1)\varepsilon\right] = 0, \tag{2.13}$$

where

$$A_{1n}=b_{11}\left(s_n^2-\frac{1}{4}\right)\left(s_n-\frac{1}{2}\right)+2b_{12}\left(s_n^2-\frac{1}{4}\right)+\left[\left(b_{12}^2+b_{12}-b_{11}b_{22}\right)\times\right.$$

$$\left.\times\left(z^2-\frac{1}{4}\right)+2b_{11}\left(G_0-1\right)\right]\left(s_n-\frac{1}{2}\right)-2b_{12}\left(G_0-1\right)\left(z^2-\frac{9}{4}\right)$$

$$A_{2n}=-b_{11}\left(s_n^2-\frac{1}{4}\right)\left(s_n+\frac{1}{2}\right)+2b_{12}\left(s_n^2-\frac{1}{4}\right)-\left[\left(b_{12}^2+b_{12}-b_{11}b_{22}\right)\times\right.$$

$$\left.\times\left(z^2-\frac{1}{4}\right)+2b_{11}\left(G_0-1\right)\right]\left(s_n+\frac{1}{2}\right)-2b_{12}\left(G_0-1\right)\left(z^2-\frac{9}{4}\right),\ (n=1,\,2)$$

$$B_{1k}=-b_{12}\left(s_n-\frac{1}{2}\right)^2+\left(b_{12}-b_{22}-b_{23}\right)\left(s_n-\frac{1}{2}\right)-b_{22}\left(z^2-\frac{9}{4}\right)$$
$$B_{2k}=-b_{12}\left(s_n+\frac{1}{2}\right)^2-\left(b_{12}-b_{22}-b_{23}\right)\left(s_n+\frac{1}{2}\right)-b_{22}\left(z^2-\frac{9}{4}\right).$$

The transcendental equation (2.13) defines a countable set of roots z_k, and the corresponding constants $C_{1k}, C_{2k}, C_{3k}, C_{4k}$ are proportional to cofactors of any row of the determinant of the system. Choosing the cofactors of the first row as the solutions of the system, we obtain

$$C_{1k}=\Delta_{11}C_k,\quad C_{2k}=-\Delta_{12}C_k,\quad C_{3k}=\Delta_{13}C_k,\quad C_{4k}=-\Delta_{14}C_k,\qquad(2.14)$$

where

$$\Delta_{11}=B_{21}\left(A_{12}B_{22}-B_{12}A_{22}\right)e^{\varepsilon s_1}+B_{12}\left(B_{21}A_{22}-A_{21}B_{22}\right)e^{-(2s_2+s_1)\varepsilon}+$$
$$+B_{22}\left(A_{21}B_{12}-B_{21}A_{12}\right)e^{(2s_2-s_1)\varepsilon},$$

$$\Delta_{12}=B_{11}\left(A_{12}B_{22}-B_{12}A_{22}\right)e^{-\varepsilon s_1}-B_{12}\left(A_{11}B_{22}-B_{11}A_{22}\right)e^{-(2s_2-s_1)\varepsilon}+$$
$$+B_{22}\left(A_{21}B_{12}-B_{21}A_{12}\right)e^{(2s_2-s_1)\varepsilon},$$

$$\Delta_{13}=B_{22}\left(A_{11}B_{21}-B_{11}A_{21}\right)e^{\varepsilon s_2}-B_{11}\left(A_{22}B_{21}-A_{21}B_{22}\right)e^{-(2s_1+s_2)\varepsilon}+$$
$$+B_{21}\left(B_{11}A_{22}-A_{11}B_{22}\right)e^{(2s_1-s_2)\varepsilon},$$

$$\Delta_{14}=B_{12}\left(A_{11}B_{21}-B_{11}A_{21}\right)e^{-\varepsilon s_2}+B_{11}\left(A_{21}B_{12}-B_{21}A_{12}\right)e^{(s_2-2s_1)\varepsilon}+$$
$$+B_{21}\left(A_{12}B_{11}-B_{12}A_{11}\right)e^{(2s_1+s_2)\varepsilon}.$$

Summing over all roots of equation (2.13) and taking into account the generalized Hooke's law, we obtain homogeneous solutions of the following form:

$$u_r=r_0\sum_{n=1}^{\infty}C_n u_n(\xi)\,m_n(\theta),$$
$$u_\theta=r_0\sum_{n=1}^{\infty}C_n v_n(\xi)\,\frac{dm_n(\theta)}{d\theta}$$

$$\begin{gathered}
\sigma_r = G_1 e^{-\frac{3}{2}\varepsilon\xi} \sum_{n=1}^{\infty} C_n Q_{rn}(\xi)\, m_n(\theta), \\
\sigma_\theta = G_1 e^{-\frac{3}{2}\varepsilon\xi} \sum_{n=1}^{\infty} C_n \left[Q_{\theta n}^{(1)}(\xi)\, m_n(\theta) + Q_{\theta n}^{(2)}(\xi)\, ctg\theta \frac{dm_n(\theta)}{d\theta} \right], \\
\sigma_\varphi = G_1 e^{-\frac{3}{2}\varepsilon\xi} \sum_{n=1}^{\infty} C_n \left[Q_{\varphi n}^{(1)}(\xi)\, m_n(\theta) + Q_{\varphi n}^{(2)}(\xi)\, ctg\theta \frac{dm_n(\theta)}{d\theta} \right], \\
\tau_{r\theta} = G_1 e^{-\frac{3}{2}\varepsilon\xi} \sum_{n=1}^{\infty} C_n T_n(\xi) \frac{dm_n(\theta)}{d\theta},
\end{gathered} \tag{2.15}$$

where

$$u_n(\xi) = e^{-\frac{1}{2}\varepsilon\xi} \left[d_1 \Delta_{11} e^{\varepsilon s_1 \xi} - d_1 \Delta_{12} e^{-\varepsilon s_1 \xi} + d_2 \Delta_{13} e^{\varepsilon s_2 \xi} - d_2 \Delta_{14} e^{-\varepsilon s_2 \xi} \right],$$

$$v_n(\xi) = e^{-\frac{1}{2}\varepsilon\xi} \left[D_{11} \Delta_{11} e^{\varepsilon s_1 \xi} + D_{21} \Delta_{12} e^{-\varepsilon s_1 \xi} + D_{12} \Delta_{13} e^{\varepsilon s_2 \xi} - D_{22} \Delta_{14} e^{-\varepsilon s_2 \xi} \right],$$

$$Q_{rn} = b_{11} u_n'(\xi) + b_{12}\varepsilon \left[2u_n(\xi) - \left(z_n - \frac{1}{4} \right) v_n(\xi) \right]$$

$$Q_{\theta n}^{(1)} = b_{12} u_n'(\xi) + \varepsilon (b_{22} + b_{23})\, u_n(\xi) - b_{22} \left(z_n - \frac{1}{2} \right) \varepsilon\, v_n(\xi),$$

$$Q_{\theta n}^{(2)} = -2G_0 \varepsilon\, ctg\theta\, \varepsilon\, v_n(\xi),$$

$$Q_{\varphi n}^{(1)} = b_{12} u_n'(\xi) + \varepsilon (b_{22} + b_{23})\, u_n(\xi) - b_{23} \left(z_n - \frac{1}{4} \right) \varepsilon\, v_n(\xi),$$

$$Q_{\varphi n}^{(2)} = 2G_0 \varepsilon\, ctg\theta\, \varepsilon\, v_n(\xi),$$

$$T_n(\xi) = v_n'(\xi) + \varepsilon \left[u_n(\xi) - v_n(\xi) \right].$$

3 Analysis of the roots of the characteristic equation

Equation (2.13) has a countable set of roots with an accumulation point at infinity. The roots of equation (2.13) can be found numerically or, as shown in [14], the asymptotic method is more effective for thin shells. Therefore, further we assume that the shell is thin-walled, ε is a small parameter.

It can be shown that the function $\Delta(z, \varepsilon)$ is an even function of its arguments. With respect to the zeros of the function $\Delta(z, \varepsilon)$ we can formulate the following statement: function $\Delta(z, \varepsilon)$ has three groups of zeros:

a) the first group of roots consists of two zeros $z_k = O(1)$ when $\varepsilon \to 0$, $(k = 1, 2)$;

b) the second group of roots consists of four zeros, which are of order O $\left(\varepsilon^{-\frac{1}{2}}\right)$;

c) the third group of roots contains a countable set of zeros that are of order O $\left(\varepsilon^{-1}\right)$.

We give a diagram of the proof of this assertion.

To this end, we expand the function in a series in ε. After very complicated calculations, we get:

$$\begin{gathered}
\Delta(z, \varepsilon) = 16 s_1 s_2 \left(s_2^2 - s_1^2\right)^2 \left(z^2 - \frac{9}{4}\right) \varepsilon^2 \left[a_1 z^2 + a_0 + \right. \\
\left. + \frac{1}{3}\varepsilon^2 \left(b_1 z^6 + b_2 z^2 + ...\right) + \frac{1}{45}\left(c_1 z^8 + c_2 z^6 + ...\right) \right] = 0,
\end{gathered} \tag{3.1}$$

where

$$a_0 = -\frac{1}{2} b_{12} \left[2b_{22} + (b_{22} + b_{23})(G_0 - 1) \right] l_0 + \left[\frac{1}{2} b_{12} b_{22} - \frac{7}{4} b_{11} b_{22} l_1 + \right.$$
$$+ 9b_{12} l_2 (G_0 - 1) + 2l_1 \left(b_{11} l_1 + 2b_{12}^2 \right)(G_0 - 1) - 2b_{12} (b_{11} + 2b_{12})(G_0 - 1) \Big] \times$$
$$\times l_1 - \frac{9}{2} (b_{22} + b_{23}) l_2 + \frac{9}{4} b_{12}^2 b_{22} (b_{11} + 4b_{22}) +$$
$$+ 2b_{12}^2 b_{22} (9b_{12} - 4b_{11})(G_0 - 1) + b_{12}^2 \left[9b_{12}^2 - 4b_{11} (b_{22} + b_{23}) \right] (G_0 - 1)^2,$$

$$a_1 = 2b_{12}^2 \left[2b_{22} + (b_{22} + b_{23})(G_0 - 1) \right] l_0 - b_{11} b_{22} l_1 (2b_{12} + l_1) + 2b_{22} (b_{22} + b_{23}) l_2 -$$

$$- 4b_{12} \left(2b_{12}^2 b_{22} + l_1 l_2 \right)(G_0 - 1) - 4b_{12}^4 (G_0 - 1)^2,$$

$$b_1 = \frac{b_{22}}{b_{11}} l_2^2 + \left[2b_{12}^3 + \left(2b_{12}^2 - 2b_{11} b_{22} \right) l_0 \right] l_0 - b_{11} b_{22} \left(2b_{12}^2 - 2b_{11} b_{22} \right),$$

$$b_2 = \left[\frac{3}{2} \frac{b_{12} b_{22}}{b_{11}} \left(2b_{11} b_{22} - b_{12}^2 \right) + \frac{11}{2} b_{12} b_{22}^2 - \frac{8b_{12}^2 b_{22}^2}{b_{11}} + b_{22} - 6b_{12}^2 b_{22} (G_0 - 1) + \right.$$
$$\left. + 8b_{11} b_{22} (G_0 - 1) - \frac{16 b_{12}^3 b_{22}}{b_{11}} (G_0 - 1) + \frac{2b_{12}^2}{b_{11}} l_2 (G_0 - 1) - \frac{8b_{12}^4}{b_{11}} (G_0 - 1) \right]^2 l_0 +$$
$$+ \left[\frac{21}{4} \frac{b_{12}^2 b_{22}}{b_{11}} + \frac{11}{2} b_{22}^2 + \frac{4b_{12}^2}{b_{11}} (b_{22} + b_{23})(G_0 - 1) \right] l_0^2 -$$
$$- \left[\frac{4b_{12}^2 b_{22}}{b_{11}} + 4\, b_{12} b_{22} + \frac{2b_{22}}{b_{11}} l_2 - 4b_{22}^2 - \frac{8b_{12}}{b_{11}} l_2 (G_0 - 1) + \right.$$
$$\left. + 4b_{12} b_{22} (G_0 - 1) \right] l_0 l_1 + \left\{ 4b_{12} b_{22} \left[b_{12} (G_0 - 1) + 2b_{22} \right] - \frac{2}{b_{11}} l_2 (G_0 - 1) \right\} l_1 +$$
$$+ \frac{51}{4} b_{12}^2 b_{22}^2 - \frac{19}{4} \frac{b_{22}}{b_{11}} l_1^2 - \frac{19}{2} b_{11} b_{22}^3 + 4b_{11} b_{22}^2 (b_{22} + b_{23})(G_0 - 1) + 6b_{12}^3 b_{22} (G_0 - 1),$$

$$l_0 = b_{11} b_{22} - b_{12}^2 - 2b_{12}, \quad l_1 = b_{12} - b_{22} - b_{23},$$

$$l_2 = b_{11} b_{22} + b_{12}^2, \quad l_3 = b_{11} b_{22} - b_{12}^2,$$

$$c_1 = -\frac{b_{12}}{b_{11}^2} (b_{12} + 2b_{22}) l_3 l_0^2 + 8 \frac{b_{12} b_{22}^2}{b_{11}} l_0^2 - 8 \frac{b_{22}^2}{b_{11}} l_3 l_0 + \frac{2b_{22}}{b_{11}} l_3^2 l_0 + \frac{b_{12}^2}{b_{11}^2} l_0.$$

Hence we see that $z = \pm\frac{3}{2}$ are zeros of the function $\Delta(z, \varepsilon)$. We note that the existence of these zeros also follows from the equilibrium condition of the sphere. Let us prove that all the remaining zeros of the function $\Delta(z, \varepsilon)$ increase unboundedly when $\varepsilon \to 0$. We proceed from the opposite, assuming that $z_k - z_k^* \neq \infty$. Then the limiting relation $\Delta(z, \varepsilon) \to \to \varepsilon^2 \Delta_0(z_k^*)$ is valid for $\varepsilon \to 0$. Thus, the limit points of the set of zeros z_k when $\varepsilon \to 0$ are determined from the equation $\Delta_0(z_k^*) = 0$. In this case

$$\Delta_0(z_k^*) = 16 s_1 s_2 \left(s_2^2 - s_1^2 \right)^2 \left(z^2 - \frac{9}{4} \right) = 0.$$

From the last equality it follows that there are no other bounded roots except $z = \pm\frac{3}{2}$.

Thus, it is proved that all the remaining zeros of the function $\Delta(z,\,\varepsilon)$ tend to infinity for $\varepsilon\to 0$. They can be divided into two groups depending on their behavior at $\varepsilon\to 0$. The following limit relationships are possible:

$$1)\ \varepsilon\, z_n\to 0\ \text{ as } \varepsilon\to 0;\qquad 2)\ \varepsilon\, z_n\to const\ \ \text{as}\,\varepsilon\to 0.$$

We define at first such z_k that $\varepsilon\, z_k\to 0$ for $\varepsilon\to 0$. For this purpose we again use the expansion (3.1) Suppose that the principal term of the asymptotic expression z_n has the form:

$$z_n=\varepsilon^{-2}\gamma_0,\ \gamma_0=O\left(1\right)\ \text{as}\,\varepsilon\to 0,\ 0<\alpha<1 \tag{3.2}$$

Substituting (3.2) into (3.1) and preserving the principal terms in it, we obtain the following limiting equation for γ_0:

$$\begin{gathered}\gamma_0^2\varepsilon^{2-2\alpha}\left\{a_1+O\left(\varepsilon^{2\alpha}\right)+\frac{1}{3}\left[b_1\gamma_0^4+O\left(\varepsilon^{2\alpha}\right)\right]\varepsilon^{2-4\alpha}\right\}+\\+O\left[\max\left(\varepsilon^{2-2\alpha},\varepsilon^{4-6\alpha}\right)\right]=0.\end{gathered} \tag{3.3}$$

We consider three cases: a) $0<\alpha<\frac{1}{2}$, b) $\alpha=\frac{1}{2}$, c) $\frac{1}{2}<\alpha<1$.
In case a) passing to the limit in (3.3) for $\varepsilon\to 0$, we obtain $\gamma_0=0$ that contradicts assumption (3.2). Similarly, in case c) we obtain $\gamma_0=0$ and we arrive at a contradiction. Finally, in case b) we have

$$\gamma_0^4+3\frac{a_1}{b_1}=0. \tag{3.4}$$

Now seeking z_n in the form of the following expansion

$$z_n=\varepsilon^{-\frac{1}{2}}\left[\alpha_n+\alpha_n^{(0)}\sqrt{\varepsilon}+\gamma_n\varepsilon+...\right],(n=1,\,2,\,3,\,4) \tag{3.5}$$

we get

$$\alpha_n=\gamma_0;\alpha_n^{(0)}=0;\gamma_n=\left(20a_1b_1^2\right)^{-1}\left(5a_0b_1^2+a_1c_1^2-3a_1b_1b_2\right).$$

To find the asymptotics of the zeros of the third group, we seek $z_k\ (k=n-4,\ n=5,\,6,...)$

$$z_k=\varepsilon^{-1}\delta_k+O\left(\varepsilon\right),\ (k=1,\,2,...)\,. \tag{3.6}$$

Substituting (3.6) into (3.12) we have

$$\tau^2-2\tilde{q}_1\delta_n^2\tau+\tilde{q}_2\delta_n^4=0, \tag{3.7}$$

where

$$2\tilde{q}_1=\frac{1}{b_{11}}l_0,\ \ \tilde{q}_2=b_{11}^{-1}b_{22};\ \ s_i=\delta_n^2\tau_i,$$

$$s_i=\sqrt{\tilde{q}_1-(-1)^i\sqrt{\tilde{q}_1^2-\tilde{q}_2}},\ \ (i=1,\,2)\ .$$

As noted in [11], depending on the characteristics of the material $\nu,\ \nu_1,\nu_2,\ G_0$, the parameters $\tilde{q}_1,\tilde{q}_2$ take different values, which entails different recording of the solutions through the function $\exp x$. This, in turn, leads to various asymptotic representations.

Consider the following possible cases:

1.$\lambda_{1,2}=\pm s_1\delta_k,\ \lambda_{3,4}=\pm s_2\delta_k,\ \tilde{q}_1>0,\ \tilde{q}_1^2-\tilde{q}_2>0,$

$$s_{1,2}=\sqrt{\tilde{q}_1\pm\sqrt{\tilde{q}_1^2-\tilde{q}_2}},\ \ \tilde{q}_1^2>\tilde{q}_2,$$

$$s_{1,2} = N + i\beta = \sqrt{\tilde{q}_1 \pm i\sqrt{\tilde{q}_2 - \tilde{q}_1^2}},\ \ \tilde{q}_1^2 < \tilde{q}_2,\ i = \sqrt{-1}.$$

2. The roots of equation (3.7) are multiple

$$\lambda_{1,2} = \ \lambda_{3,4} = \pm\delta_k\rho,\ \ \tilde{q}_1 > 0,\ \ \tilde{q}_1^2 - \tilde{q}_2 = 0,\ \ p = \sqrt{\tilde{q}_1},$$

$$s_{1,2} = \sqrt{\tilde{q}_1 \pm \sqrt{\tilde{q}_1^2 - \tilde{q}_2}},\ \ \tilde{q}_1^2 > \tilde{q}_2,$$

3.$\lambda_{1,2} = \pm i s_1 \delta_k,\ \ \lambda_{3,4} = \pm i s_2 \delta_k,\ \ \tilde{q}_1 < 0,\ \ \tilde{q}_1^2 - \tilde{q}_2 \neq 0,$

$$s_{1,2} = \sqrt{|\tilde{q}_1| \pm \sqrt{\tilde{q}_1^2 - \tilde{q}_2}},\ \ \tilde{q}_1^2 > \tilde{q}_2,$$

$$s_{1,2} = \sqrt{|\tilde{q}_1| \pm i\sqrt{\tilde{q}_2 - \tilde{q}_1^2}},\ \ \tilde{q}_1^2 < \tilde{q}_2.$$

4.$\lambda_{1,2} = \lambda_{3,4} = \pm i\delta_n p,\ \ \tilde{q}_1 < 0,\ \ \tilde{q}_1^2 - \tilde{q}_2 = 0,\ \ p = \sqrt{|\tilde{q}_1|}.$

In cases 1.2, after substituting (3.6) into (2.13) and transforming it for δ_n, we obtain:

$$(s_2 - s_1)\, sh\,(s_1 + s_2)\,\delta_n \pm (s_1 + s_2)\, sh\,(s_2 - s_1)\,\delta_n = 0, \tag{3.8}$$

$$N sh 2\beta\delta_n \pm \beta \sin 2N\delta_n = 0, \tag{3.9}$$

$$sh 2p\delta_n \pm 2p\delta_n = 0. \tag{3.10}$$

As for cases 3 and 4, the results for them are obtained from cases 1 and 2 by a formal replacement of s_1, s_2 with is_1, is_2. These equations coincide with the equations that determine the exponents of the Saint Venant boundary effects in the theory of transversely isotropic plates.

4 Construction of asymptotic formulas for displacements and stresses

In this section, assuming that ε is a small parameter, we give an asymptotic construction of homogeneous solutions corresponding to three groups of zeros. Substituting $z = \frac{3}{2}$ into (2.15), we obtain the following expressions:

$$\begin{aligned} u_r &= C_0 P_1(\cos\theta) = C_0 \cos\theta, \\ u_\theta &= C_0 \frac{dP_1(\cos\theta)}{d\theta} = -C_0 \sin\theta, \\ \sigma_r &= \sigma_\theta = \sigma_\varphi = \tau_{r\theta} = 0. \end{aligned} \tag{4.1}$$

Thus, this solution corresponds to the displacement of the shell, as a rigid body.

Similarly, for $z = -\frac{3}{2}$ we get

$$\begin{aligned} u_r &= r_0 \left(\cos\theta \ln ctg\frac{\theta}{2} - 1 \right) A, \\ u_\theta &= -r_0 \left(\sin\theta \ln ctg\frac{\theta}{2} + ctg\theta \right) A, \\ \sigma_r &= 0,\ \ \sigma_\theta = -\sigma_\varphi = \frac{G_1 A}{r\sin^2\theta},\ \ \tau_{r\theta} = 0. \end{aligned} \tag{4.2}$$

Here C_0, A are arbitrary constants.

The above formulas (2.15) are exact. On the basis of these formulas, it is easy to obtain approximate formulas by expanding all the expressions in powers of the parameter ε.

We now turn to the study of homogeneous solutions corresponding to the second group of roots. As follows from expression (3.4), this group of roots corresponds to four solutions. Substituting (3.5) into (2.15) and expanding in powers of the parameter ε, we obtain:

$$\begin{aligned}
&u_r = r_0 \sum_{k=1}^{4} C_k u_k(\xi)\, m_k(\theta),\\
&u_\theta = r_0 \sum_{k=1}^{4} C_k v_k(\xi)\, \frac{dm_k(\theta)}{d\theta}\\
&\sigma_r = O(\varepsilon), \quad \tau_{r\theta} = O\left(\sqrt{\varepsilon}\right),\\
&\sigma_\theta = G_1 \sigma(\xi) \sum_{k=1}^{4} C_k Q_{\theta k}(\xi)\, \frac{dm_k(\theta)}{d\theta},\\
&\sigma_\varphi = G_1 \sigma(\xi) \sum_{k=1}^{4} C_k Q_{\varphi k}(\xi)\, \frac{dm_k(\theta)}{d\theta},\\
&\sigma(\xi) = 1 - \varepsilon\xi + \frac{1}{2}\varepsilon^2\xi^2 + O\left(\varepsilon^3\right),
\end{aligned} \tag{4.3}$$

where C_k are arbitrary constants;

$$u_k(\xi) = 1 - \nu - 2\nu_1\nu_2 + O(\varepsilon),$$

$$v_k(\xi) = 4G_0^2 E_0 \left(1+\nu^2\right) - \left(1 - \nu - 2\nu_1\nu_2\right)\varsigma + O(\varepsilon),$$

$$Q_{rk} = O(\varepsilon),$$

$$Q_{\theta k} = b_{11}^{-1}\left(b_{11}b_{22} - b_{12}^2\right)\alpha_k^2\xi - \left(b_{22}+b_{23}\right) + O(\varepsilon),$$

$$Q_{\varphi k} = \left(b_{11}b_{23} - b_{12}^2\right)\left[b_{11}^{-1}\alpha_k^2\xi - \left(b_{11}b_{22} - b_{12}^2\right)\left(b_{22}+b_{23}\right) + O(\varepsilon)\right].$$

Here we should pay attention to the following circumstance. The solution of equation (2.7), generally speaking, can be written in terms of a Legendre function.

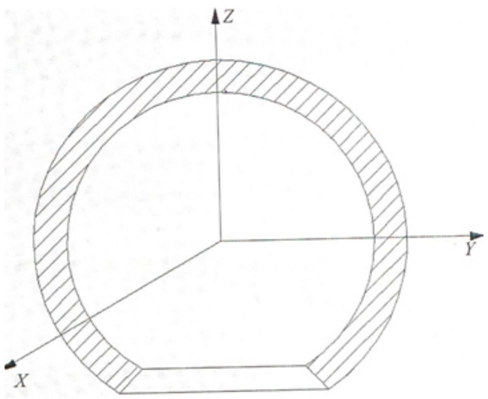

Fig. 3.

However, as shown in [16], it is more convenient to use approximate methods. Two cases should be considered separately:

1) The shell does not contain any of the poles $0,\ \pi$.

2) The shell contains at least one of the poles.

In the first case, it is convenient to use asymptotic methods for approximate integration, as detailed in [13]. As for the second case (Fig. 3.), here the asymptotic method of integration can not give an approximate solution of the problem for any relative thickness ε of the shell. The point is that the asymptotic approximations lose their accuracy in the vicinity of the vertex $\theta = 0$. In this case, it is necessary to select only those solutions of equation (2.7) that remain bounded for $\theta = 0$. These solutions were constructed in [16], where approximate methods of their calculation are given. Therefore, we will not dwell on them.

We assume that the shell does not contain any of the poles $0,\ \pi$. We give the final result:

$$m\left(\theta\right) = A_1 e^{\frac{z_k}{\sqrt{\varepsilon}}\theta} + B_1 e^{-\frac{z_k}{\sqrt{\varepsilon}}\theta}. \tag{4.4}$$

It follows from (4.4) that, for sufficiently small ε, the quantity $m\left(\theta\right)$ has the character of an edge effect that varies as an exponential function with exponent $\frac{1}{\sqrt{\varepsilon}}$. Thus, the second group of roots defines edge effects analogous to the edge effect of the applied theory of shells.

In the case of the third group of roots for displacements and stresses, we obtain two classes of solutions, the first of which corresponds to the zeros of the function

$$\left(s_2 - s_1\right) sh\left(s_1 + s_2\right)\delta_n + \left(s_1 + s_2\right) sh\left(s_2 - s_1\right)\delta_n,$$

and the second one - to the zeros of the function.

$$\left(s_2 - s_1\right) sh\left(s_1 + s_2\right)\delta_n + \left(s_1 + s_2\right) sh\left(s_2 - s_1\right)\delta_n.$$

They have the same structure and can be represented by the following expressions:

$$\begin{gathered}
u_r = r_0 \sum_{n=1,3,\ldots}^{\infty} B_n \delta_n \left[\left(s_1^2 - b_{22}\right)\left(b_{12}s_2^2 + b_{22}\right) chs_2\delta_n chs_1\delta_n\xi - \right. \\
\left. - \left(s_2^2 - b_{22}\right)\left(b_{12}s_1^2 + b_{22}\right) chs_1\delta_n chs_2\delta_n\xi + O\left(\varepsilon\right)\right] m_n\left(\theta\right), \\
u_\theta = -\left(b_{12} + 1\right)\varepsilon r_0 \sum_{n=1,3,\ldots}^{\infty} B_n \left[s_1\left(b_{12}s_2^2 + b_{22}\right) chs_2\delta_n chs_1\delta_n\xi - \right. \\
\left. - s_2\left(b_{12}s_1^2 + b_{22}\right) chs_1\delta_n chs_2\delta_n\xi + O\left(\varepsilon\right)\right] \frac{dm_n\left(\theta\right)}{d\theta},
\end{gathered} \tag{4.5}$$

$$\sigma_r = G_1\left(b_{12}s_1^2 + b_{22}\right)\left(b_{12}s_2^2 + b_{22}\right) \sum_{n=1,3,\ldots}^{\infty} B_n\delta_n^2 \times$$

$$\times\left[s_1 chs_2\delta_n chs_1\delta_n\xi - s_2 chs_1\delta_n chs_2\delta_n\xi + O\left(\varepsilon\right)\right] m_n\left(\theta\right),$$

$$\sigma_\varphi = G_1\varepsilon^{-1} \sum_{n=1,3,\ldots}^{\infty} B_n\delta_n^2 \left[s_1\left(b_{12}s_2^2 + b_{22}\right)\left(b_{12}s_1^2 - 2b_{12}G_0 + b_{23}\right) chs_2\delta_n chs_1\delta_n\xi - \right.$$

$$\left. - s_2\left(b_{12}s_1^2 + b_{22}\right)\left(b_{12}s_2^2 - 2b_{12}G_0 + b_{23}\right) chs_1\delta_n chs_2\delta_n\xi + O\left(\varepsilon\right)\right] m_n\left(\theta\right),$$

$$\sigma_\theta = G_1\varepsilon^{-1} \sum_{n=1,3,\ldots}^{\infty} B_n\delta_n^2 \left[s_1\left(b_{12}s_2^2 + b_{22}\right)\left(b_{11}s_1^2 - b_{12}^2 + b_{12} - b_{11}b_{22}\right) chs_2\delta_n chs_1\delta_n\xi - \right.$$

$$\left. - s_2\left(b_{12}s_1^2 + b_{22}\right)\left(b_{11}s_2^2 - b_{12}^2 + b_{12} - b_{11}b_{22}\right) chs_1\delta_n chs_2\delta_n\xi + O\left(\varepsilon\right)\right] m_n\left(\theta\right).$$

In the case $q_1 > 0$ when the roots of the quadratic equation (4.5) are also multiple, the solutions have the form:

$$u_r = r_0\left(p^2 - b_{22}\right)\sum_{n=1,3,\ldots}^{\infty} D_n\delta_n\left\{\left[\frac{2b_{22}\left(b_{12}+1\right)p}{\left(p^2-b_{22}\right)}chp\delta_n - \delta_n\left(b_{12}p^2+b_{22}\right)shp\delta_n\right]\times\right.$$
$$\left.\times chp\delta_n + \delta_n\xi\left(b_{12}p^2+b_{22}\right)chp\delta_n shp\delta_n\xi + O\left(\varepsilon\right)\right\} m_n\left(\theta\right),$$
$$u_\theta = \left(b_{12}+1\right)\varepsilon r_0\sum_{n=1,3,\ldots}^{\infty} D_n\delta_n\left\{\left[\left(b_{12}p^2-b_{22}\right)chp\delta_n + p\delta_n\left(b_{12}p^2+b_{22}\right)shp\delta_n\right]\times\right.$$
$$\left.\times shp\delta_n\xi - p\delta_n\xi\left(b_{12}p^2+b_{22}\right)chp\delta_n shp\delta_n\xi + O\left(\varepsilon\right)\right\}\frac{dm_n\left(\theta\right)}{d\theta},$$

$$\sigma_r = G_1\varepsilon^{-1}\left(b_{12}p^2+b_{22}\right)^2\sum_{n=1,3,\ldots}^{\infty} D_n\delta_n^2\left[\left(chp\delta_n - p\delta_n chp\delta_n\right)shp\delta_n\xi+\right.$$
$$\left.+p\delta_n\xi chp\delta_n chp\delta_n\xi + O\left(\varepsilon\right)\right]m_n\left(\theta\right),$$

$$\sigma_\varphi = G_1\varepsilon^{-1}\sum_{n=1,3,\ldots}^{\infty} D_n\delta_n^2\left\langle\left\{\left[2b_{12}b_{22}\ \left(b_{12}+1\right)p^2\left(b_{12}p^2-b_{22}\right)\times\right.\right.\right.$$
$$\times\left(b_{12}p^2+b_{22}\right) - b_{23}\left(b_{12}+1\right)\left(b_{12}p^2-b_{22}\right)\big]chp\delta_n - p\delta_n\left(b_{12}p^2+b_{22}\right)\times \qquad (4.6)$$
$$\times\left(b_{12}p^2 - 2b_{12}G_0 + b_{23}\right)shp\delta_n\big\} + p\delta_n\left(b_{12}p^2+b_{22}\right)\times$$
$$\times\left(b_{12}p_2 - 2b_{12}G_0 + b_{23}\right)\xi chp\delta_n chp\delta_n\xi + O\left(\varepsilon\right)\big\rangle m_n\left(\theta\right),$$

$$\sigma_\theta = G_1\varepsilon^{-1}\left(b_{12}p^2+b_{22}\right)\left(b_{12}p^2+b_{12}^2+b_{12}-b_{11}b_{22}\right)\sum_{n=1,3,\ldots}^{\infty} D_n\delta_n^2\times$$
$$\times\left[\left(chp\delta_n + p\delta_n shp\delta_n\right)shp\delta_n\xi - p\delta_n\xi chp\delta_n chp\delta_n\xi + O\left(\varepsilon\right)\right]m_n\left(\theta\right),$$

$$\tau_{r\theta} = G_1\varepsilon^{-1}\left(b_{12}p^2+b_{22}\right)^2\sum_{n=1,3,\ldots}^{\infty} D_n\delta_n^2\left[shp\delta_n chp\delta_n\xi - \xi chp\delta_n shp\delta_n\xi + O\left(\varepsilon\right)\right]\times$$
$$\times\frac{dm_n\left(\theta\right)}{d\theta}$$

$$sh2p\delta_n + 2p\delta_n = 0.$$

Similarly, in the case of $Nsh2\beta\delta_n + 2\beta\delta_n = 0$,

$$u_r = r_0\sum_{n=1,3,\ldots}^{\infty} E_n\left[\left(a_1\varDelta_{1n} - a_2\varDelta_{2n}\right)ch\beta\delta_n\xi\cos N\delta_n\xi-\right.$$
$$\left.-\left(a_1\varDelta_{2n} + a_2\varDelta_{1n}\right)sh\beta\delta_n\xi\sin N\delta_n\xi + O\left(\varepsilon\right)\right]m_n\left(\theta\right),$$
$$u_\theta = \left(b_{12}+1\right)\varepsilon r_0\sum_{n=1,3,\ldots}^{\infty} E_n\left[\left(\beta\varDelta_{2n} - N\varDelta_{1n}\right)ch\beta\delta_n\xi shN\delta_n\xi+\right.$$
$$\left.+\left(N\varDelta_{2n} + \beta\varDelta_{1n}\right)ch\beta\delta_n\xi\cos N\delta_n\xi + O\left(\varepsilon\right)\right]\frac{dm_n\left(\theta\right)}{d\theta},$$

$$\sigma_r = G_1\varepsilon^{-1}\sum_{n=1,3,...}^{\infty} E_n\delta_n^2\left\langle\{[\beta b_{12}a_2 - Nb_{12}a_1 - \beta Nb_{22}(b_{12}+1)]\ \varDelta_{2n}-\right.$$
$$-\left[\beta b_{12}a_1 + Nb_{12}a_2 + \beta^2 Nb_{22}(b_{12}+1)\right]\varDelta_{1n}\big\}\ sh\beta\delta_n chN\delta_n\xi+$$
$$+\{[Nb_{12}a_1 - \beta b_{12}a_2 - Nb_{22}(b_{12}+1)]\ \varDelta_{1n}-$$
$$\left.[Nb_{12}a_2 + \beta b_{12}a_1 + \beta b_{22}(b_{12}+1)]\ \varDelta_{2n}\}\cos\beta\delta_n\xi shN\delta_n\xi + O(\varepsilon)\right\rangle m_n(\theta), \tag{4.7}$$
$$\sigma_\varphi = G_1\varepsilon^{-1}\sum_{n=1,3,...}^{\infty} E_n\delta_n^2\left\langle\{[\beta b_{12}a_2 - Nb_{12}a_1 - \beta Nb_{23}(b_{12}+1)]\ \varDelta_{2n}-\right.$$
$$-\left[\beta b_{12}a_1 + Nb_{12}a_2 + \beta^2 Nb_{23}(b_{12}+1)\right]\varDelta_{1n}\big\}\ sh\beta\delta_n\xi chN\delta_n\xi+$$
$$+\{[Nb_{12}a_1 - \beta b_{12}a_2 + Nb_{23}(b_{12}+1)]\ \varDelta_{1n}-$$
$$\left.-[Nb_{12}a_2 + \beta b_{12}a_1 + \beta b_{23}(b_{12}+1)]\ \varDelta_{2n}\}\cos\beta\delta_n\xi shN\delta_n\xi + O(\varepsilon)\right\rangle m_n(\theta),$$

$$\sigma_\theta = G_1\varepsilon^{-1}\sum_{n=1,3,...}^{\infty} E_n\delta_n^2\left\langle\{[\beta b_{11}a_2 - Nb_{11}a_1 - \beta Nb_{12}(b_{12}+1)]\ \varDelta_{2n}-\right.$$
$$-\left[\beta b_{11}a_1 + Nb_{11}a_2 + \beta^2 Nb_{12}(b_{12}+1)\right]\varDelta_{1n}\big\}\ sh\beta\delta_n\xi chN\delta_n\xi+$$
$$+\{[Nb_{11}a_1 - \beta b_{11}a_2 + Nb_{12}(b_{12}+1)]\ \varDelta_{1n} - [Nb_{11}a_2 + \beta b_{11}a_1 + \beta b_{12}(b_{12}+1)]\times$$
$$\left.\times\varDelta_{2n}\}\cos\beta\delta_n\xi shN\delta_n\xi + O(\varepsilon)\right\rangle m_n(\theta),$$

$$\tau_{r\theta} = G_1\left(N^2+\beta^2\right)\left[b_{12}^2 + 2b_{22}\left(N^2-\beta^2\right) + b_{22}^2\left(N^2+\beta^2\right)^2\right]\times$$
$$\times\sum_{n=1,3,...}^{\infty} E_n\delta_n\left[\cos\beta\delta_n shN\delta_n\sin\beta\delta_n\xi chN\delta_n\xi-\right.$$
$$\left.-\sin\beta\delta_n chN\delta_n\cos\beta\delta_n\xi shN\delta_n\xi + O(\varepsilon)\right]\frac{dm_n(\theta)}{d\theta},$$

where

$$a_1 = 1 - b_{22}\left(N^2-\beta^2\right),\quad a_2 = 2b_{22}\beta N,$$

$$\varDelta_{1n} = N\left[b_{12} + b_{22}\left(N^2+\beta^2\right)\right]\sin\beta\delta_n chN\delta_n + \beta\left[b_{12} - b_{22}\left(N^2+\beta^2\right)\right]\cos\beta\delta_n shN\delta_n,$$
$$\varDelta_{2n} = -\beta\left[b_{12} - b_{22}\left(N^2+\beta^2\right)\right]\sin\beta\delta_n chN\delta_n + N\left[b_{12} - b_{22}\left(N^2+\beta^2\right)\right]\cos\beta\delta_n shN\delta_n.$$

Here $B_n,\ D_n,\ E_n$ are arbitrary constants.

$$\frac{d^2m_n(\theta)}{d\theta^2} + \frac{\delta_n^2}{\varepsilon^2}m_n(\theta) = 0.$$

Expressions for $n = 2,\ 4,\ 6, ...$ are obtained from formulas (4.5), (4.6), (4.7)by replacing chx with shx and shx with $-chx$, $\cos x$ with $\sin x$ and $\sin x$ with $-\cos x$, respectively. In the formulas (4.5), (4.6), (4.7) replacing s_1, s_2, p with is_1, is_2, ip, respectively, we obtain the solution of cases 3 and 4.

In [10], the roots of equations (3.8), (3.9), (3.10) were investigated and a method for their calculation was developed. The nature of these roots significantly affects the overall picture of the stress-strain state of the shell.

In the case of substantial anisotropy, which occurs at large values of G_0, the Saint-Venant boundary layer weakens very weakly and solutions (4.5), (4.6), (4.7) should be listed in terms of penetrating solutions. Therefore, in this case, the stress-strain states of the transversely isotropic and isotropic shell are qualitatively different.

Let us consider the connection between homogeneous solutions and the principal stress vector P acting in the section $\theta = const$. We have:

$$P = \int_0^{2\pi} \int_{R_1}^{R_2} \left(-\sigma_\theta \sin\theta + \tau_{r\theta} \cos\theta\right) \, r \sin\theta \, d\varphi \, dr. \tag{4.8}$$

Imagine the stress σ_θ and $\tau_{r\theta}$ in the form:

$$\begin{gathered} \sigma_\theta = \sigma_\theta^0 + \sum_{n=1}^{\infty} \left[Q_1^{(n)} m_n(\theta) + Q_2^{(n)} ctg\theta \frac{dm_n(\theta)}{d\theta}\right], \\ \tau_{r\theta} = \tau_{r\theta}^0 + \sum_{n=1}^{\infty} T_n(\xi) \frac{dm_n(\theta)}{d\theta}. \end{gathered} \tag{4.9}$$

The terms $\sigma_\theta^0, \tau_{r\theta}^0$ correspond to eigenvalues $z = -\frac{3}{2}$. The second term includes the stresses of the second and third groups of solutions. Substituting (4.9) into (4.8), we obtain

$$P = P_0 + 2\pi r_0^2 \sum_{n=1}^{\infty} \left[-b_{1n} \sin\theta \, m_n(\theta) + b_{2n} \cos\theta \frac{dm_n(\theta)}{d\theta}\right], \tag{4.10}$$

where

$$\begin{gathered} b_{1n} = \int_{-1}^{1} Q_1^{(n)}(\xi) \, e^{2\varepsilon\xi} d\xi, \\ b_{2n} = \int_{-1}^{1} \left(T_n(\xi) - Q_2^{(n)}(\xi)\right) e^{2\varepsilon\xi} d\xi. \end{gathered}$$

Let us prove that all b_{1n} and b_{2n} $(n = 1, 2, 3, ...)$ are equal to zero. For this we consider the following boundary value problem for $\theta = \theta_j$ $(j = 1, 2)$.

$$\begin{gathered} \sigma_\theta = Q_1^{(k)} m_k(\theta_j) + Q_2^{(k)} ctg\theta_j \frac{dm_k(\theta_j)}{d\theta}, \\ \tau_{r\theta} = T_k(\xi) \frac{dm_k(\theta_j)}{d\theta}. \end{gathered} \tag{4.11}$$

The solution of this problem is the k-th terms in the sums of formulas (4.9).

$$\sigma_\theta = Q_1^{(k)} m_k(\theta) + Q_2^{(k)} ctg\theta_j \frac{dm_k(\theta)}{d\theta},$$

$$\tau_{r\theta} = T_k(\xi) \frac{dm_k(\theta)}{d\theta}.$$

The principal vector, which corresponds to the stress state of the problem (4.9) in the cross section $\theta = const$, is reduced to the following form:

$$P_k = 2\pi r_0^2 \sin\theta \left[-b_{1k} \sin\theta \, m_k(\theta) + b_{2k} \cos\theta \frac{dm_k(\theta)}{d\theta}\right]. \tag{4.12}$$

According to the solvability condition for the elasticity problem, the vector P_k must not depend on the variable θ. However, in the relation (4.12), the right-hand side, because of the linear independence $\sin\theta \, m_k(\theta)$ and $\cos\theta \frac{dm_k(\theta)}{d\theta}$, depends on θ. Hence it follows that $P_k = 0$ and $b_{1k} = b_{2k} = 0$ for any $(k = 1, 2, ...)$. Thus, we obtain $P = 2\pi r_0^2 G_1 A\varepsilon$ for the principal vector P.

The stress state corresponding to the zeros of the second and third groups is self-balancing in each section $\theta = const$.

We calculate the bending moment M and the cutting force Q in section $\theta = const$ for the second and third groups of solutions.

Consider solution (4.3). It has the character of an edge effect with a damping index of order $O\left(\varepsilon^{-\frac{1}{2}}\right)$ with respect to ε. We represent the constants $C_k\ (k = 1, 2, 3, 4)$ in the form

$$C_k = C_{k0} + \varepsilon C_{k1} + ...$$

We have

$$\begin{gathered} M = \int\limits_{R_1}^{R_2} \sigma_\theta (r, \theta) (r - R_0) r \sin\theta dr, \quad R_0 = \tfrac{1}{2} (R_1 + R_2), \\ Q = \int\limits_{R_1}^{R_2} \tau_{r\theta} (r, \theta) r \sin\theta dr . \end{gathered} \tag{4.13}$$

Substituting the stresses into formulas (4.13), for the second group of solutions we obtain expressions in variables ξ, θ.

$$\begin{gathered} M_2 = r_0^3 \sin\theta \sum_{k=1}^{4} C_{k0} m_k (\theta) \int\limits_{-1}^{1} Q_{\theta k} (\xi) (1 - Rch\varepsilon) d\xi + O(\varepsilon), \\ Q_2 = r_0^3 \sin\theta \sum_{k=1}^{4} C_{k0} \tfrac{dm_k(\theta)}{d\theta} \int\limits_{-1}^{1} T_k (\xi) d\xi + O(\varepsilon). \end{gathered} \tag{4.14}$$

Similarly, for the stresses of the third group, we obtain

$$M_3 \sim O(\varepsilon), \quad Q_3 =\sim O(\varepsilon).$$

Thus, the main part of the bending moment M_j $(j = 1,\ 2)$ and the shearing force Q_j determine the solution of the second group. Expanding the bending moments and the shearing forces Q_j acting on conical surfaces $\theta = \theta_k$, into series in ε

$$M_j = M_{j0} + \varepsilon M_{j1} + ..., \quad Q_j = Q_{j0} + \varepsilon Q_{j1} + ...,$$

to determine the constants $C_{k0}\ (k = 1, 2, 3, 4)$, we obtain the system

$$\begin{gathered} M_{j0} = r_0^3 \sin\theta_j \sum_{k=1}^{4} C_{k0} m_k (\theta_j) \int\limits_{-1}^{1} Q_{\theta k} (\xi) (1 - Rch\varepsilon) d\xi, \\ Q_{j0} = r_0^3 \sin\theta_j \sum_{k=1}^{4} C_{k0} \tfrac{dm_k(\theta_j)}{d\theta} \int\limits_{-1}^{1} T_k (\xi) d\xi, \quad (j = 1,\ 2). \end{gathered}$$

Thus, the constants C_{k0} are determined through the principal parts of the bending forces on the lateral surface of the layer.

The solution (4.2), (4.3) determines the internal stress-strain state of the shell. The first terms of their asymptotic expansions with respect to the thin-walled parameter ε determine the momentless stress state. In the first term, the asymptotics can be considered as a solution in the applied theory of shells.

The stress state corresponding to solutions (4.5), (4.6), (4.7) has the character of a boundary layer. The first terms of its asymptotic form are completely equivalent to the Saint-Venant edge effect of a transversally isotropic plate [8], [9].

5 Generalized conditions for the orthogonality of boundary conditions on the lateral surface of a sphere

As is well known, the Schiff-Papkovich-orthogonality relations have played an important role in the development of methods for solving the basic boundary-value problems of the theory of elasticity.

It is shown in [13] that these relationships always hold in the problems of elasticity theory, since they are a corollary of the general theorem of reciprocity of Betty's papers. Below, these relationships are established for a transversely isotropic spherical layer which allow one to solve exactly an elasticity problem for a transversely isotropic spherical layer under mixed boundary conditions on the face surfaces.

$$\sigma_r = 0,\ \tau_{r\theta} = 0 \quad \text{as} \ \ r = R_n \ \ (n = 1,\ 2), \tag{5.1}$$

$$u_r = 0,\ u_\theta = 0 \quad \text{as} \ \ r = R_n \ \ (n = 1,\ 2) \tag{5.2}$$

$$u_r = 0,\ \tau_{r\theta} = 0 \quad \text{as} \ \ r = R_n \ \ (n = 1,\ 2) \tag{5.3}$$

$$\sigma_{1k}\left(r\right) = 0. \tag{5.4}$$

According to (2.15), homogeneous solutions have the form:

$$\begin{gathered} u_r = u_k\left(r\right) m_k\left(\theta\right), \\ u_\theta = v_k\left(r\right) \frac{dm_k\left(\theta\right)}{d\theta}, \\ \sigma_\theta = G_1\left[\sigma_{1k}\left(r\right) m_k\left(\theta\right) - \sigma_{2k}\left(r\right)\frac{dm_k\left(\theta\right)}{d\theta}\right], \\ \tau_{r\theta} = G_1 T_k\left(R\right)\frac{dm_k\left(\theta\right)}{d\theta}, \end{gathered} \tag{5.5}$$

where

$$\sigma_{1k} = b_{12}u_k^{'}\left(r\right) + \frac{1}{r}\left(b_{22}+b_{23}\right)u_k\left(r\right) - \frac{b_{22}}{r}\left(z_k^2 - \frac{1}{4}\right)v_k\left(r\right),$$

$$\sigma_{2k} = 2G_0\, v_k\left(r\right), \quad T_k\left(r\right) = \frac{1}{r}u_k\left(r\right) + v_k'\left(r\right) - \frac{1}{r}v_k\left(r\right).$$

Let $u_r^i, u_\theta^i, \sigma_\theta^i, \tau_{r\theta}^i$ $(i = 1, 2)$ be the displacements and stresses of the first and second states. Then by Betty's theorem, for any θ, the equality holds true:

$$\int\limits_{R_1}^{R_2}\left(u_\theta^1\sigma_\theta^2 + u_r^1\tau_{r\theta}^2\right) r\sin\theta dr = \int\limits_{R_1}^{R_2}\left(u_\theta^2\sigma_\theta^1 + u_r^2\tau_{r\theta}^1\right) r\sin\theta dr. \tag{5.6}$$

Substituting (5.5) into (5.6), we obtain

$$\begin{gathered} \cos\theta \tfrac{dm_k(\theta)}{d\theta}\tfrac{dm_n(\theta)}{d\theta}\int\limits_{R_1}^{R_2}\left(\sigma_{2k}v_n - \sigma_{2n}v_k\right) rdr + \\ \sin\theta\, m_n\left(\theta\right)\tfrac{dm_k(\theta)}{d\theta}\int\limits_{R_1}^{R_2}\left(\sigma_{1n}v_k - u_nT_k\right) rdr + \\ +\sin\theta\, m_k\left(\theta\right)\tfrac{dm_n(\theta)}{d\theta}\int\limits_{R_1}^{R_2}\left(u_kT_n - \sigma_{1k}v_n\right) rdr = 0. \end{gathered} \tag{5.7}$$

Since (5.7) holds true for any θ, it is achieved only under the condition

$$\int\limits_{R_1}^{R_2} (\sigma_{2k} v_n - \sigma_{2n} v_k)\, r dr = 0, \tag{5.8}$$

$$\int\limits_{R_1}^{R_2} (\sigma_{1n} v_k - u_n T_k)\, r dr = 0, \tag{5.9}$$

$$\int\limits_{R_1}^{R_2} (u_k T_n - \sigma_{1k} v_n)\, r dr = 0. \tag{5.10}$$

Relationship (5.8) is identically satisfied, conditions (5.9), (5.10) are equivalent. Thus, we arrive at the equality

$$\int\limits_{R_1}^{R_2} (u_k T_n - \sigma_{1k} v_n)\, r dr = 0, \quad k \neq n. \tag{5.11}$$

From (5.11) we obtain the following orthogonality condition for the functions $u_n(r), v_n(r)$

$$\begin{aligned} &\int\limits_{R_1}^{R_2} \left\{ \left(\tfrac{1}{r} u_n + v_n' - \tfrac{1}{r} v_n\right) u_k + v_n \left[b_{12} u_k' + \tfrac{1}{r}(b_{22} + b_{23}) u_k - \right.\right. \\ &\left.\left. - \tfrac{b_{22}}{r}\left(z_k^2 - \tfrac{1}{4}\right) v_k \right] \right\} r dr = 0, \;\; k \neq n. \end{aligned} \tag{5.12}$$

However, when boundary conditions are satisfied on the lateral surface of the shell, generalized orthogonality conditions for inhomogeneous solutions, as shown in [13], do not completely solve the question of exact satisfaction of the boundary conditions on the lateral surface of the sphere. Apparently, in the general case, besides the reduction to infinite systems of linear equations, nothing can be proposed here. Nevertheless, under special conditions of support of the shell edge, generalized orthogonality conditions for homogeneous solutions make it possible to represent the solution in the form of a series whose coefficients are determined exactly. In addition, condition (5.11) can be useful in solving infinite systems of equations, since it allows to always satisfy one of the boundary conditions on the lateral surface of a spherical layer exactly.

Using the generalized orthogonality conditions, let us consider the following problem: let the face surfaces $r = R_s$ $(s = 1, 2)$ be free of stress, and on the conical surfaces $\theta = \theta_j$ $(j = 1, 2)$ the following boundary conditions be given

$$\begin{aligned} &u_\theta(\xi, \theta) = a(\xi), \;\; \tau_{r\theta} = \tau(\xi) \;\; \text{for}\, \theta = \theta_1, \\ &u_\theta(\xi, \theta) = 0, \;\; \tau_{r\theta} = 0 \;\text{for}\, \theta = \theta_1. \end{aligned} \tag{5.13}$$

We seek the solution in the form of a sum of elementary solutions

$$\begin{aligned} &u_r(\xi, \theta) = r_0 e^{\varepsilon\xi} \sum_{k=1}^{\infty} u_k(\xi)\, m_{z_k - \frac{1}{2}}(\cos\theta), \\ &u_\theta(\xi, \theta) = r_0 e^{\varepsilon\xi} \sum_{k=1}^{\infty} v_k(\xi)\, m^{(1)}_{z_k - \frac{1}{2}}(\cos\theta). \end{aligned} \tag{5.14}$$

Here $z_k,\ [u_k(\xi), v_k(\xi)]$ are the eigenvalues and the eigenvalue pair of functions of spectral problem (2.8),(2.9).

$$m_\theta^{(1)} = \frac{dm_n(\theta)}{d\theta},\ \ m^{(1)}_{z_k-\frac{1}{2}}(\theta) = A_k P_{z_k-\frac{1}{2}}(\cos\theta) + B_k Q_{z_k-\frac{1}{2}}(\cos\theta),$$

where $P_n(\cos\theta)$, $Q_n(\cos\theta)$ are Legendre functions of the first and second series, respectively A_n, B_n are unknown constants.

The summation of the series (5.14) is carried out over the roots located in the upper half-plane $(Jm\, z_k \geq 0)$. In accordance with relationships (5.14), elementary stress states can be represented in the form:

$$\begin{aligned} &\sigma_\theta^{(k)}(\xi,\theta) = \sigma_{1k}(\xi)\, m_{z_k-\frac{1}{2}}(\theta) + \sigma_{2k}(\xi)\, ctg\theta m^{(1)}_{z_k-\frac{1}{2}}(\theta), \\ &\tau_{r\theta}^{(k)}(\xi,\theta) = \sigma_{3k}(\xi)\, T^{(1)}_{z_k-\frac{1}{2}}(\theta) \end{aligned} \tag{5.15}$$

where

$$\begin{aligned} &\sigma_{1k}(\xi) = G_1\varepsilon^{-1}\left[b_{12}u_k'(\xi) + (b_{22}+b_{23})\,\varepsilon u_k(\xi) - b_{22}\varepsilon\left(z_k^2 - \tfrac{1}{4}\right) v_k(\xi)\right], \\ &\sigma_{2k}(\xi) = -2G_0\varepsilon^{-1}v_k(\xi), \quad \sigma_{3k}(\xi) = G_1\varepsilon^{-1}\left[v_k'(\xi) + \varepsilon u_k(\xi)\right]. \end{aligned}$$

Then, satisfying boundary conditions (5.13), using generalized orthogonality conditions (5.11), we obtain the following algebraic equations for A_k, B_k

$$\begin{aligned} &A_k P^{(1)}_{z_k-\frac{1}{2}}(\cos\theta_1) + B_k P'^{(1)}_{z_k-\frac{1}{2}}(\cos\theta_1) = H_k, \\ &A_k P^{(1)}_{z_k-\frac{1}{2}}(\cos\theta_2) + B_k Q^{(1)}_{z_k-\frac{1}{2}}(\cos\theta_2) = 0, \end{aligned} \tag{5.16}$$

where

$$\begin{aligned} &H_k(\xi) = \Delta_H^{-1}\int_{-1}^{1} e^{2\varepsilon\xi}\left\{a(\xi)\, G_1\left[b_{12}u_k'(\xi) + (b_{22}+b_{23})\,\varepsilon u_k(\xi) - \right.\right. \\ &\left.\left. -b_{22}\varepsilon\left(z_k^2 - \tfrac{1}{4}\right) v_k(\xi)\right] - \varepsilon e^{\varepsilon\xi}\tau_k(\xi)\, u_k(\xi)\right\} d\xi, \\ &\Delta_H = \int_{1}^{1} G_1 e^{3\varepsilon\xi}\left\{\left[b_{12}u_k'(\xi) + (b_{22}+b_{23})\,\varepsilon u_k(\xi) - \right.\right. \\ &\left.\left. -b_{22}\varepsilon\left(z_k^2 - \tfrac{1}{4}\right) v_k(\xi)\right] - \varepsilon u_k(\xi)\, v_k'(\xi) - \varepsilon u_k^2(\xi)\right\} d\xi. \end{aligned}$$

From linear systems (5.16), constants A_k and B_k are defined for any

$$A_k = H_k\Delta^{-1}Q^{(1)}_{z_k-\frac{1}{2}}(\cos\theta_2),\ \ B_k = -H_k\Delta^{-1}P^{(1)}_{z_k-\frac{1}{2}}(\cos\theta_2), \tag{5.17}$$

where

$$\Delta = P^{(1)}_{z_k-\frac{1}{2}}(\cos\theta_1)\, Q^{(1)}_{z_k-\frac{1}{2}}(\cos\theta_2) - P^{(1)}_{z_k-\frac{1}{2}}(\cos\theta_2)\, Q^{(1)}_{z_k-\frac{1}{2}}(\cos\theta_1).$$

The summation of the necessary number of elementary solutions in the increasing order $(Jm\, z_k > 0)$ makes it possible to find the characteristics of a stress-strain state with a given accuracy.

As noted above, in the general case the generalized orthogonality conditions do not allow to accurately satisfy the boundary conditions on the lateral surface of the spherical layer. Here, to satisfy the boundary conditions on the lateral surface, it is convenient to use Lagrange's variational principle. According to this principle, one can write

$$\delta R = \int_V (F_x\delta u + F_y\delta v + F_z\delta w)\, dV dz + \int_S (F_x\delta u + F_y\delta v + F_z\delta w)\, dS = 0, \tag{5.18}$$

where F_x, F_y, F_z are the components of the mass forces, f_x, f_y, f_z are the components of the surface forces, u, v, w are the components of the displacement vector.

Suppose that for $\theta = \theta_j \ \ (j = 1,\, 2)$ the following boundary conditions are given:

$$\sigma_\theta = Q_j\left(r\right), \ \tau_{r\theta} = \tau_j\left(r\right), \tag{5.19}$$

where $Q_j\left(r\right), \ \tau_{r\theta} = \tau_j\left(r\right)$ are sufficiently smooth functions and satisfy the equilibrium conditions

$$\begin{gathered} 2\pi \sin\theta_1 \int\limits_{-1}^{1} \left(\tau_1 \cos\theta_1 - Q_1 \sin\theta_1\right) r dr = \\ = 2\pi \sin\theta_2 \int\limits_{-1}^{1} \left(\tau_2 \cos\theta_2 - Q_2 \sin\theta_2\right) r dr. \end{gathered} \tag{5.20}$$

Since homogeneous solutions satisfy the equation of equilibrium and the boundary conditions on the face surface, the variational principle takes the following form

$$\int\limits_{R_1}^{R_2} \left[\left(\sigma_\theta - Q_j\right) \delta u_\theta + \left(\tau_{r\theta} - \tau_j\right) \delta u\right]_{\theta=\theta_j} r dr = 0. \tag{5.21}$$

Substituting (5.15) into (5.21) for the definition of C_k, whose variations are assumed to be independent, we obtain the following infinite system

$$\sum_{k=1}^{\infty} H_{nk}^{j} C_k = N_n^j, \ \ (n = 1,\, 2, ...). \tag{5.22}$$

Here

$$\begin{gathered} H_{nk}^{j} = \left[2m_k\left(\theta\right) \tfrac{dm_n(\theta)}{d\theta} \int\limits_{R_1}^{R_2} \sigma_{1k}\left(r\right) v_n\left(r\right) r dr - 2ctg\theta \tfrac{dm_k(\theta)}{d\theta} \tfrac{dm_n(\theta)}{d\theta} \times \right. \\ \left. \times \int\limits_{R_1}^{R_1} \sigma_{2k}\left(r\right) v_n\left(r\right) r dr + m_n\left(\theta\right) \tfrac{dm_k(\theta)}{d\theta} \int\limits_{R_1}^{R_2} T_k\left(r\right) u_n\left(r\right) r dr \right]_{\theta=\theta_j}, \end{gathered} \tag{5.23}$$

$$N_n^j = \left[\frac{dm_n\left(\theta\right)}{d\theta} \int\limits_{R_1}^{R_2} Q_j\left(r\right) v_n\left(r\right) r dr + m_n\left(\theta\right) \int\limits_{R_1}^{R_2} T_j\left(r\right) u_n\left(r\right) r dr\right]_{\theta=\theta_j}.$$

Using the smallness of the shell thin-wall parameter ε, we can construct an asymptotic solution of the system (5.22). To do this, we must substitute the asymptotic expressions for $u_r, u_\theta, \sigma_\theta, \tau_{r\theta}$, corresponding to different groups of zeros of the characteristic equation, into formulas (2.13).

As was noted above, the non-self-balanced part of stresses (5.19) can be removed with the help of the penetrating solution (4.2), and the connection between the constant A and the principal vector P is given by equality $P = 2\pi r_0^2 G_1 \varepsilon A$. The constants $C_k \ (k = 1, 2, 3, 4)$ are determined through the principal parts of the bending moment and the shear forces on the lateral surface of the layer. Therefore, we will assume below that $A = 0, \ C_k = 0$.

For simplicity, we suppose that the middle surface is a sphere with one circular hole.

Further, using (4.5), (4.6), (4.7) the unknowns B_n, D_n, E_n will be sought in the form

$B_n = B_{n0}+\varepsilon B_{n1}+...,\quad D_n = D_{n0}+\varepsilon D_{n1}+...,\quad E_n = E_{n0}+\varepsilon E_{n1}+...,\quad (n = 1, 3, 5, ...)$

Based on the variational principle, we obtain the following system of equations with respect to B_{n0}, D_{n0}, E_{n0}:

$$\begin{gathered}\sum_{n=1,3,...}^{\infty} \prod_{nk} B_{n0} = H_k, \quad (k = 1, 3, ...)\,, \\ \sum_{n=1,3,...}^{\infty} g_{nk} D_{n0} = h_k,\ (k = 1, 3, ...)\,, \\ \sum_{n=1,3,...}^{\infty} l_{nk} E_{n0} = F_k, \quad (k = 1, 3, ...)\,, \end{gathered} \tag{5.24}$$

where

$$\begin{gathered} \varPi_{nk} = G_1 \left(b_{12}s_1^2 + b_{22}\right)\left(b_{12}s_2^2 + b_{22}\right)\exp\left(\frac{\delta_n + \delta_k}{\varepsilon}\theta_1\right)\times \\ \times\int\limits_{-1}^{1}\left\{-\left(b_{12}+1\right)\delta_n^2\left[s_1 chs_2\delta_n chs_1\delta_n\xi - s_2 chs_1\delta_n chs_2\delta_n\xi\right]\times\right. \\ \times\left[s_1\left(b_{12}s_2^2+b_{22}\right)chs_2\delta_k shs_1\xi - s_2\left(b_{12}s_1^2+b_{22}\right)chs_1\delta_k shs_2\xi\right] + \\ +\delta_n\delta_k\left[chs_2\delta_n chs_1\delta_n\xi - chs_1\delta_n chs_2\delta_n\xi\right]\left[\left(s_1^2-b_{22}\right)\left(b_{12}s_2^2+b_{22}\right)\times\right. \\ \left.\times chs_2\delta_k chs_1\delta_k\xi - \left(s_2^2-b_{22}\right)\left(b_{12}s_1^2+b_{22}\right)chs_1\delta_k chs_2\delta_k\xi\right]d\xi, \end{gathered} \tag{5.25}$$

$$\begin{gathered} H_k = \exp\left(\frac{\delta_k}{\varepsilon}\theta_1\right)\int\limits_{-1}^{1}\left\{\tau_1(\xi)\left(s_1^2-b_{22}\right)\left(b_{12}s_2^2+b_{22}\right)chs_2\delta_k chs_1\delta_k\xi - \right. \\ -\left(s_2^2-b_{22}\right)\left(b_{12}s_1^2+b_{22}\right)chs_1\delta_k chs_2\delta_k\xi\left]\delta_k - \left(b_{12}+1\right)\sigma_1(\xi)\times\right. \\ \times\left[s_1\left(b_{12}s_2^2+b_{22}\right)chs_2\delta_k shs_1\delta_k\xi - s_2\left(b_{12}s_1^2+b_{22}\right)chs_1\delta_k shs_1\delta_k\xi\right]\left.\right\}d\xi, \end{gathered}$$

$$\begin{gathered} g_{nk} = \delta_n\left(b_{12}p^2+b_{22}\right)^2\exp\left(\frac{\delta_n+\delta_k}{\varepsilon}\theta_1\right)\int_{-1}^{1}\langle\delta_n\left(b_{12}+1\right)\left[\left(chp\delta_n - p\delta_n shp\delta_n\right)\times\right. \\ \times shp\delta_n\xi + p\delta_n\xi chp\delta_n chp\delta_n\xi\left]\left\{\left[\left(b_{12}p^2-b_{22}\right)chp\delta_k + p\delta_k\left(b_{12}p^2+b_{22}\right)shp\delta_k\right]\times\right.\right. \\ \times shp\delta_k - p\delta_k\xi\left(b_{12}p^2+b_{22}\right)chp\delta_k chp\delta_k\xi\right\} + \delta_k\left[shp\delta_n chp\delta_n\xi - \xi chp\delta_n shp\delta_n\xi\right]\times \\ \times\left\{\left[\frac{2b_{22}\left(b_{12}+1\right)}{p^2-b_{22}}chp\delta_k - \delta_k\left(b_{12}p^2+b_{22}\right)shp\delta_k\right] + \delta_k\xi\left(b_{12}p^2+b_{22}\right)chp\delta_k chp\delta_k\xi\right\}\times \\ \times\left(p^2-b_{22}\right)\rangle\,d\xi, \end{gathered}$$

$$\begin{gathered} h_k = \exp\left(\frac{\delta_k}{\varepsilon}\theta_1\right)\int_{-1}^{1}\langle\left(b_{12}+1\right)\sigma(\xi)\left\{\left[\left(b_{12}p^2-b_{22}\right)chp\delta_k + \right.\right. \\ \left.+p\delta_k\left(b_{12}p^2+b_{22}\right)shp\delta_k\right]shp\delta_k\xi - p\delta_k\xi\left(b_{12}p^2+b_{22}\right)chp\delta_k chp\delta_k\xi\right\} + \\ +\delta_k\left(p^2-b_{22}\right)\tau(\xi)\left\{\left[\frac{2b_{22}\left(b_{12}+1\right)}{p^2-b_{22}}chp\delta_k - \delta_k\left(b_{12}p^2+b_{22}\right)shp\delta_k\right]\times\right. \\ \left.\times chp\delta_k\xi + \delta_k\xi\left(b_{12}p^2+b_{22}\right)chp\delta_k shp\delta_k\xi\right\}\rangle\,d\xi, \end{gathered} \tag{5.26}$$

$$l_{nk} = \exp\left(\frac{\delta_n + \delta_k}{\varepsilon}\theta_1\right)\int\limits_{-1}^{1} \left[\left\langle (b_{12}+1)\,\delta_n^2 \left\{\left[\beta b_{12}a_2 - N b_{12} a_1 - \beta N b_{22}\right.\right.\right.\right. \times$$
$$\times (b_{12}+1)]\,\Delta_{2n} - \left[\beta b_{12}a_1 + N b_{12}a_2 + \beta^2 b_{22}(b_{12}+1)\right]\Delta_{1n}\right\} -$$
$$- sh\beta\delta_n\xi chN\delta_n\xi + [N b_{12}a_1 - \beta b_{12}a_2 + \beta N b_{22}(b_{12}+1)]\,\Delta_{1n} -$$
$$- [N b_{12}a_2 + \beta b_{12}a_1 + \beta b_{22}(b_{12}+1)]\,\Delta_{2n}\}\cos\beta\delta_n\xi shN\delta_n\xi\rangle \times$$
$$\times [(a_1\Delta_{1k} - a_2\Delta_{2k})\,ch\beta\delta_k\xi\cos N\delta_n\xi - (a_1\Delta_{2k} + a_2\Delta_{1k}) \times \tag{5.27}$$
$$\times sh\beta\delta_k\xi\sin N\delta_k\xi] + (b_{12}+1)\left(N_2^2+\beta^2\right)\delta_n\left[b_{12}^2 + 2b_{22}\left(N_2^2-\beta^2\right) + \right.$$
$$\left.+ b_{22}^2\left(N^2+\beta^2\right)^2\right](\cos\beta\delta_n\xi shN\delta_n\xi\,\sin\beta\delta_n\xi chN\delta_n\xi -$$
$$- \sin\beta\delta_n chN\delta_n\cos\beta\delta_n\xi shN\delta_n\xi)\left[(\beta\Delta_{2k} - N\Delta_{1k})\,ch\beta\delta_k\xi \times\right.$$
$$\times shN\delta_k\xi\,(N\Delta_{2k} + \beta\Delta_{1k})\,sh\beta\delta_k\xi\cos N\delta_n\xi]\,d\xi,$$

$$F_k = \exp\left(\frac{\delta_k}{\varepsilon}\theta_1\right)\int_{-1}^{1}\{(b_{12}+1)\,\sigma_k(\xi)\,[(\beta\Delta_{2k} - N\Delta_{1k})\,ch\beta\delta_k\xi \times$$
$$\times shN\delta_k\xi\,(N\Delta_{2k} + \beta\Delta_{1k})\,sh\beta\delta_k\xi\cos N\delta_k\xi + \tau(\xi) \times$$
$$\times [(a_1\Delta_{1k} - a_2\Delta_{2k})\,ch\beta\delta_k\xi\cos N\delta_n\xi - (a_1\Delta_{2k} + a_2\Delta_{1k}) \times$$
$$\times sh\beta\delta_k\xi\sin N\delta_k\xi]\}\,d\xi.$$

For $n,k = 2,4,...$, the corresponding expressions $\prod_{nk}$,g_{nk},l_{nk},H_k,h_k,F_k are obtained from (5.25),(5.26), (5.27) by replacing chx with shx and shx with $-chx$, $\cos x$ with $\sin x$ and $\sin x$ with $-\cos x$, respectively. The definition of $B_{ni},\ D_{ni}, E_{ni}\ \ (i = 1,2,...)$ is invariably reduced to the inversion of the same matrices that coincide with the matrices (5.24). In turn, the elements of these matrices do not depend on the type of the load on the tapered sections of the layer, and, therefore, the inversion can be made once and for all. It is pertinent to note that similar systems (5.24) have already been encountered in the theory of thick plates and on their basis a numerical analysis of various problems has already been carried out repeatedly. The solvability and convergence of the reduction method for these systems was proved in [15].

The general solution of the problem of determining the strained and deformed state of the shell can be obtained by a superposition of solutions corresponding to different groups of roots. For simplicity, we shall consider only those cases when the roots of quadratic equation (3.7) are real and $s_1 = s_2$:

$$u_r = r_0\left\langle\left(\cos\theta\ln ctg\frac{\theta}{2} - 1\right)A + \sum_{k=1}^{4} C_{k0}u_k(\xi)\exp\left(\frac{a_k}{\sqrt{\varepsilon}}\theta\right) + \right.$$
$$+ \sum_{n=1,3,...}^{\infty} B_{n0}\delta_n\left[\left(s_1^2 - b_{22}\right)\left(b_{12}s_2^2 + b_{22}\right)chs_2\delta_n chs_1\delta_n\xi - \right.$$
$$\left.\left. - \left(s_2^2 - b_{22}\right)\left(b_{12}s_1^2 + b_{22}\right)chs_1\delta_n chs_2\delta_n\xi\right]\exp\left(\frac{\delta_n}{\varepsilon}\theta\right)\right\rangle,$$

$$u_\theta = r_0 \left\langle - \left(\sin\theta \ln ctg\frac{\theta}{2} + ctg\theta \right) A + \varepsilon^{-\frac{1}{2}} \sum_{k=1}^{4} C_{k0} v_k(\xi) \exp\left(\frac{a_k}{\sqrt{\varepsilon}}\theta \right) - \right.$$

$$- (b_{12}+1) \sum_{n=1,3,\ldots}^{\infty} B_{n0} \left[s_1 \left(b_{12}s_2^2 + b_{22} \right) chs_2\delta_n chs_1\delta_n\xi - \right. \tag{5.28}$$

$$\left. \left. -s_2 \left(b_{12}s_1^2 + b_{22} \right) chs_1\delta_n chs_2\delta_n\xi \right] \exp\left(\frac{\delta_n}{\varepsilon}\theta \right) \right\rangle,$$

$$\sigma_r = \varepsilon^{-1} G_1 \sum_{n=1,3,\ldots}^{\infty} B_{n0}\delta_n^2 \left[s_1 \left(b_{12}s_1^2 + b_{22} \right) \left(b_{12}^2 s_1^2 + b_{12}^2 + b_{12} - b_{11}b_{22} \right) \times \right.$$

$$\times chs_2\delta_n chs_1\delta_n\xi - s_2 \left(b_{12}s_1^2 + b_{22} \right) \left(b_{12}^2 s_2^2 + b_{12}^2 + b_{12} - b_{11}b_{22} \right) chs_1 \times$$

$$\left. \times \delta_n chs_2\delta_n\xi \right] \exp\left(\frac{\delta_n}{\varepsilon}\theta \right),$$

$$\sigma_\theta = \varepsilon^{-1} G_1 \left\langle \frac{2\pi G_1 P}{\sin^2\theta} + \sqrt{\varepsilon} \sum_{k=1}^{4} C_{k0} Q_{\theta k}(\xi) \exp\left(\frac{a_k}{\sqrt{\varepsilon}}\theta \right) + \left(b_{12}s_1^2 + b_{22} \right) \times \right.$$

$$\left. \times \left(b_{12}s_2^2 + b_{22} \right) \varepsilon^{-1} \sum_{n=1,3,\ldots}^{\infty} B_{n0}\delta_n^2 \left[s_1 chs_2\delta_n shs_1\delta_n\xi - s_2 chs_1\delta_n shs_2\ \delta_n\xi \right] \exp\left(\frac{\delta_n}{\varepsilon}\theta \right) \right\rangle,$$

$$\sigma_\varphi = \varepsilon^{-1} G_1 \left\langle \frac{2\pi G_1 P}{\sin^2\theta} + \sqrt{\varepsilon} \sum_{k=1}^{4} C_{k0} Q_{\varphi k}(\xi) \exp\left(\frac{a_k}{\sqrt{\varepsilon}}\theta \right) + \right.$$

$$+ \sum_{n=1,3,\ldots}^{\infty} B_{n0}\delta_n^2 \left[s_1 \left(b_{12}s_2^2 + b_{22} \right) \left(b_{12}s_1^2 - 2b_{12}G_0 + b_{23} \right) chs_2\delta_n shs_1\delta_n\xi - \right. \tag{5.29}$$

$$\left. \left. - s_2 \left(b_{12}s_1^2 + b_{22} \right) \left(b_{12}s_2^2 - 2b_{12}G_0 + b_{23} \right) chs_1\delta_n shs_2\ \delta_n\xi \right] \exp\left(\frac{\delta_n}{\varepsilon}\theta \right) \right\rangle,$$

$$\tau_{r\theta} = \varepsilon^{-1} G_1 \left(b_{12}s_1^2 + b_{22} \right) \left(b_{12}s_1^2 + b_{22} \right) \sum_{n=1,3,\ldots}^{\infty} B_{n0}\delta_n^2 \left[chs_2\delta_n chs_1\delta_n\xi - \right.$$

$$\left. -chs_1\delta_n chs_2\delta_n\xi \right] \exp\left(\frac{\delta_n}{\varepsilon}\theta \right).$$

In formulas (5.28), (5.29), the first and second terms of the right-hand sides correspond to the applied theory of shells, the following are additions to solutions in applied theory. Near the boundary $\theta_j = const$, the additional terms in the stresses $\sigma_\theta, \sigma_\varphi$ are of the same order as in the applied theory of shells, while the additional terms in the stresses $\sigma_r, \tau_{r\theta}$ for $\varepsilon \to 0$ begin to play the main role.

6 Conclusion

Thus, the above analysis shows that the stress-strain state of an anisotropic spherical shell consists of three types: an internal stress state, a simple edge effect and a boundary layer.

Constructed homogeneous solutions not only reveal the qualitative structure of the three-dimensional solution of an anisotropic spherical shell but they can serve as an effective apparatus for solving specific problems, as well as a basis for the evaluation of simplified theories and the construction of refined applied theories.

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