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Variational methods of stability analysis of elastic, thin, multilayered, inhomogeneous, anisotropic plates of variables thickness under temperature and strength load

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Abstract. *The article presents the study of transform theories of variational problems and revealed that the most effective tool for determining deflections and solving problems of stability of elastic thin multilayer inhomogeneous anisotropic plates of variable thickness is use of their methods. The accurate methods to solve the bending problem can be applied only in some special cases, when the plate is of simple configuration and constant thickness. This problem is also limited by boundary conditions, and can be solved only under certain types of such conditions.*
The article deals with variational methods of stability analysis of elastic, thin, multilayered, inhomogeneous, anisotropic plates of variables thickness under temperature and strength load. It was proposed the mathematical description of the increments of total free energy of elastic thin multilayered inhomogeneous anisotropic plate of variable thickness when transferring to a close, bending state under temperature and strength load with the use of different variational functionals.

Keywords. stability · oscillation · bending · elastic thin multilayered inhomogeneous anisotropic plate · variable thickness · variational method · functional

Mathematics Subject Classification (2010): 74K20, 35A15

1 Introduction

Most modern structures and buildings are consisting of elements that can be classified as plates. These plates can be of different complex geometric shape with or without cutouts, and they can be affected not only by force, but also by temperature factors. From the mechanical point of view inhomogeneous anisotropic multilayered plates with variable thermos-elastic and elastic characteristics of material, which are subjected to thermal and strength loading can be a possible structural model for such structural elements.

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The important point in the design of the entire system is the forecasting of stability of strength thin-walled structural elements (plates).

In many cases the variational (energy) methods are the most effective in solving the problems of plate stability research.

The variational method involves solving mathematical problems using the minimization of a certain functional. A test function with sufficient criteria of assumptions and depending on a small number of parameters can also be applied here.

For the first time the stability problems of isotropic plates solved by the variational formulation method were presented by Brian. In the following different variational formulations to study the stability of the plates were proposed by other researchers: N.A. Alfutov, L.I. Balabuhov, S.P. Timoshenko and others. They did not require preliminary investigations and solutions of the plane problem of elasticity research in contrast to Brian's method. Nevertheless, all the proposed formulations do not clearly define the necessity of fulfilling the preconditions, and they are only valid for the study of the stability of single-layer plates [1], [5].

The main goal of this work is to study and build a generalized system of variational methods for the study of stability of an elastic thin layered inhomogeneous anisotropic plate of variable thickness with temperature strength loading.

2 Variational methods of stability research

Variational methods aim to carry out the calculation program for all possible functions by the use of comparison functions that contain one or more parameters, called variational parameters or functionals which values should be defined.

The specific form adopted is dictated by some a prior ideas about the nature of the exact decision [2], [4].

The increment of the total free energy of the elastic thin multilayer inhomogeneous anisotropic plate of variable thickness when transferring to a close, bending with the proximity parameter $\propto$ from two-dimensional state can be determined by the following functional [1]:

$$\varPi - \varPi^0 \cong \propto^2 \left[\frac{1}{2} \int\limits_{\varOmega} D_{ijkl} u_{3,ij} u_{3,kl} d\varOmega + \right.$$

$$\left. + \int\limits_{V} \sigma^0_{ij} \varepsilon''_{ij} dV - \int\limits_{\varOmega} h u_i F^0_i d\varOmega - \int\limits_{S_\tau} h u_i P^0_i ds \right], \tag{2.1}$$

where: $\propto$ - is a proximity parameter of bending and two-dimensional state;

$\varOmega$, S_τ, V – inseparable contour limiting a certain plane;

ε''_{ij} - a function, that determines additional deformations of the median plane (x_3=0) of the plate, which are appeared by the loss of stability;

F^0_i – body force;

σ_{ij} – strain;

σ^0_{ij} – pre-critical strain;

index "0"– designation of the generalized plane state;

$D_{ijkl} u_{3,ij}$– tensor of bending stiffness of a plane;

u_3 – function of deflections;

$u_{3,ij}$ – the coefficient that is determined by the geometric boundary conditions imposed on the deflection function.

3 Brian's functionality

According to [1] the increment of total free energy when transferring to bending state for elastic thin multilayered inhomogeneous anisotropic plate of variable thickness, provided, that the plane stress condition of a plate is equilibrial and is written as:

$$J\left[u_3\right]=\left[\frac{1}{2}\int\limits_{\Omega} D_{ijkl}u_{3,ij}u_{3,kl}d\Omega+\int\limits_{\Omega} T^0_{ij}u_{3,i}u_{3,j}d\Omega\right], \tag{3.1}$$

According to the variational equation (3.1) if $\delta J=0$ we have the following equations:

1 stability equation:

$$\left(D_{ijkl}u_{3,kl}\right)_{,ij}-T^0_{ij}u_{3,ij}+hF^0_iu_{3,i}=0, \tag{3.2}$$

$$x_i\in\Omega,$$

2 equations of static boundary conditions:

$$M_{ij,j}n_i+T^0_{ij}u_{3,i}n_i+M_{ij,k}t_in_jt_k=0, u_3\neq 0, \tag{3.3}$$

$$M_{ij}n_in_j=0, u_{3,n}\neq 0, \tag{3.4}$$

$$\left(M_{ij}t_in_j\right)\Big|^+_-=0, u_3\neq 0, \tag{3.5}$$

where: n_i – is directional cosines of outer normal unit vector to the area boundary.

Equation (3.5) is used for the corner points.

Therefore, the problem of finding of extrema of Brian's function with the additional requirement of a minimum external load is equivalent to the variational problem of stability and it is the energy stability criterion of the plates in the form of Brian.

4 Tymoshenko's functional

We can assume that generalized plane stressed condition is equilibrial and the following equations are made:

$$T''_{ij,j}=0, x_i\in\Omega, \tag{4.1}$$

$$M_{ij,j}+T^0_{ij}u_{3,ij}-hF^0_iu_{3,i}=0, x_i\in\Omega, \tag{4.2}$$

$$T''_{ij}n_j=0, u_i\neq 0, \tag{4.3}$$

Then, the increments of total free energy of elastic thin multiconnected inhomogeneous anisotropic plate of variable thickness will be as follows [6]:

$$J_i\left[u_3\right]=W\left[u_3\right]-\frac{1}{2}\int\limits_{\Omega} h\beta_{ij}\theta^0u_{3,i}u_{3,j}d\Omega+$$

$$+\int\limits_{\Omega}\left[hF^0_i-\left(h\beta_{ij}\theta^0\right)_{,j}\right]u_id\Omega-\int\limits_{S_e}\left(hu_iP^0_i-h\beta_{ij}\theta^0u_in_j\right)ds+$$

$$+\int\limits_{S_k} T''_{ij}n_j\overline{u}^0_ids, \tag{4.4}$$

where: θ^0 – temperature;

$\theta^0 = \theta^0(x_1, x_2)$ – given temperature field in a plate;

β_{ij} – symmetric tensor of elastic constants.

Independence from the pre-critical stress value is a characteristic feature of this functional..

Variational equation $\delta J = 0$ is equivalent to $\delta J_1 = 0$.

This functional takes a classic view under the following conditions [3]:

1 temperature changes equals zero $\theta^0 = 0$;
2 volume force equals zero $F_i^0 = 0$;
3 boundary conditions $S_k = 0$ occur at the entire boundary of the area Ω.

5 Generalized functional

Rewrite the expression (2.1) excluding $\propto^2$:

$$J[u_1, u_2, u_3] = \frac{1}{2}\int\limits_{\Omega} D_{ijkl} u_{3,ij} u_{3,kl} d\Omega + \tag{5.1}$$

$$+\int\limits_{V} \sigma_{ij}^0 \varepsilon_{ij}^{''} dV - \int\limits_{\Omega} h u_i F_i^0 d\Omega - \int\limits_{S_\tau} h u_i P_i^0 ds,$$

Determine the extrema of this functional, taking into account the following kinematic boundary conditions:

$$u_i^0|_{S_k} = \overline{u}_i^0, \;\; u_i|_{S_k} = 0, \;\; i = 1, 2; \; x_i \in S_k. \tag{5.2}$$

$$u_3|_K = u_3^*, \;\; u_{3,n}|_K = u_{3,n}^*, \;\; K \text{ and } K' \in S \tag{5.3}$$

and the following ratio:

1 the precritical stress-strain state from the linear elastic theory:

$$\overline{u}^0 = u_i^0(x_1, x_2)\overline{e}_i, \tag{5.4}$$

$$\varepsilon_{i,j}^0 = \frac{1}{2}(u_{i,j}^0 + u_{j,i}^0), \tag{5.5}$$

$$\sigma_{ij}^0 = a_{ijkl}\sigma_{kl}^0 - \beta_{ij}\theta^0, \tag{5.6}$$

$$\varepsilon_{ij}^0 = b_{ijkl}\sigma_{kl}^0 + \vartheta_{ij}\theta^0, \tag{5.7}$$

$$i, j, k, l = 1, 2,$$

where: $a_{ijkl}, \;\; b_{ijkl}, \beta_{ij}, \vartheta_{ij}$ – are the symmetric tensors of elastic and thermoelastic constants:

1 stress and deformation tensors with accuracy $\propto^2$:

$$\sigma_{ij}^0 = \sigma_{ij}^0 + \propto \sigma_{ij}^{'} + \propto^2 \sigma_{ij}^{''}, \tag{5.8}$$

$$\sigma_{ij}^{'} = a_{ijkl}\varepsilon_{kl}^{'}, \tag{5.9}$$

$$\sigma_{ij}^{''} = a_{ijkl}\varepsilon_{kl}^{''}, \tag{5.10}$$

$$\varepsilon_{ij}^{0}=\varepsilon_{ij}^{0}+\propto\varepsilon_{ij}^{'}+\propto^{2}\varepsilon_{ij}^{''}, \tag{5.11}$$

$$\varepsilon_{ij}^{'}=-x_3u_{3,ij}, \tag{5.12}$$

$$\varepsilon_{ij}^{''}=\frac{u_{i,j}+u_{j,j}+u_{3,i}u_{3,j}}{2}, \tag{5.13}$$

$$i,j=1,2,$$

These conditions are natural for the following generalized functional [3]:

$$J_{ob}\left[u_1,u_2,u_3,\varepsilon_{ij}^{''},u_1^0,u_2^0,\sigma_{ij}^0,\varepsilon_{ij}^0\right]=\frac{1}{2}\int\limits_{\Omega}D_{ijkl}u_{3,ij}u_{3,kl}d\Omega+$$

$$+\frac{1}{2}\int\limits_{\Omega}\sigma_{ij}^0u_{3,i}u_{3,j}d\Omega+\int\limits_{\Omega}h\sigma_{ij}^0u_{i,j}d\Omega+\int\limits_{\Omega}ha_{ijkl}\varepsilon_{kl}^{''}u_{i,k}^{0}d\Omega-$$

$$-\int\limits_{\Omega}h\sigma_{ij}^0\varepsilon_{ij}^{''}d\Omega-\int\limits_{\Omega}h\beta_{ij}\varepsilon_{ij}^{''}d\Omega-\int\limits_{\Omega}hu_iF_i^0d\Omega-$$

$$-\int\limits_{S_k}hu_iP_i^0ds-\int\limits_{S_k}ha_{ijkl}\varepsilon_{kl}^{''}n_j\left(u_i^0-\overline{u}_i^0\right)ds-\int\limits_{S_k}h\sigma_{ij}^0n_ju_id\Omega+$$

$$+\int\limits_{K}[(D_{ijkl}u_{3,kl})_{,j}n_k+(D_{ijkl}u_{3,kl}n_lt_k)_s-h\sigma_{ij}^0n_{3,i}n_j](u_3-u_3^*)ds+$$

$$+\int\limits_{K}D_{ijkl}u_{3,kl}n_kn_i\left(u_{3,n}-u_{3,n}^*\right)ds-\sum_{y}^{k}\left(D_{ijkl}u_{3,ij}n_lt_k\right)|_{-}^{+}\left(u_3-u_3^*\right),$$

$$i,j,k,l=1,2. \tag{5.14}$$

Thus, according to the variational equation $\delta J_{ob}=0$ the kinematic boundary conditions serve as natural conditions (5.2), (5.3) and ratio (1.14 - 1.17), (1.7 -1.9).

6 Conclusions

This paper presents the study and the construction of a generalized system of variational methods for the study of stability of elastic thin layered inhomogeneous anisotropic plates of variable thickness when subjected to a temperature and strength loading. It was presented the increment of the total free energy of the elastic thin multilayer inhomogeneous anisotropic plate of variable thickness when transferring to a close, bending state in the form of Brian's and Tymoshenko's functional as well as generalized functional.

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