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Electro-mechanical energies of the PZT/Elastic/PZT sandwich circular plate with penny-shaped interface cracks under action of the normal opening forces on the cracks' edges

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Abstract. *This paper studies the electro-mechanical energies of the PZT/Elastic/PZT sandwich circular plate with interface penny-shaped cracks under action on the cracks edges opening uniformly distributed normal forces. The axisymmetric stress-strain state is considered and the investigations are made by utilizing the exact field equations and relations of electro-elasticity for piezoelectric materials. The corresponding boundary value problem is solved numerically by employing the finite element method (FEM) and aforementioned energies are calculated for various PZT materials of the face layers and for various elastic materials of the core layer. It is made attempt to indicate the influence of the coupling effect of the mechanical and electrical fields on the total electro-mechanical potential energies. Moreover, the effect of the geometrical parameters such as face layers thickness, crack's radius and etc. on the energies under consideration is also analyzed.*

Keywords. Electro-mechanical energies · piezoelectric material · penny-shaped interface crack · sandwich circular plate.

Mathematics Subject Classification (2010): 74H55

1 Introduction.

The information and knowledge on the electro-mechanical strain energies of the piezoelectric + metal layered systems with interface cracks have a great significance for estimation of the load carried capacity of these systems. Moreover, the values of the stress intensity factors (SIF) and energy release rate (ERR) at crack tips are calculated through these energies, according to which, the fracture of the piezoelectric or piezoelectric + metal layered materials are determined. It is evident that for determination of the aforementioned energies it is necessary to solve the corresponding boundary value problems for the PZT + Metal layered system with interface cracks. Note that under formulation and solution to these problems one of the main question is the construction of the electrical conditions across the crack's edges. Consider a brief review of the related investigations and the formulation of the aforementioned conditions on the crack edges. We begin this review with the paper by Kudryatsev et al. (1975) in which a special solution of the stress and displacement fields is

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obtained for the penny shaped crack embedded in a piezoelectric material. Under mathematical formulation of the problem, the so-called permeable condition on the crack edges is considered, i.e. it is assumed that the electrical potential and the normal components of the electrical displacements are continuous across the crack edge surfaces. In the papers by Parton (1976), Yang (2004) and other ones listed therein the same type of conditions on the crack's edge are also used. Analyses of the various types of conditions on the crack edges in the piezoelectric materials, which is differ from the permeable condition, are discussed in the papers by Li, McMeeking and Landis (2008) and Li, Feng and Xu(2009),

An axisymmetric penny-shaped crack problem for the infinite piezoelectric layer in the case where the crack is in the middle plane of the layer is studied in the paper by Li and Lee (2012).It is assumed that on the crack's edge surfaces the impermeable boundary conditions are satisfied, according to which, the electric displacements on the crack's edge surfaces are equal to zero. Moreover, in the crack problems for the piezoelectric materials considered in the papers by Zhong (2012), Eskandari et al. (2010) and others, the energetically consistent boundary condition, which was proposed by Landis (2004), is also used.

In all the works reviewed above the case where the penny-shaped crack is embedded completely in a piezoelectric material is considered. Therefore formulation of the permeable, impermeable, energetically consistent, semi-consistent and other types of conditions for the electrical quantities across the crack's edge surfaces, becomes necessary. However, in the cases where the penny-shaped crack is in the interface between piezoelectric and elastic mediums such conditions do not have any meaning. Therefore, in later cases on the crack's edge face which relate to the piezoelectric medium, the ordinary "electrically-open" (or "open-circuit") and "electrically-shorted" (or "short-circuit") conditions are satisfied. Note that the "electrically-open" (or "open-circuit") condition coincides with the aforementioned impermeable condition. In connection with this, we note that the first attempt to study the problem related to the interface penny-shaped crack between the piezoelectric layer and elastic half-space is made in the paper by Ren et al. (2014). This study is carried out for the opening mode in the case where on the crack face, which is in the piezoelectric layer, the "open-circuit" condition is satisfied.

This completes the consideration of the review of the investigations related to the penny-shaped crack in piezoelectric materials and carried out during the last 10 years. The review of the regarding works carried out in earlier years can be found in the papers by Kuna (2006) and Kuna (2010).

Thus, it follows from the foregoing review that all the investigations carried out for the penny-shaped cracks in piezoelectric materials and in the interface between piezoelectric and elastic materials have been made within the scope of the linear piezoelectric fracture mechanics and within the scope of the assumptions that the layers' dimensions are infinite in the plane on which this crack lies. Namely, this infinities allows to use the Hankel integral transformation method for solution to the corresponding boundary value problems.

However, in the cases where the dimensions of the layers in the planes on which the cracks are located are finite the methods based on the integral transformations, in general, is not applicable. As an example for such cases, it can be considered the sandwich PZT/Metal/PZT circular plate-disc the radius of which is commensurable with the radius of the penny-shaped crack. Namely this case is considered in the present paper and for solution to the corresponding boundary value problem the numerical method, i.e. the finite element method (FEM) is employed. Note that corresponding buckling delamination problems were considered in the papers by Cafarova et al. (2017), Akbarov et al. (2017) and Cafarova and Rzayev (2016). Moreover, note that the corresponding buckling delamination and crack problems for the plane-strain state were considered in the papers by Aklbarov and Yahnioglu (2013, 2016).

2 Formulation of the problem

We consider a circular PZT/Metal/PZT sandwich plate with geometry illustrated in Fig. 1a and assume that the thicknesses and piezoelectric materials of the face layers are the same, and the material of the middle (core) layer is an elastic one. Moreover, assume that between the core and face layers there are penny-shaped cracks whose locations are illustrated in Fig. 1b. At the same time, in Fig. 1b the geometric parameters and the external opening forces acting on the cracks edge surfaces are also indicated.

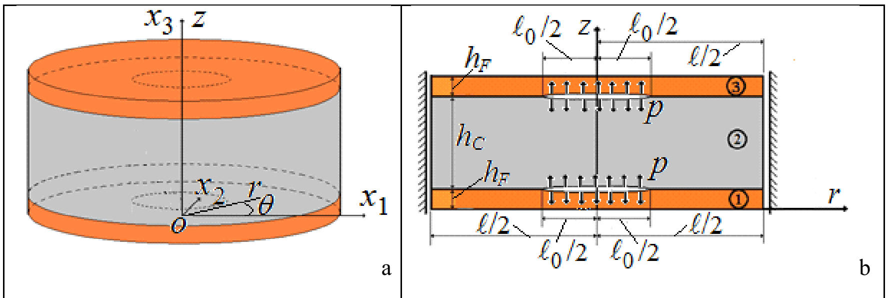

Fig. 1. The sketches of the PZT/Metal/PZT plate-disc (a), the geometry of this disc, interface cracks and external opening forces

As shown in Fig. 1a, we associate with the lower face plane of the plate the cylindrical coordinate system $Or\theta z$ (Fig. 1), according to which, the plate occupies the region $\{0 \le r \le l/2; 0 \le \theta \le 2\pi; 0 \le z \le h\}$ $(h = 2h_F + h_c)$ and the penny-shaped cracks occur in $\{z = h_F \pm 0\ ;\ 0 \le r \le l_0/2\}$ and in $\{z = h_C + h_F \pm 0\ ; 0 \le r \le l_0/2\}$.

Thus, within this framework, we suppose that on the cracks' edges the uniformly rotational symmetric distributed normal opening forces with intensity p act and it is required to determine the electro-mechanical energies accumulated in the sandwich plate caused with this mechanical forces. For this purpose, first, we consider formulation of the problems for determination of the electromechanical quantities which appear in the plate as a result of the action of the aforementioned mechanical forces.

As we are considering the rotationally axisymmetric deformation case, therefore under the mathematical formulation of the corresponding problem we will use the corresponding field equations related to this case. Moreover, below we will denote the values related to the upper and lower face layers by upper indices (2.3) and (2.1), respectively, whereas the values related to the core layer are denoted by upper index (2.2).

Assuming that the electro-mechanical state in the sandwich plate under consideration appears within the scope of the linear theory of piezoelectricity for the face layers and the linear theory of elasticity for the core layer, the corresponding field equations, according to the monograph by Yang (2005), can be written as follows.

Equilibrium and electrostatic equations:

$$\frac{\partial \sigma_{rr}^{(j)}}{\partial r} + \frac{\partial \sigma_{zr}^{(j)}}{\partial z} + \frac{1}{r}(\sigma_{rr}^{(j)} - \sigma_{\theta\theta}^{(j)}) = 0, \frac{\partial \sigma_{rz}^{(j)}}{\partial r} + \frac{\partial \sigma_{zz}^{(j)}}{\partial z} + \frac{1}{r}\sigma_{rz}^{(j)} = 0, j = 1, 2, 3,$$

$$\frac{\partial D_r^{(k)}}{\partial r} + \frac{1}{r}D_r^{(k)} + \frac{\partial D_z^{(k)}}{\partial z} = 0, k = 1, 3 \qquad (2.1)$$

The electro-mechanical constitutive relations for piezoelectric materials:

$$\sigma_{rr}^{(k)} = c_{1111}^{(k)} s_{rr}^{(k)} + c_{1122}^{(k)} s_{\theta\theta}^{(k)} + c_{1133}^{(k)} s_{zz}^{(k)} - e_{111}^{(k)} E_r^{(k)} - e_{311}^{(k)} E_z^{(k)},$$

$$\sigma_{\theta\theta}^{(k)} = c_{2211}^{(k)} s_{rr}^{(k)} + c_{2222}^{(k)} s_{\theta\theta}^{(k)} + c_{2233}^{(k)} s_{zz}^{(k)} - e_{122}^{(k)} E_r^{(k)} - e_{322}^{(k)} E_z^{(k)},$$

$$\sigma_{zz}^{(k)} = c_{3311}^{(k)} s_{rr}^{(k)} + c_{3322}^{(k)} s_{\theta\theta}^{(k)} + c_{3333}^{(k)} s_{zz}^{(k)} - e_{133}^{(k)} E_r^{(k)} - e_{333}^{(k)} E_z^{(k)},$$

$$\sigma_{rz}^{(k)} = c_{1311}^{(k)} s_{rz}^{(k)} - e_{113}^{(k)} E_r^{(k)} - e_{313}^{(k)} E_z^{(k)},$$

$$D_r^{(k)} = e_{111}^{(k)} s_{rr}^{(k)} + e_{122}^{(k)} s_{\theta\theta}^{(k)} + e_{133}^{(k)} s_{zz}^{(k)} + \varepsilon_{11}^{(k)} E_r^{(k)} + \varepsilon_{13}^{(k)} E_z^{(k)},$$

$$D_z^{(k)} = e_{311}^{(k)} s_{rr}^{(k)} + e_{322}^{(k)} s_{\theta\theta}^{(k)} + e_{333}^{(k)} s_{zz}^{(k)} + \varepsilon_{31}^{(k)} E_r^{(k)} + \varepsilon_{33}^{(k)} E_z^{(k)},$$

$$E_r^{(k)} = -\frac{\partial \phi^{(k)}}{\partial r}, E_z^{(k)} = -\frac{\partial \phi^{(k)}}{\partial z}. \tag{2.2}$$

Elasticity relations for the core layer material.

$$\sigma_{rr}^{(2)} = \lambda^{(2)} s^{(2)} + 2\mu^{(2)} s_{rr}^{(2)},$$
$$\sigma_{\theta\theta}^{(2)} = \lambda^{(2)} s^{(2)} + 2\mu^{(2)} s_{\theta\theta}^{(2)}, \sigma_{zz}^{(2)} = \lambda^{(2)} s^{(2)} + 2\mu^{(2)} s_{zz}^{(2)},$$
$$\sigma_{rz}^{(2)} = 2\mu^{(2)} s_{rz}^{(2)}, s^{(2)} = s_{rr}^{(2)} + s_{\theta\theta}^{(2)} + s_{zz}^{(2)},\ \text{k=1,3}. \tag{2.3}$$

Strain-displacement relations:

$$s_{rr}^{(j)} = \frac{\partial u_r^{(j)}}{\partial r}, s_{\theta\theta}^{(j)} = \frac{u_r^{(j)}}{r}, s_{zz}^{(j)} = \frac{\partial u_z^{(j)}}{\partial z}, s_{rz}^{(j)} = \frac{1}{2}\left(\frac{\partial u_r^{(j)}}{\partial z} + \frac{\partial u_z^{(j)}}{\partial r}\right), j = 1, 2, 3. \tag{2.4}$$

Note that in (2.1) – (2.4) the following notation is used: $\sigma_{rr}^{(j)}, \ldots, \sigma_{rz}^{(j)}$ and $s_{rr}^{(j)}, \ldots, s_{rz}^{(j)}$ are the components of the stress and strain tensors, respectively, $u_r^{(j)}$ and $u_z^{(j)}$ are the components of the displacement vector, $D_r^{(k)}$ and $D_z^{(k)}$ are the components of the electrical displacement vector, $E_r^{(k)}$ and $E_z^{(k)}$ are the components of the electrical field vector, $\phi^{(k)}$ is the electric potential, $\lambda^{(2.2)}$and $\mu^{(2.2)}$ are Lame constants of the core layer material, and $c_{ijkl}^{(k)}$, $e_{nij}^{(k)}$ and $\varepsilon_{nj}^{(k)} (k = 1, 2, 3)$ are the elastic, piezoelectric and dielectric constants, respectively.

We recall that the piezoelectric material exhibits the characteristics of orthotropic materials with the corresponding elastic symmetry axes and becomes electrically polarized under mechanical loads or mechanical deformation placed in an electrical field. According to the monograph by Yang (2005) and other related references, the polled direction of the piezoelectric material will change according to the position of the material constants in the constitutive relations in (2.2). In the present paper, under numerical calculations, it is assumed that the $O\,z$ axis direction is the polarized direction. Moreover, in general, in the theory of the piezoelectricity for simplicity the following notation is used.

$$c_{1111}^{(k)} = c_{11}^{(k)}, c_{2211}^{(k)} = c_{1122}^{(k)} = c_{12}^{(k)}, c_{3311}^{(k)} = c_{1133}^{(k)} = c_{13}^{(k)}, c_{2222}^{(k)} = c_{22}^{(k)},$$

$$c_{3322}^{(k)} = c_{2233}^{(k)} = c_{23}^{(k)}, c_{3333}^{(k)} = c_{33}^{(k)}, c_{1313}^{(k)} = c_{55}^{(k)}, e_{111}^{(k)} = e_{11}^{(k)}, e_{311}^{(k)} = e_{31}^{(k)},$$

$$e_{122}^{(k)} = e_{12}^{(k)}, e_{322}^{(k)} = e_{32}^{(k)}, e_{133}^{(k)} = e_{13}^{(k)}, e_{333}^{(k)} = e_{33}^{(k)}, e_{313}^{(k)} = e_{35}^{(k)}, e_{113}^{(k)} = e_{15}^{(k)}. \tag{2.5}$$

Thus, with equations and relations (2.1) – (2.5) the writing of the field equations completes. Now we consider mathematical formulation of the boundary conditions.

Boundary conditions on the cracks' edges:

$$\sigma_{zr}^{(3)}\Big|_{z=h_F+h_C+0}=0,\ \sigma_{zz}^{(3)}\Big|_{z=h_F+h_C+0}=-p,$$

$$\sigma_{zr}^{(2)}\Big|_{z=h_F+h_C-0}=0,\ \sigma_{zz}^{(2)}\Big|_{z=h_F+h_C-0}=-p,$$

$$\sigma_{zr}^{(2)}\Big|_{z=h_F+0}=0,\ \sigma_{zz}^{(2)}\Big|_{z=h_F+0}=-p,\ \sigma_{zr}^{(1)}\Big|_{z=h_F-0}=0,\ \sigma_{zz}^{(1)}\Big|_{z=h_F-0}=-p,$$

$$D_z^{(3)}\Big|_{z=h_F+h_C+0}=0,\ D_z^{(1)}\Big|_{z=h_F-0}=0,\quad \text{for}\, 0\le r\le l_0/2. \tag{2.6}$$

Contact conditions between the layers in the areas which are out of the cracks:

$$\sigma_{zz}^{(3)}\Big|_{z=h_F+h_C}=\sigma_{zz}^{(2)}\Big|_{z=h_F+h_C},\ \sigma_{zr}^{(3)}\Big|_{z=h_F+h_C}=\sigma_{zr}^{(2)}\Big|_{z=h_F+h_C},$$

$$u_z^{(3)}\Big|_{z=h_F+h_C}=u_z^{(2)}\Big|_{z=h_F+h_C},$$

$$u_r^{(3)}\Big|_{z=h_F+h_C}=u_r^{(2)}\Big|_{z=h_F+h_C},\ \sigma_{zz}^{(2)}\Big|_{z=h_F}=\sigma_{zz}^{(1)}\Big|_{z=h_F},\ \sigma_{zr}^{(2)}\Big|_{z=h_F}=\sigma_{zr}^{(1)}\Big|_{z=h_F},$$

$$u_r^{(2)}\Big|_{z=h_F}=u_r^{(1)}\Big|_{z=h_F},\ D_z^{(3)}\Big|_{z=h_F+h_C}=0,\ D_z^{(1)}\Big|_{z=h_F}=0,\ \text{for}\ l_0/2\le r\le l/2. \tag{2.7}$$

Boundary conditions on the face planes of the plate:

$$\sigma_{zz}^{(3)}\Big|_{z=2h_F+h_C}=0,\ \sigma_{zr}^{(3)}\Big|_{z=2h_F+h_C}=0,\ \sigma_{zz}^{(1)}\Big|_{z=0}=0,$$

$$\sigma_{zr}^{(3)}\Big|_{z=0}=0,\ D_z^{(3)}\Big|_{z=2h_F+h_C}=0,\ D_z^{(1)}\Big|_{z=0}=0,\ \text{for}\ 0\le r\le l/2. \tag{2.8}$$

Conditions on the lateral boundary of the plate:

$$\sigma_{rr}^{(j)}\Big|_{r=l/2}=0,\ u_z^{(j)}\Big|_{r=l/2}=0, for j=1,2,3;\ \phi^{(k)}\Big|_{r=l/2}=0 \tag{2.9}$$

for $k=1,3$, under $0\le z\le 2h_F+h_C$.

This completes the formulation of all the boundary and contact conditions for the problem under consideration.

3 Method of solution. FEM modeling of the problem

As noted above the analytical or approximate analytical solution of the problem under consideration is impossible and therefore the formulated problem is solved numerical by employing FEM. For FEM modeling of the problem, according to Yang (2005) and others, the following functional is introduced.

$$\Pi(u_r^{(1)}, u_r^{(2)}, u_r^{(3)}, u_z^{(1)}, u_z^{(2)}, u_z^{(3)}, \phi^{(1)}, \phi^{(3)}) =$$

$$\frac{1}{2}2\pi \sum_{k=1}^{3} \iint\limits_{\Omega^{(k)}} \left[\sigma_{rr}^{(k)} \frac{\partial u_r^{(k)}}{\partial r} + \sigma_{\theta\theta}^{(k)} \frac{u_r^{(k)}}{r} + \sigma_{rz}^{(k)} \frac{\partial u_z^{(k)}}{\partial r} + \sigma_{zr}^{(k)} \frac{\partial u_r^{(k)}}{\partial z} + \sigma_{zz}^{(k)} \frac{\partial u_z^{(k)}}{\partial z} \right] rdrdz$$

$$+\frac{1}{2}2\pi \iint\limits_{\Omega^{(1)}} \left[E_r^{(1)} D_r^{(1)} + E_z^{(1)} D_z^{(1)} \right] rdrdz + \frac{1}{2}2\pi \iint\limits_{\Omega^{(3)}} \left[E_r^{(3)} D_r^{(3)} + E_z^{(3)} D_z^{(3)} \right] rdrdz -$$

$$-2\pi \int_0^{l_0/2} p u_z^{(1)} \Big|_{z=h_F} rdr - 2\pi \int_0^{l_0/2} p u_z^{(2)} \Big|_{z=h_F} rdr -$$

$$-2\pi \int_0^{l_0/2} p u_z^{(2)} \Big|_{z=h_F+h_C} rdr - 2\pi \int_0^{l_0/2} p u_z^{(3)} \Big|_{z=h_F+h_C} rdr, \tag{3.1}$$

where

$$\Omega^(1) = \{0 \le r \le l_0/2; 0 \le z \le h_F\} - \{z = h_F - 0; 0 \le r \le l_0/2\};$$

$$\begin{aligned} \Omega^{(2)} &= \{0 \le r \le l_0/2; h_F \le z \le h_F + h_C\} - \{z = h_F + 0; 0 \le r \le l_0/2\}; \\ &- \{z = h_F + h_C - 0; 0 \le r \le l_0/2\}; \\ \Omega^{(3)} &= \{0 \le r \le l/2; h_F + h_C \le z \le 2h_F + h_C\} - \\ &\{z = h_F + h_C + 0; 0 \le r \le l_0/2\}. \end{aligned} \tag{3.2}$$

From equating to zero the first variation of the functional (3.1), i.e. from the relation

$$\delta\Pi = \sum_{k=1}^{3} \frac{\partial \Pi}{\partial u_r^{(k)}} \delta u_r^{(k)} + \sum_{k=1}^{3} \frac{\partial \Pi}{\partial u_z^{(k)}} \delta u_z^{(k)} + \frac{\partial \Pi}{\partial \phi^{(1)}} \delta\phi^{(1)} + \frac{\partial \Pi}{\partial \phi^{(3)}} \delta\phi^{(3)} = 0 \tag{3.3}$$

and after well-known mathematical manipulations we obtain the equations in (2.1) and all the corresponding boundary and contact conditions in (2.6) – (2.9) with respect to the forces and electrical displacements. In this way it is proven that the equations in (2.1) are the Euler equations for the functional (3.1), and the boundary and contact conditions in (2.6) – (2.9) which are given with respect to the forces and electrical displacements, are the related natural boundary and contact conditions.

According to FEM modelling, the solution domains indicated in (3.2) are divided into a finite number of finite elements. For the considered problem, each of the finite elements is selected as a standard rectangular Lagrange family quadratic finite element with nine nodes and each node has three degrees of freedom, i.e. radial displacement $u_r^{(j)}$, transverse displacement $u_z^{(j)}$ $(j = 1, 2, 3)$ and electric potential $\phi^{(k)}$ $(k = 1, 2)$. We note that under FEM modelling of the region containing the crack's tip, as did our predecessors, we use ordinary (not singular) finite elements. This is because up to now finite elements with oscillating singularity which appear at the interface crack tips have not been found. Furthermore, as shown in the references Akbarov (2013), Akbarov and Yahnioglu (2016), Akbarov and Turan (2009), Henshell and Shaw (1975) and other ones listed therein, under calculation of the global characteristics of the element of construction (such as the critical forces, electro-mechanical energies, ERR, etc.) the results obtained by the use of the "ordinary" singular finite elements coincide, with very high accuracy, with the results obtained by the use of the ordinary finite elements.

The algorithm and the programs to obtain the numerical results are coded within the foregoing assumptions by the author in the FORTRAN programming language (FTN77). Employing the standard Ritz technique detailed in many references, for instance, in the book by Zienkiewicz and Taylor (1989), we determine the displacements and electrical potential at the selected nodes. After this determination, according to the relation

$$U = \frac{1}{2}2\pi \sum_{k=1}^{3} \iint\limits_{\Omega^{(k)}} \left[\sigma_{rr}^{(k)} \frac{\partial u_r^{(k)}}{\partial r} + \sigma_{\theta\theta}^{(k)} \frac{u_r^{(k)}}{r} + \sigma_{rz}^{(k)} \frac{\partial u_z^{(k)}}{\partial r} + \sigma_{zr}^{(k)} \frac{\partial u_r^{(k)}}{\partial z} + \sigma_{zz}^{(k)} \frac{\partial u_z^{(k)}}{\partial z} \right] r dr dz +$$

$$\frac{1}{2}2\pi \iint\limits_{\Omega^{(1)}} \left[E_r^{(1)} D_r^{(1)} + E_z^{(1)} D_z^{(1)} \right] r dr dz + \frac{1}{2}2\pi \iint\limits_{\Omega^{(3)}} \left[E_r^{(3)} D_r^{(3)} + E_z^{(3)} D_z^{(3)} \right] r dr dz, \quad (3.4)$$

the electro-mechanical strain enegy is calculated. Moreover, through the expression

$$\gamma = \frac{\partial U}{\pi l_0 \partial l_0} \quad (3.5)$$

the energy release rate γ (ERR) is also determined.

Table 1. The values of the mechanical, piezoelectrical and dielectrical constants of the selected piezoelectric materials: here $c_{11}^{(r_1)}$,...,$c_{66}^{(r_1)}$ are the elastic constants, $e_{31}^{(r_1)}$,..., $e_{15}^{(r_1)}$ are the piezoelectric constants, and $\varepsilon_{11}^{(r_1)}$ and $\varepsilon_{33}^{(r_1)}$ are the dielectric constants

Mater. (Source Ref.)	$c_{11}^{(r_1)}$	$c_{12}^{(r_1)}$	$c_{13}^{(r_1)}$	$c_{33}^{(r_1)}$	$c_{44}^{(r_1)}$	$c_{66}^{(r_1)}$	$e_{31}^{(r_1)}$	$e_{33}^{(r_1)}$	$e_{15}^{(r_1)}$	$\varepsilon_{11}^{(r_1)}$	$\varepsilon_{33}^{(r_1)}$
PZT-4 (Yang, 2005)	13.9	7.78	7.40	11.5	2.56	3.06	-5.2	15.1	12.7	0.646	0.562
PZT-5H (Yang, 2005)	12.6	7.91	8.39	11.7	2.30	2.35	-6.5	23.3	17.0	1.505	1.302
	$\times 10^{10} N/m$						C/m^2			$\times 10^{-8} C/Vm$	

4 Numerical results and discussions

In the present paper we consider only the numerical results related to the electromechanical energy $U$and all the numerical results are made for the piezoelectric materials PZT - 4 and

PZT - 5H which are selected for the face layers, however the metal materials - aluminum (Al) and steel (St) are taken as the core layer materials. The values of the elastic, piezoelectric and dielectric constants of the selected piezoelectric materials and the references used are given in Table 1. According to the monograph by Guz (2004), the values of Lame's constants of the core layer material are selected as follows: for the Al: $\lambda = 48.1\,GPa$ and $\mu = 27.1\,GPa$; and for the St: $\lambda = 92.6\,GPa$ and $\mu = 77.5\,GPa$.

In order to analyze the coupling effects of the electro-mechanical fields on the energies, the numerical results are obtained for the following two cases:

$$\text{Case 1. } e_{ij}^{(r_n)} = 0,\ \varepsilon_{ii}^{(r_n)} = 0, \tag{4.1}$$

$$\text{Case 2. } e_{ij}^{(r_n)} \neq 0, \varepsilon_{ii}^{(r_n)} \neq 0. \tag{4.2}$$

Numerical results obtained in Case 1 (15) relate to the pure mechanical energies, however the numerical results obtained in Case 2 (16) relate to the total electro-mechanical energies and comparison of the results obtained in Case 2 with the corresponding ones obtained in Case 1 will give the information for estimation of the influence of the coupling electro-mechanical effect on the studied quantities.

Under obtaining all the numerical results illustrated in the present paper, we assume that the piezoelectric materials are polarized along the plate thickness, i.e. the polarized direction of the PZT materials coincides with the Oz axis. Moreover, all the numerical results are obtained in the case where $h/l = 0.2$.

We do not consider here the testing of the algorithm and PC programs used for obtaining numerical results because the corresponding testings are already made in the papers Cafarova et al. (2017), Akbarov et al. (2017), Cafarova and Rzayev (2016).

Thus, within the foregoing assumptions, we consider numerical results related to the energies and under this consideration the following energies in the Case 2 are distinguished:

i) total electro-mechanical energy under calculation of which all the terms in the expression (3.4) are taken into consideration,

ii) pure mechanical energy under calculation of which the last two integral terms in the expression (3.5) are ignored,

iii) interaction energy under calculation of which only the terms containing the mechanical and electrical quantities simultaneously in the expression (3.5) are taken into consideration, and

iv) pure electrical energy under calculation of which only the terms in the expression (3.5) containing only the electrical quantities are taken into consideration.

Numerical results illustrating the influence of the crack's radius on the values of the foregoing energies appearing in the PZT-5H/Al/PZT-5H and PZT-5H/St/PZT-5H plates are given in Fig. 2a and Fig. 2b, respectively. Under obtaining these results, it is assumed that $h_F/l = 0.025$. These results show that for all the considered lengths of the penny-shaped interface crack's radius, the values of the total electro-mechanical, pure mechanical and interaction energies are positive numbers and these values increase monotonically with this radius. However, in all the considered lengths of the crack radius, the values obtained for the pure electrical energy are negative and the absolute values of this energy also increase monotonically with the penny-shaped crack's radius. Note that in the qualitative sense, similar results are also obtained in the paper by Akbarov and Yahnioglu (2016) for the sandwich plate-strip in the plane-strain state.

Consider also the influence of the change of the piezoelectric face layers thickness on the foregoing energies. For this purpose analyze the graphs given in Fig. 3 which illustrate the dependencies among the pure electrical (Fig. 3a), the interaction (Fig. 3b), the pure mechanical (Fig. 3c), the total electro-mechanical (Fig. 3d) and the dimensionless radius of the crack (l_0/l) obtained for various thicknesses of the piezoelectric face layers of the PZT-5H/Al/PZT-5H plate. These results show that for a fixed thickness of the whole

PZT-5H/Al/PZT-5H plate, a decrease of the face layers' thickness causes an increase in the absolute values of all the energies under consideration.

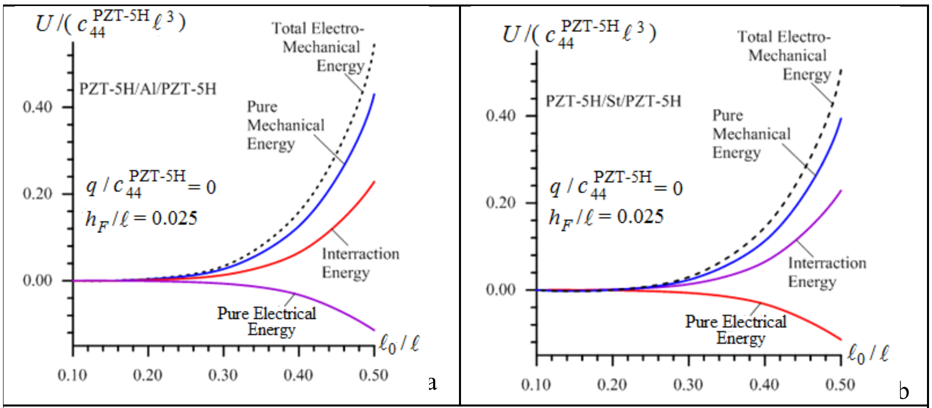

Fig. 2. Graphs of the dependencies between energies and the ratio ℓ_0/ℓ in Case 2 for the PZT-5H/Al/PZT-5H (a) and PZT-5H/St/PZT-5H (b) plates

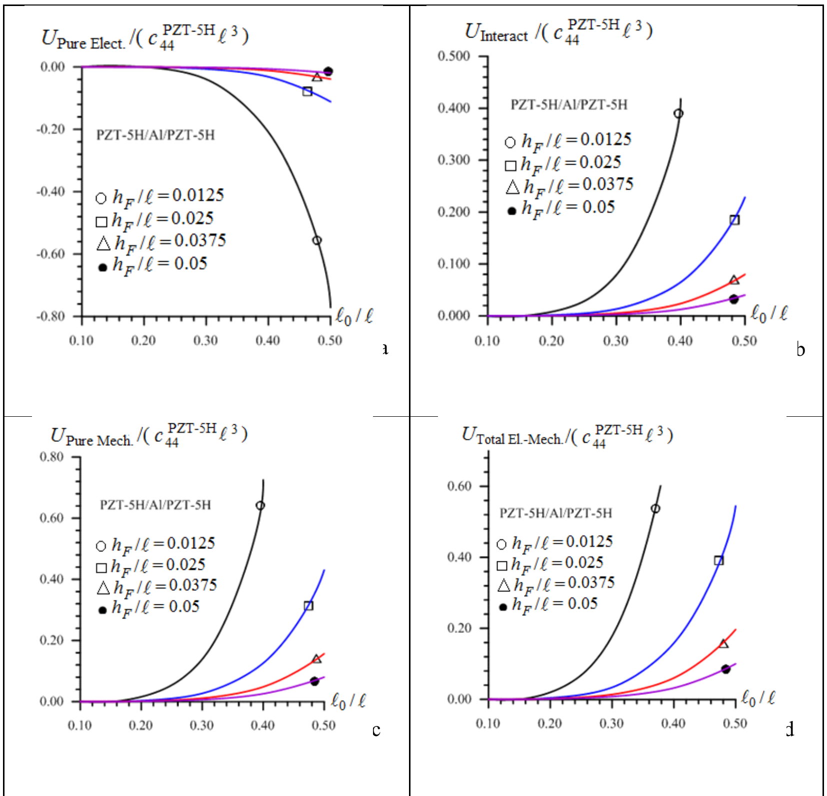

Fig. 3 Dependence of various pure electrical (a), interaction (b), pure mechanical (c) and total electro-mechanical (d) energies on the ratio ℓ_0/ℓ under various h_F/ℓ for the PZT-5H/Al/PZT-5H plate.

For estimation of the electro-mechanical coupling effect of the piezoelectric materials on the total electro-mechanical energy, we consider the graphs given in Fig. 4 which show the dependence between U determined by expression (3.5) and the dimensionless length l_0/l of the crack radius under various thicknesses of the face layers for PZT-5H/Al/PZT-5H (Fig. 4a) and PZT-5H/St/PZT-5H (Fig. 4b). Note that in Fig. 4, the graphs are constructed for Case 2 and Case 1 simultaneously. The difference between the corresponding results obtained Case 1 and in Case 2 namely causes the coupling electro-mechanical effect of the piezoelectric material. We recall that, according to the expressions (4.1) and (4.2) under obtaining the results related to Case 1, the coupling effect is not taken into consideration, however, under obtaining the results related to Case 2 this effect is taken into consideration completely. Thus, it follows from the results given in Fig. 4 that for all the selected values of the layers' thickness and crack radius the piezoelectricity of the face layers' materials causes to decrease the total electro-mechanical energy of the plates under consideration.

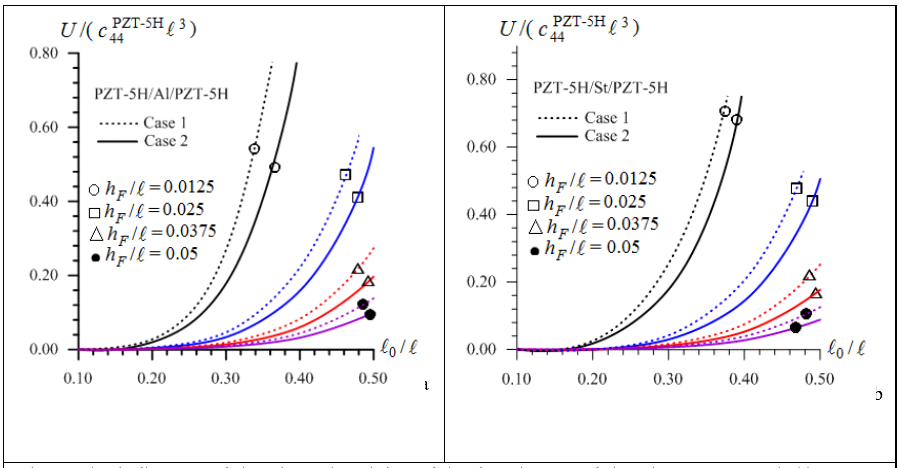

Fig. 4 The influence of the piezoelectricity of the face layers of the plates PZT-5H/Al/PZT-5H (a) and PZT-5H/St/PZT-5H (b) on the values of the total electro-mechanical energies obtained for various values of the ratios ℓ_0/ℓ and h_F/ℓ.

This completes the consideration of the numerical results presented in the paper. We recall that these results relate to the various type energies indicated above.

5 Conclusions

Thus, in the present paper, the rotationally symmetic interface penny-shaped crack problem for the PZT/Elastic/PZT sandwich circular plate is considered by utilizing the corresponding exact field equations and relations of the theory of elecro-elasticity for the piezoelectric material. It is assumed that on the cracks edges uniformly distributed normal opening forces act and corresponding boundary value problem is solved numerically by employing FEM. Numerical results are presented and discussed for the various types energies determined by expressions (13) for the PZT-5H/Al/PZT-5H, PZT-5H/St/PZT-5H and PZT-4/Al/PZT-4 plates.According to analyses of the results, it is established that the piezoelectricity of the face layers materials causes to decrease the total electro-mechanical energies and the magnitude of this influence increases with increasing of the ratio l_0/l and with decreasing of the ratio h_F/l, where $l_0/2$ ($l/2$) is the radius of the penny-shaped crack (circular disc), and h_F

is the thickness of the piezoelectric face layer. Consequently, the parameters l_0/l and h_F/l characterize not only the dimensions of the penny-shaped crack and face layer thickness, but also the dimension of the circular plate.

Numerical results obtained in the present paper in the qualitative sense agree with the corresponding onesobtained in the related investigations carried out, for instance, in the papers by Yang (2004), Li and Lee (2012), Akbarov and Yahnioglu (2016), and others.

At the same time, for estimation of the interface cracks grow (or fracture) in the sandwich plate it is necessary to consider and analyze also the numerical results related ERR determined through the expression (14) which will be made in future papers by author.

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