Transactions of NAS of Azerbaijan, Issue Mechanics, **37** (7), 72–77 (2017).
Series of Physical-Technical and Mathematical Sciences.

A finite length rectangular bar under the action of axial impact forces

Nazila B. Rasulova

Received: 10.09.2017 / Revised: 07.10.2017 / Accepted: 02.11.2017

Abstract. *In the article, the dynamics of an elastic parallelepiped, subjected to the action of the impact forces, is studied. An exact analytical solution for constant loading along the end section areas, and in the presence of free side surfaces of parallelepiped, is received.*

Keywords. parallelepiped · Lame equation nonstationary waves · Laplace transform

Mathematics Subject Classification (2010): 74B05

1 Introduction

The paper is logical continuation of a cycle of works [1]-[3] of the author devoted to dynamic of rectangular bars.

The peculiarities of these works is that they were performed by a specially developed method on the basis of which an exact analytic solutions of the famous three-dimensional problems of theory of elasticity were constructed.

In the paper [3] the analytic solution on wave propagation in a semi-infinite rectangular prism with free lateral surfaces, was constructed. The present paper is similar to this, except that the prism's length is finite. Emergence of two additional obstacles in the path of wave propagation (in the form of end areas) undoubtedly even aggravates the already complicated wave picture of the movement even more. Since in the case under consideration the body has finite dimensions, for the existence of a unique solution the opposite ends are exposed to opposite impact loads so that the resultant force was zero.

The exact analytic solution of the stated problem is found for the case of constancy of impact loads in the domain of end areas. Recall that the similar statical problem on equilibrium of a parallelepiped (a problem unsolved up to day) was stated in 1852 by G. Lame and bears his name. Then the Paris Academy of Sciences established a prize for the author of this solution.

N.B. Rasulova
Institute of Mathematics and Mechanics,
Baku, AZ 1143, str. B. Vahabzadeh 9.
E-mail: rasulova@gmail.com

2 Statement and solution of the problem

A rectangular, elastic, finite length bar, a parallelepiped is exposed to the action of opposite axial impact forces applied to the end areas. The lateral sides are free from forces and this complicates the process of constructing solutions.

Obviously, the problem under consideration requires the solution of the initial boundary value problem of mathematical physics, the motion is described by the system of three-dimensional Lame equations that in the vector form has the form:

$$\frac{\partial^2 \bar{U}}{\partial t^2} = (\lambda + \mu)\, graddiv\overline{U} + \mu \mathit{\Delta} \overline{U}, \quad \overline{U} = \overline{U}\,(u, v, w) \tag{2.1}$$

and it happens in the domain of the space occupied by a parallelepiped: $-a \le x \le a$, $-b \le y \le b$, $-l \le z \le l$ for $t > 0$. To this system we join the following initial boundary conditions:

$$U = 0; \quad \dot{U} = 0 \text{ for } t = 0 \tag{2.2}$$

$$\sigma_{zz} = \sigma_0 f\,(t) \quad u = 0, \;\; v = 0 \;\text{ for } z = \pm l \tag{2.3}$$

$$\sigma_{xx} = \sigma_{xy} = \sigma_{xz} = 0 \;\text{ for } x = \pm a, \;\; z > 0$$

$$\sigma_{yx} = \sigma_{yy} = \sigma_{yz} = 0 \;\text{ for } y = \pm b, \;\; z > 0 \tag{2.4}$$

where U is a displacement vector, $\{\sigma\}$ is a stress tensor, λ, μ are the Lame coefficients, t is time, ρ is the material's density.

Following [1], [2] the problem (2.1)-(2.4) is reduced to integration of a simpler, but this time to a homogeneous system:

$$\begin{cases} \mu \frac{\partial}{\partial z} H_1^* \psi_2 - (\lambda + 2\mu)\, H_1^* \varphi = 0 \\ H_2^* \psi_1 = 0 \\ \mathit{\Delta} H_2^* \psi_2 = 0 \end{cases}$$

where H_i^* and $\mathit{\Delta}$ are three-dimensional Helmholts and Laplace operators, respectively:

$$H_i^* = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{p^2}{c_i^2};$$

$$\mathit{\Delta} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2};$$

and the three new functions φ, ψ_1, ψ_2 are connected with transformations of three components of displacement by the Laplace operator by the formulas:

$$\overline{u} = \frac{\partial \varphi}{\partial x} + \frac{\partial \psi_1}{\partial y} - \frac{\partial^2 \psi_2}{\partial z \partial x}$$

$$\overline{v} = \frac{\partial \varphi}{\partial x} - \frac{\partial \psi_1}{\partial y} - \frac{\partial^2 \psi_2}{\partial z \partial y}$$

$$\overline{w} = \frac{\partial \varphi}{\partial z} - \frac{\partial^2 \psi_2}{\partial x^2} - \frac{\partial^2 \psi_2}{\partial y}$$

This system is equivalent to the following system:

$$\begin{cases} \mathit{\Delta} H_1^* \varphi = 0 \\ H_1^* \psi_1 = 0 \\ \mathit{\Delta} H_r^* \psi_2 = 0 \end{cases}. \tag{2.5}$$

The solutions of the similar problem of an infinite prism will guide to determine the form of the solutions.
We can look for the solution in the form:

$$\varphi = \varphi(z). \tag{2.6}$$

Having substituted (2.5) in the first equation, taking into account considerations on symmetry, we get:

$$\varphi = Ach\frac{p}{c_1}z. \tag{2.7}$$

The constant A is determined from conditions (2.3) after their transformation by the Laplace operator:

$$(\lambda + 2\mu)\frac{\partial^2\varphi}{\partial z^2} = \sigma_0 f(p) \quad z = \pm l.$$

Hence

$$A = \frac{c_1^2\sigma_0 f(p)}{p^2(\lambda + 2\mu)\, ch\frac{p}{c_1}l}. \tag{2.8}$$

Note that allowing for (2.8), solution (2.7) satisfies all end conditions (2.3).

This condition corresponds to longitudinal waves that simultaneously start from the ends and propagate along the prism's axis, in opposite directions.

Undoubtedly, interaction of these waves with free lateral surfaces causes first of all transverse wave motions.

At first we note that all the solutions corresponding to these waves satisfy the zero end conditions; consequently,

$$\left.\begin{aligned} u &= 0 \\ v &= 0 \\ \sigma_{zz} &= 0 \end{aligned}\right\} \text{ for } z = \pm l. \tag{2.9}$$

These waves, reflected from the lateral surfaces may be sought as the solutions of the third equation of the system (2.5) subject to conditions (2.9). It is appropriate to note that the conditions (2.9) can be automatically satisfied by the choice of the solution in the following form:

$$\psi_2 = \sum_{m=0}^{\infty} \bar{\psi}_{2m}(x, y)\sin\frac{\pi m z}{l}. \tag{2.10}$$

Having substituted this function in the above-mentioned equation, we get

$$H_{0m}^{*}H_{2m}^{*}\bar{\psi}_{2m} = 0 \tag{2.11}$$

where

$$H_0^* = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{y^2} - \frac{\pi^2 m^2}{l^2}$$

$$H_r^* = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{y^2} - \left(\frac{\pi^2 m^2}{l^2} + \frac{p^2}{c_1}\right)$$

to determine each $\bar{\psi}_{2m}(x, y)$ separately.

Following [3], we shall look for the functions $\bar{\psi}_{2m}(x, y)$ in the class of solutions $f_1(x)+$ $+f_2(y)$. The simplest solution of this form will be:

$$\bar{\psi}_{2m}(x, y) = B_{1m}ch\sqrt{\frac{p^2}{c_2^2} + \frac{\pi^2 m^2}{l^2}}x + B_{2m}ch\sqrt{\frac{p^2}{c_2^2} + \frac{\pi^2 m^2}{l^2}}y.$$

The constants B_{1m} and B_{2m} may be determined from the condition that tensile stresses on the surfaces $x=\pm a$ and $y=\pm b$ are zero, more exactly:

$$\sigma_{xx}|_{x=\pm a}=\sigma_{xx}(\varphi)+\sigma_{xx}(\psi_2)=0$$

$$\sigma_{yy}|_{y=\pm b}=\sigma_{yy}(\varphi)+\sigma_{yy}(\psi_2)=0$$

$$2\mu\frac{\sigma_0 f(p)}{\lambda+2\mu}\cdot\frac{ch\frac{p}{c_1}z}{ch\frac{p}{c_1}l}=$$

$$=-\lambda\sum_{m=0}^{\infty}B_{im}\frac{\pi m}{l}\left[\frac{p^2}{c_2}+\left(\frac{\pi^2m^2}{l^2}\right)\right]ch\sqrt{\frac{p^2}{c_2}+\frac{\pi^2m^2}{l^2}}\cos\frac{\pi mz}{l},\quad i=1,2. \tag{2.12}$$

We divide the left hand side of equation (2.12) by the functions $\cos\frac{\pi mz}{l}$, and then comparing the coefficients for the same functions $\cos\frac{\pi mz}{l}$ we get:

$$B_{1m}=\frac{4\mu\cdot\sigma_0 f(p)}{\pi\lambda(\lambda+2\mu)m}\cdot\frac{(-1)^m\alpha_1 th\alpha_1 l}{\nu_{1m}^2(p)\cdot\nu_{2m}^2(p)\cdot ch\nu_{2m}(p)\frac{a}{c_2}};$$

$$B_{2m}=\frac{4\mu\cdot\sigma_0 f(p)}{\pi\lambda(\lambda+2\mu)m}\cdot\frac{c_1^2c_2^2(-1)^m\alpha_1 th\alpha_1 l}{\nu_{1m}^2(p)\cdot\nu_{2m}^2(p)\cdot ch\nu_{2m}(p)\frac{b}{c_2}}; \tag{2.13}$$

Here:

$$\alpha_1=\frac{p}{c_1};\quad \nu_{im}(p)=\sqrt{\rho^2+\frac{\pi^2m^2c_2^2}{l^2}}.$$

Thus, the solutions (2.10) corresponding to the first group of reflected transverse waves are completely determined. On the lateral surfaces these solutions cause tangential stresses $x=\pm a$ and $y=\pm b$. that must be canceled by additional solutions. It turns out that as it was noted in [3], from all possible solutions of the system of equation (2.5), only the solutions of the form

$$\bar{\psi}_{2m}^*=\sum_k C_{1mk}\cos\left(\frac{1}{2}+k\right)\frac{\pi x}{a}+C_{2mk}\left(\frac{1}{2}+k\right)\frac{\pi y}{b} \tag{2.14}$$

are capable to neutralize the above-mentioned stresses.

Let us consider to the equations $H_{0m}^*H_{2m}^*\bar{\psi}^*=D_m$ or

$$\left(\frac{\partial^2}{\partial x^2}-\frac{\pi^2m^2}{l^2}-\frac{p^2}{c_2^2}\right)\left(\frac{\partial^2}{\partial x^2}-\frac{\pi^2m^2}{l^2}\right)\bar{\psi}_{2m}^*=D_{1m}$$

$$\left(\frac{\partial^2}{\partial y^2}-\frac{\pi^2m^2}{l^2}-\frac{p^2}{c_2^2}\right)\left(\frac{\partial^2}{\partial y^2}-\frac{\pi^2m^2}{l^2}\right)\bar{\psi}_{2m}^*=D_{2m} \tag{2.15}$$

where the unknown constants $D_m=D_{1m}+D_{2m}$ be defined. Without going into details, we give the ready formulas for these constants that were determined from the condition that all tangential σ_{xz} and σ_{yz} stresses equal zero on the surfaces $x=\pm a$ and $y=\pm b$:

$$C_{1km}=2D_{1m}\frac{(-1)^k}{\pi\left[\eta_{1k}^2+\frac{\nu_{2m}^2(p)}{c_2^2}\right]\left[\eta_{1k}^2+\frac{\pi^2m^2}{l^2}\right]\eta_{1k}}$$

$$C_{2km}=2D_{2m}\frac{(-1)^k}{\pi\left[\eta_{2k}^2+\frac{\nu_{2m}^2(p)}{c_2^2}\right]\left[\eta_{2k}^2+\frac{\pi^2m^2}{l^2}\right]\eta_{2k}} \tag{2.16}$$

$$D_{1m}=\frac{2}{\pi}\frac{2\mu\sigma_0c_1^2}{(\lambda+2\mu)\lambda}\frac{f(p)}{\nu_{1m}^2(p)}\frac{(-1)^m\left(\frac{p^2}{c_2^2}+2\frac{\pi^2m^2}{l^2}\right)(th\frac{\nu_{2m}}{c_2}a)\frac{p}{c_1}th\frac{p}{c_1}l}{\sum\limits_{k=0}^{\infty}\left[\left(\eta_{2k}^2+\frac{\nu_{2m}^2}{c_2^2}\right)\frac{\frac{\pi^2m^2}{l^2}-\eta_{2k}^2}{\frac{\pi^2m^2}{l_2}+\eta_{2k}^2}\right]^{-1}} \tag{2.17}$$

$$D_{2m} = \frac{2}{\pi}\frac{2\mu\sigma_0 c_1^2}{(\lambda+2\mu)\lambda}\frac{f(p)}{\nu_{1m}^2(p)}\frac{(-1)^m\left(\frac{p^2}{c_2^2}+2\frac{\pi^2m^2}{l^2}\right)(th\frac{\nu_{2m}}{c^2}b)\frac{p}{c_1}th\frac{p}{c_1}l}{\sum\limits_{k=0}^{\infty}\left[\left(\eta_{2k}^2+\frac{\nu_{2m}^2}{c_2^2}\right)\frac{\frac{\pi^2m^2}{l^2}-\eta_{2k}^2}{\frac{\pi^2m^2}{l_2}+\eta_{2k}^2}\right]^{-1}}$$

$$\eta_{1k} = \frac{\pi}{a}\left(\frac{1}{2}+k\right)$$

$$\eta_{2k} = \frac{\pi}{b}\left(\frac{1}{2}+k\right)$$

$$\nu_{im} = \sqrt{p^2+\frac{\pi^2m^2c_i^2}{l^2}}$$

Note that the equation $H_{0m}^* H_{1m}^* \psi_{2m} = -Dm$ also has an obvious solution $\psi_{2m} = -\frac{D_m}{\left(\frac{\pi^2m^2}{l^2}+\frac{p^2}{c_2^2}\right)\frac{\pi^2m^2}{l^2}}$; that compensates the solution (2.14) of inhomogeneous equation (2.15) and simultaneously does not cause any displacement in the body.

The obtained solutions by their form are identical with the solutions given in [3] with only difference that the integrals over continuous parameters, $q \in (0,\infty)$ in solutions of [3], (representing originals of Fourier transform) are replaced by an infinite sum for discrete values $q = \frac{\pi m}{l}$ $(m = 1,2,3,...)$ of the similar parameter. Therefore, the methods of inversion of the solutions-transformations by the Laplace operator, stated in [3], are also identical and can easily be applied here too. Without dwelling on the details, we only give the solutions for axial velocities $\left(\overline{\dot{W}} = \dot{W}\frac{(\lambda+2\mu)\lambda}{2\mu\cdot\sigma_0\cdot c_1}\right)$ of the parallelepiped under consideration:

$$f(p) \doteq H(t)$$

$$\overline{\dot{W}} = \frac{\lambda}{2\mu}\sum_k \frac{(\sin\pi(\frac{1}{2}+k)\frac{z}{l})(\sin\pi(\frac{1}{2}+k)\frac{c_1t}{l})}{(\frac{1}{2}+k)(-1)^k}+$$

$$+\left[\int_0^t \sum_m \frac{(-1)^m}{m}F_m(t-\tau)\cdot M\left(\tau,\frac{2nl}{c_1}\right)d\tau\right]\frac{\sin\pi mz}{l}$$

where:

$$F_m(\tau) = F_{am}(\tau)+F_{bm}(\tau);$$

$$F_{am}(\tau) = \frac{\alpha_{1m}\left(\cos\frac{B_m x}{c_2}\right)\sin\alpha_{1m}t}{\cos\frac{\beta_m a}{c_2}}+$$

$$+\sum_{k=0}^{\infty}\frac{(-1)^k\sqrt{(\alpha_{1m}^2+\eta_{1k}^2c_2^2)}\eta_{1k}\cdot c_2^2(\cos\eta_{1k}c_2x)\cos\eta_{1k}c_2t}{\alpha\left(\beta_m^2-\eta_{1k}^2c_2^2\right)}$$

$$F_{ab}(\tau) = \frac{\alpha_{1m}\left(\cos\frac{B_m y}{c_2}\right)\sin\alpha_{1m}t}{\cos\frac{\beta_m b}{c_2}}+$$

$$+\sum_{k=0}^{\infty}\frac{(-1)^k\sqrt{(\alpha_{1m}^2+\eta_{2k}^2c_2^2)}\eta_{2k}\cdot c_2^2(\cos\eta_{2k}c_2x)\cos\eta_2c^2t}{b\left(\beta_m^2-\eta_{2k}^2c_2^2\right)}$$

$$M\left(\tau, \frac{2nl}{c_1}\right) = H\left(\tau\right) + 2\sum_{1}^{\infty}\left(-1\right)^n H\left(\tau - \frac{2nl}{c_1}\right)$$

$$\alpha_{im} = \frac{c_i m\pi}{l}$$

$$\beta_m = \sqrt{c_1^2 - c_2^2}\frac{\pi m}{l} = \sqrt{\alpha_{1m}^2 - \alpha_{2m}^2}.$$

3 Conclusion

The exact analytic solution of non stationary dynamic problem of wave propagation in a short length rectangular prism, known as G. Lame problem in mechanics, is found.

References

1. Rassulova, N.B.: *Wave propagation in a prismatic bar exposed to the action of axial forces*, Izv. RAN MTT, **6**, 171–176 (1997).
2. Rasulova, N.B.:*On dynamic of bar rectangular cross section*, Trans. Of ASME, J. of Appl. Mech. **68**, 662–666 (2001).
3. Rasulova, N.B., Shamilova, G.R.: *Propagation of stress wave in a rectangular bar*, Izv. RAN MTT, **4**, 144–152 (2016).