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Perturbations in stratified flow of viscous liquids

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Abstract. *Dynamics of disturbances on the surface of stratified flow of two incompressible viscous liquids in narrow channel is simulated mathematically. The evolution equation with regard to friction on the interface of liquids and free surface was obtained for small but finite amplitudes of the wave. It is shown that two type waves propagate in the system. The first wave propagates on the interface of liquids. The second wave propagates on the free surface of the stratified liquid. Influence of depth of viscosities and densities of liquids on wave characteristics was studied.*

Keywords. dispersion · damping decrement · viscous velocity · viscous liquid · surface waves · two-layer liquid

Mathematics Subject Classification (2010): 76A05, 76E05

1 Introduction

Currently, there is a growing interest in mathematical modeling of the dynamics of the wave movements of various heterogeneous natural stratified mediums, due to problems in geophysics, oceanology, physics of atmosphere, protection and study of the environment, operation of complex hydraulic structures, including the marine oil producing complexes, and a number of other actual problems of science and technology. This interest is caused not only by practical requirements but also a large theoretical content problem of mathematical modeling. The study of wave processes in inhomogeneous stratified environments has become a rapidly evolving field, and the results of these studies are important both from

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a fundamental point of view and for technical applications. New experimental and technical ability stimulate work on mathematical modeling and asymptotic study of the dynamics of non-harmonic wave packets of internal gravity waves. In the basis "analysis, as a rule, are asymptotic methods based on the study of the unperturbed equations to generate the appropriate asymptotic expansion, taking into account the inhomogeneity and nonstationarity of the natural stratified mediums.

Issues of dynamics of internal gravity waves in stratified media devoted considerable number of works. The main attention is being paid to experimental investigation of internal wave dynamics, a detailed theoretical consideration of the dynamics of linear internal gravity waves in media with model distributions of density and direct numerical simulations of the relevant hydrodynamics equations

The relative simplicity of the solution of linear equations compared to the full nonlinear problem, the modern development of the appropriate mathematical apparatus and computing machinery allow answering many of the demands of practice.

For a detailed description; a wide range of physical phenomena associated with wave dynamics of the stratified heterogeneous in the horizontal; and non-stationary environments must be based on a sufficiently developed mathematical models, which tend to be highly complex, nonlinear, multivariable, and their full study are only effective numerical methods. However, in some cases, an appropriate initial qualitative picture of the investigated phenomena can be obtained on the basis of: simple asymptotic models and analytical methods of research. In this respect, it is characteristic of the problem of mathematical simulation of dynamics of non-harmonic packets of internal gravity waves. Even within linear models of their solutions are of independent mathematical interest.

It is known that the force of gravity and the stratification of liquids strongly modify the propagating waves in them. Modeling such waves devoted many works [2-5]. In most of them the conclusion about the necessity to take into accounts both dispersion and nonlinearity of gravitational perturbations in the fluid for an adequate description of the investigated phenomena.

In recent years, more attention is attracted to the nonlinear dispersion model of shallow or deep water. In particular, in [1] considered a mathematical model of propagation of long-wavelength disturbances in two-layer flow of an ideal stratified fluid with a free boundary.

Models based on the Korteweg - de Vries, do not consider the impact of the perturbation decay due to the influence of liquid viscosity. For a detailed study of the perturbations introduced into the KDV equation, describes the various properties of fluids [2], considered the direct method of perturbation theory and inverse scattering for the solution of the modified perturbed KDV equation.

In contrast to [1] consider a model of disturbance propagation, taking into account the viscosity of the fluid.

2 Statement of a problem

Let in a narrow channel $0 \leq y \leq h(x,t)$ the flow of a viscous incompressible fluid (Fig. 1). Layer stratified fluid of variable depth $h(x,t)$ consists of fluid density ρ_1, viscosity μ_1, depth $h_1(x,t)$ and fluid density$\rho_2(\rho_1 > \rho_2)$, viscosity μ_2, depth $h(x,t) = h_1(x,t) + h_2(x,t)$.

Free surface and interface fluid in equilibrium in a gravity field is flat, i.e.

$$h_1(x,0) = h_{10}, h_1(x,0) + h_2(x,0) = h_{10} + h_{20} = h_0 \tag{2.1}$$

If the result of any external influences the surface of the liquid at any location is deduced from its equilibrium position, there is a movement spreading across the surface and the interface of the liquid in the form of waves, called gravity, as they are caused by the action of the field of gravity.

We assume that the flow is characterized by variables x,y and time dependent t. Thus, it is believed that the fluid velocity is non-zero components u v. In addition, it is believed that the levels $h_1(x,t)$ and $h_1(x,t)+h_2(x,t)$ depend only on x and t.

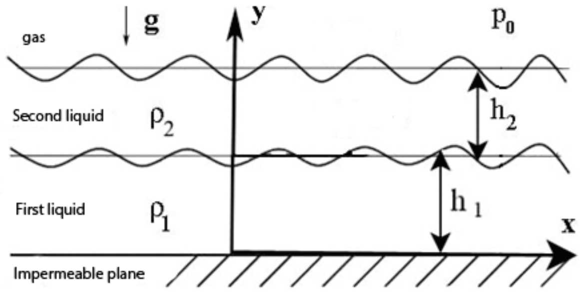

Fig. 1. Schematic image showing a layer of stratified fluid.

Then a plane-parallel motion of a viscous incompressible fluid over a smooth bottom with a free boundary in the gravity field will be described by a system of Navier-Stokes equations in the form:

$$\begin{gathered}\frac{\partial u_1}{\partial t}+u_1\frac{\partial u_1}{\partial x}+v_1\frac{\partial u_1}{\partial y}-\frac{\mu_1}{\rho_1}\left(\frac{\partial^2 u_1}{\partial x^2}+\frac{\partial^2 u_1}{\partial y^2}\right)+\frac{1}{\rho_1}\frac{\partial p_1}{\partial x}=0;\\ \frac{\partial v_1}{\partial t}+u_1\frac{\partial v_1}{\partial x}+v_1\frac{\partial v_1}{\partial y}-\frac{\mu_1}{\rho_1}\left(\frac{\partial^2 v_1}{\partial x^2}+\frac{\partial^2 v_1}{\partial y^2}\right)+\frac{1}{\rho_1}\frac{\partial p_1}{\partial y}=-g;\\ \frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}=0.\end{gathered} \tag{2.2}$$

$$\begin{gathered}\frac{\partial u_2}{\partial t}+u_2\frac{\partial u_2}{\partial x}+v_2\frac{\partial u_2}{\partial y}-\frac{\mu_2}{\rho_2}\left(\frac{\partial^2 u_2}{\partial x^2}+\frac{\partial^2 u_2}{\partial y^2}\right)+\frac{1}{\rho_2}\frac{\partial p_2}{\partial x}=0;\\ \frac{\partial v_2}{\partial t}+u_2\frac{\partial v_2}{\partial x}+v_2\frac{\partial v_2}{\partial y}-\frac{\mu_2}{\rho_2}\left(\frac{\partial^2 v_2}{\partial x^2}+\frac{\partial^2 v_2}{\partial y^2}\right)+\frac{1}{\rho_2}\frac{\partial p_2}{\partial y}=-g;\\ \frac{\partial u_2}{\partial x}+\frac{\partial v_2}{\partial y}=0.\end{gathered} \tag{2.3}$$

The boundary conditions of the problem are the condition of adhesion and the absence of the fluid flowing through the duct wall

$$u_1(x,0,t)=0, v_1(x,0,t)=0$$

The coincidence of the velocity of the surface and the particles of the liquid and pressures in $y=h_1(x,t)$

$$\begin{array}{c}\frac{\partial h_1}{\partial t}+u_1(x,h_1,t)\cdot\frac{\partial h_1}{\partial x}=v_1(x,h_1,t),\\ \frac{\partial h_1}{\partial t}+u_2(x,h_1,t)\cdot\frac{\partial h_1}{\partial x}=v_2(x,h_1,t),\end{array} \tag{2.4}$$

$$\begin{array}{c}p_1(x,h_1,t)=p_2(x,h_1,t),\\ u_1(x,h_1,t)=u_2(x,h_1,t), v_1(x,h_1,t)=v_2(x,h_1,t)\end{array} \tag{2.5}$$

The coincidence of the velocity of the surface and the particles of the liquid at $y=h_1(x,t)+h_2(x,t)$

$$\frac{\partial\left(h_1+h_2\right)}{\partial t}+u_2\left(x,h_1+h_2,t\right)\cdot\frac{\partial\left(h_1+h_2\right)}{\partial x}=v_2\left(x,h_1+h_2,t\right) \tag{2.6}$$

the lack of friction on the free surface of the channel and the coincidence of the pressure of liquid and gas (Fig.1) where $y=h_1\left(x,t\right)+h_2\left(x,t\right)$

$$\frac{\partial u_2\left(x,h_1+h_2,t\right)}{\partial y}=0, p\left(x,h_1+h_2,t\right)=p_0 \tag{2.7}$$

In addition, at the interface between the fluids must be equal to the tangential stresses:

$$\tau_{1xy}=\tau_{2xy} \text{ or } \mu_1\left(\frac{\partial u_1}{\partial y}+\frac{\partial v_1}{\partial x}\right)=\mu_2\left(\frac{\partial u_2}{\partial y}+\frac{\partial v_2}{\partial x}\right) \tag{2.8}$$

If you go to dimensionless parameters and variables:

$$U_i\left(X,Y,\tau\right)=u_i/\sqrt{gh_0}, V_i\left(X,Y,\tau\right)=v_i/\left(\varepsilon\sqrt{gh_0}\right) \tag{2.9}$$

$$P_i\left(X,Y,\tau\right)=p_i/\rho_i g h_0, H_i\left(X,\tau\right)=h_i\left(x,t\right)/h_0 \tag{2.10}$$

$$X=x/\lambda, Y=y/h_0, \tau=t\sqrt{gh_0}/\lambda, \varepsilon=h_0/\lambda \tag{2.11}$$

and to assume that the surface of the liquid there is a long wave of small amplitude, i.e., the dimensionless parameter $\varepsilon=h_0/\lambda$ is small, then the system of equations becomes:

$$\frac{\partial U_1}{\partial\tau}-\frac{1}{Re_1\cdot\varepsilon}\cdot\frac{\partial^2 U_1}{\partial Y^2}+\frac{\partial H_1}{\partial X}+\frac{1}{\rho}\cdot\frac{\partial H_2}{\partial X}=0 \tag{2.12}$$

$$\frac{\partial U_1}{\partial X}+\frac{\partial V_1}{\partial Y}=0 \tag{2.13}$$

$$\frac{\partial H_1}{\partial\tau}=V_1\left(X,H_1,\tau\right), \left.\frac{\partial U_1}{\partial Y}\right|_{Y=H_1}=\mu\left.\frac{\partial U_2}{\partial Y}\right|_{Y=H_1} V_1\left(X,H_1,\tau\right)=V_2\left(X,H_1,\tau\right) \tag{2.14}$$

$$U_1\left(X,0,\tau\right)=0, V_1\left(X,0,\tau\right)=0 \tag{2.15}$$

$$\frac{\partial U_2}{\partial\tau}-\frac{1}{Re_2\cdot\varepsilon}\cdot\frac{\partial^2 U_2}{\partial X^2}+\frac{\partial H_1}{\partial X}+\frac{\partial H_2}{\partial X}=0 \tag{2.16}$$

$$\frac{\partial U_2}{\partial X}+\frac{\partial V_2}{\partial Y}=0 \tag{2.17}$$

$$\frac{\partial\left(H_1+H_2\right)}{\partial\tau}=V_2\left(X,H_1+H_2,\tau\right) \tag{2.18}$$

$$\frac{\partial U_2\left(X,H_1+H_2,\tau\right)}{\partial Y}=0 \tag{2.19}$$

Consider a plane harmonic in time wave:

$$U_i=U_i^0\left(Y\right)\exp\left[i\left(k_*X+\omega\tau\right)\right], V_i=V_i^0\left(Y\right)\exp\left[i\left(k_*X+\omega\tau\right)\right], \tag{2.20}$$

$$H_i = H_i^0 \exp\left[i\left(k_* X + \omega\tau\right)\right],\ i = 1, 2, \tag{2.21}$$

when $\omega_* = \omega$ a real positive number, $(\omega > 0, \omega_{**} = 0)$ that is, when we seek the solution of the system (2.10) - (2.17) in the form of the real part of the complex expressions (2.18).After substituting (2.18) into the system of equations and the boundary conditions, we obtain the required dispersion equation relating the wave k_* to the frequency ω

$$\left(\frac{k_*}{\omega}\right)^4 \cdot D_0\psi_0 + \left(\frac{k_*}{\omega}\right)^2 \cdot \sqrt{\mu} a\varphi_0 \left(D_0 - \psi_0\right) - \mu a^2 \varphi_0^2 = 0 \tag{2.22}$$

$$a = \chi\left(1+i\right), \chi = \sqrt{\frac{\omega Re_2 \varepsilon}{2}}, \varphi_0 = \sqrt{\mu} \cdot sh\left[\sqrt{\mu} a k_0\right] \cdot sh\left[a\left(1-k_0\right)\right] +$$
$$+ch\left[\sqrt{\mu} a k_0\right] \cdot ch\left[a\left(1-k_0\right)\right], \tag{2.23}$$

$$\psi_0 = e^{\sqrt{\mu} a k_0} \cdot \left\{ch\left[a\left(1-k_0\right)\right] + \sqrt{\mu} sh\left[a\left(1-k_0\right)\right]\right\} -$$
$$\varphi_0\left(1+\sqrt{\mu} a\right),\ k_0 = h_{10}/h_0 \tag{2.24}$$

$$D_0 = \left(1 - \frac{1}{\rho}\right) \cdot \left\{2\sqrt{\mu} sh\left[a\left(1-k_0\right)\right] \cdot \left[1 - ch\left(\sqrt{\mu} a k_0\right)\right] +\right.$$
$$\left.\sqrt{\mu} a k_0 \varphi_0 - sh\left(\sqrt{\mu} a k_0\right) \cdot ch\left[a\left(1-k_0\right)\right]\right\} \tag{2.25}$$

3 Solution of the dispersion equation. Investigation of the structure of pulsations.

The dispersion equation (2.19) in special cases, when $k_0 = 0$ or, $\rho = 1, \mu = 1, 0 < k_0 < 1$ (a two-layer liquid consisting of two identical liquids) is a dispersion equation for a single-layer liquid. In these particular cases, the fourth-order equation becomes square (quadratic)

In contrast to the single-layer flow, in the case of a two-layer flow, we obtain a fourth-order dispersion equation that has solutions:

$$\left(\frac{k_*}{\omega}\right)_I^2 = \frac{\sqrt{\mu} a \varphi_0}{D_0}$$
$$\left(\frac{k_*}{\omega}\right)_{II}^2 = -\frac{\sqrt{\mu} a \varphi_0}{\psi_0} \tag{3.1}$$

Thus, the theory predicts the existence of two types of waves. The first wave, denoted by the subscript below, propagates at the interface between the liquids. The second wave, denoted by the subscript below, extends to the free surface of the stratified fluid. For the numerical calculation, water ($\nu = 1.31 * 10^{-6} <^2 /A$) was chosen as a viscous liquid. The width of the canal $h_0 = 0.1 <$, $\varepsilon = 0, 1$, $Re_2 = 2400$, $\mu = \mu_2/\mu_1 = 4$ (the more viscous liquid is, the more on the upper layer it will be located) $\rho = \rho_1/\rho_2 = 2$, $k_0 = 0, 5$,. Pic. 2 shows the dependences of the logarithmic damping decrement Λ (a), phase velocity v_D (b), and the linear damping decrement $\sigma = k_{**}$on the cyclic frequency ω (c).

The behavior of the curves allows us to draw the following conclusions about the nature of the two-layer motion. Regardless of the frequency of the perturbations, because of the frictional forces between liquids having different viscosities, the internal waves propagate with a lower phase velocity compared to the waves on the free surface. Moreover, the damping of internal waves is more intense than surface damping. As the frequency increases, this difference becomes insignificant for damping decrements Λ , while it increases for linear decrements of wave σ, and the phase velocities reach asymptotic values.

The results of calculations for various ratios of the densities of layers of a two-layer liquid, shown in pic. 3 (a) $(\mu = 2; k_0 = 0,5)$showed that the parameter ρ has the greatest influence on the phase velocity. The damping decrement Λ is practically independent from ρ. In this case, the density ratio has practically no effect on the characteristics of the second (surface) wave.

An increase in the density of the lower layer, of a two-layer liquid leads to an increase in the phase velocity.

Calculations for different viscosities of layers of a two-layer liquid in pic. 3 (b) $(\rho = 1,5; k_0 = 0,5)$ showed that the parameter μ has the greatest influence on the phase velocity and the damping decrement Λ of the first wave (pic. 4,) $\rho = 1,5 : k_0 = 0,5$. In this case, the viscosity ratio has practically no effect on the characteristics of the second (surface) wave. With increasing viscosity of the upper layer of the liquid, the phase velocity of internal waves increases, and the damping decrements decreases.

Figure 5 shows the dependence of the characteristics of the waves on the cyclic frequency ω for different depths of layers of a two-layer liquid $(\rho = 1,5 : \mu = 4)$.

Calculations showed that the greatest influence of the parameter k_0 is on the characteristics of the first wave and it practically does not affect the characteristics of the second (surface) wave. As the depth of the heavier liquid increases, the phase velocity of the internal waves increases, and the damping decrements decrease.

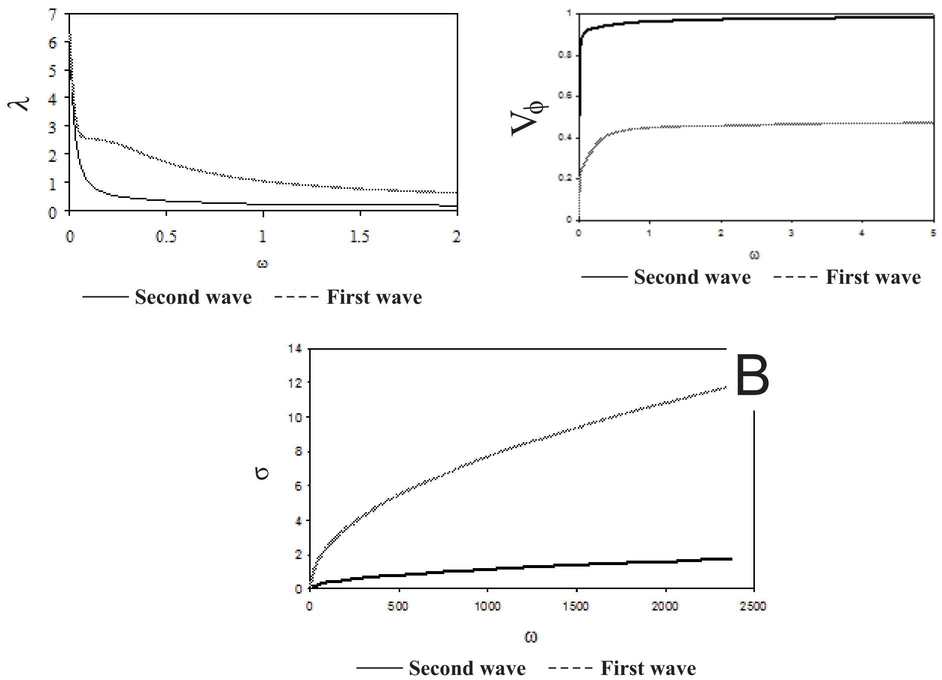

Fig. 2. Dependences of the logarithmic damping decrement (a), phase velocity (b) and linear damping decrement (c) on the cyclic frequency

The behavior of the phase velocities at high frequencies can be estimated using the following asymptotics. In case $|a| >> 1$ from formulas (2.20) and (2.21) we obtain: $(v_D)_I \approx \sqrt{k_0 \left(1 - \frac{1}{\rho}\right)}\ (v_D)_{II} \approx 1$

Thus, it follows from the asymptotics obtained that the value of the phase velocity of the internal wave at high frequencies does not depend on the viscosities of the two-layer liquid, but is completely determined by the ratio of the densities and the depth of the first layer. The value of the phase velocity of the surface wave is determined only by the depth of the canal.

Both relations are consistent with the results of the numerical calculations shown in pic.2.3.5.

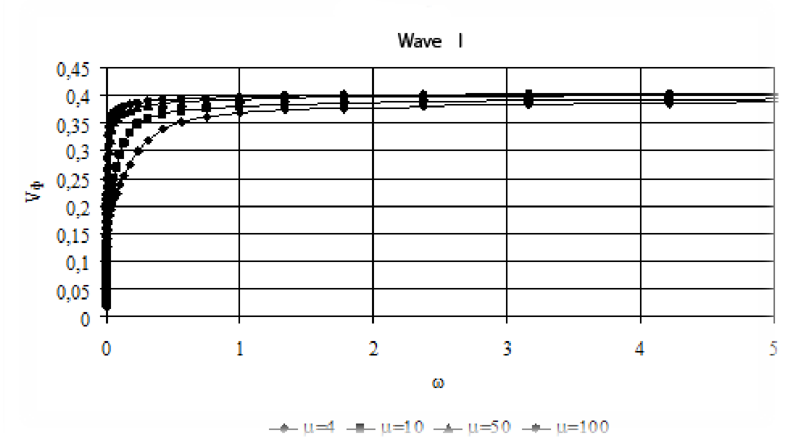

Fig. 3. Dependence of the phase velocity on the frequency for different values of the ratio of the densities (a) and viscosities (b) of a two-layer liquid

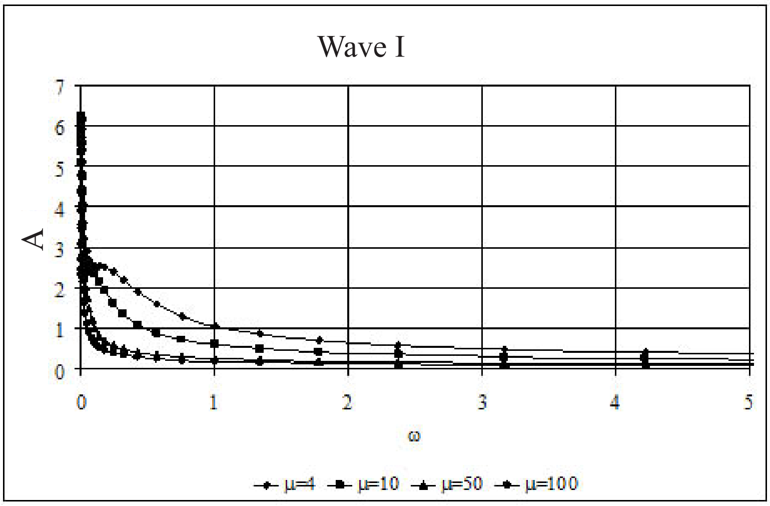

Fig. 4. Dependence of the logarithmic decrement of the wave damping on the cyclic frequency for different values of the viscosity ratio of a two-layer fluid

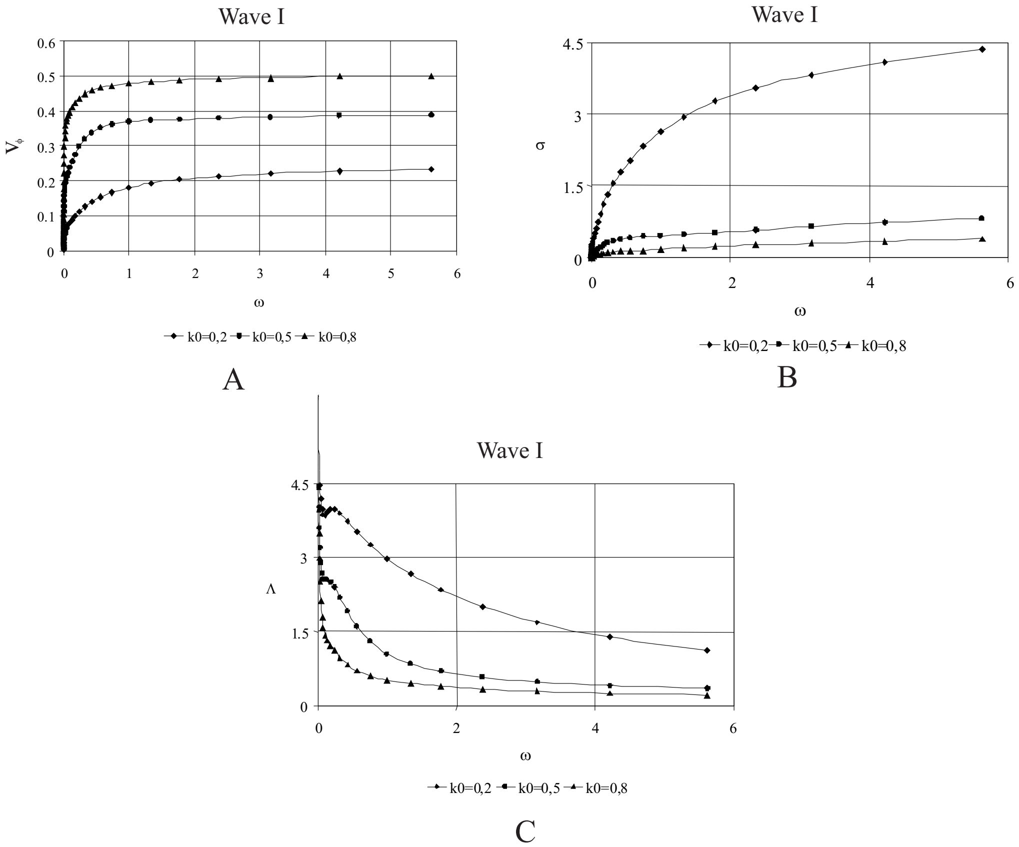

Fig. 5. Dependences of the phase velocity (a), the linear decrement of damping (b), and the logarithmic decrement of damping (c) on the cyclic frequency

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