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Limit state of a cylindrical shell under the action of nouniform external pressure

Magomed F. Mekhtiyev · Nina I. Fomina · Laura F. Fatullayeva

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Abstract. *The goal of the paper is to study of limit state of a cylindrical shell subjected to non-uniform external pressure and made of elastic and non-linear elastic material. In this case, it is very significant to calculate the critical buckling force depending on the parameter that characterizes non-uniformity of external pressure. The solution of the stated problem is based on the mixed type variational method in combination with the Rayleigh-Rits method.*
The values of critical strength of stability at elasticity and nonlinear - elasticity of the material we renumerically calculated and influence of nonlinearity index and the parameter characterizing non uniformity of compressible pressure was revealed.

Keywords. cylindrical shell · nonlinear elasticity · critical force · variation method.

Mathematics Subject Classification (2010): 35Q74

1 Introduction and statement of the problem

The problems of stability of cylindrical shells have the greatest practical significance among the problems of stability of thin elastic and nonlinear-elastic shells. This is due to a wide spread and use of compressible structural elements in various areas of engineering and in construction.

Relevance and importance of such problems are material saving with simultaneous increase of load bearing capacity of the construction.

In the considered work we study loss of stability of a long cylindrical shell of radius R and thickness $2h$. It is accepted that the shell is compressed by a nonuniformly distributed radial load that varies in size and direction according to the law

$$q = q_0 \left(1 + \lambda \sin^2 \varphi\right), \tag{1.1}$$

M.F. Mehtiyev, N.I. Fomina, L.F. Fatullayeva,
Baku State University, Azerbaijan,
23, Z.I.Khalilov str., Baku AZ 1148, Azerbaijan
E-mail: laura_far@rambler.ru

where the parameter λ characterizes not hydrostatic character of the compressible ($\lambda > 0$). Such simplification peruses a goal to get apparent dependence that allows to reveal both qualitatively and quantitatively the influence of the parameter λ on the critical force of stability.

Here, considering q_0 as a load control parameter we will study the process of bulged ring as q_0 increases from zero value.

Neglecting the influence of fixing the ends , the initial problem is reduced to analysis of loss of stability of load bearing capacity of a unit width ring isolated from this shell. Then we accept the simplest geometrical theory at which the tangential stress can be neglected, to take into account nonlinearity of only the defection n and to consider the inequality $w/R << 1$.

2 Solution of the stated problem

Because of mathematical complexity associated with the need of integration of nonlinear boundary value problem with variable coefficients, here the approximate solution is realized by means of mixed type variation method [1]. The advantage of this approach is the ability to determine the critical load not solving Euler's differential equations that in the case under consideration are essentially linear [2].

As a consideration are essentially linear [3]

$$\varepsilon = \varepsilon_0 + kz. \tag{2.1}$$

Taking into consideration the polar coordinates $(z\,,\ \varphi)$, the value ε_0 and flexure κ by the formulas of nonlinear theory of thin shells we determine

$$\varepsilon_0 = \frac{w}{R} + \frac{1}{2R^2}\left(\frac{\partial w}{\partial \varphi}\right)^2, \kappa = -\frac{1}{R^2}\left(\frac{\partial^2 w}{\partial \varphi^2}\right).$$

The equation of the state of body under elasticity has the form

$$e^v = \frac{\sigma}{E}.$$

Then the expression of the functional has the following form under elasticity [3]:

$$K_{non/el} = R\int\limits_{-h}^{h}\int_0^{2\pi}\left\{\dot{\sigma}\dot{\varepsilon} + \frac{\sigma}{2R^2}\left(\frac{\partial \dot{w}}{\partial \varphi}\right)^2\right\}d\varphi dz$$

$$-\frac{R}{2}\int\limits_{-h}^{h}\int\limits_{0}^{2\pi}\frac{\dot{\sigma}^2}{E}d\varphi dz + R\int\limits_{0}^{2\pi}\dot{q}\dot{w}d\varphi. \tag{2.2}$$

In different constructions, thin walled rings whose material has non linear elasticity are often used as load-bearing elements. Based on this model we can describe the behavior of long cylindrical shells in the area where the influence of fastening may be neglected. In this case we write the equation of the state by the equality

$$e^v = \frac{\sigma}{E}\left\{1 + \left(\frac{\sigma}{\sigma_0}\right)^n\right\}, \tag{2.3}$$

which represents the dependence "stress-starin" for many structural materials for engineering purposes accuracy.

Here σ is stress , E and σ_0 are modulus of elasticity and limit of proportionality, respectively, $n$is nonlinearity index. If n is even, then dependence (2.3) is valid both for compression and tension. Note that for $n = 2$ equation (2.3) is a sufficiently good approximation of elasticity theory for reinforced plastics, aluminum alloy, duralumin. The value $n = 4$ corresponds to diagram of linear hardening. For sufficiently large n, we relation (2.3) approximately describes the law of ideal plasticity (Prandtl diagram).

Taking into account formula (2.3), the expression for the corresponding functional with nonlinear elasticity of the material is written as follows [3]:

$$K_{nl/el} = R\int_{-h}^{h}\int_{0}^{2\pi}\left\{\dot{\sigma}\dot{\varepsilon} + \frac{\sigma}{2R^2}\left(\frac{\partial\dot{w}}{\partial\varphi}\right)^2\right\}d\varphi dz -$$

$$-\frac{R}{2}\int_{-h}^{h}\int_{0}^{2\pi}\left\{\frac{\dot{\sigma}^2}{E}\left[1+(n+1)\left(\frac{\sigma}{\sigma_0}\right)^n\right]\right\}d\varphi dz + R\int_{0}^{2\pi}\dot{q}\dot{w}d\varphi. \qquad (2.4)$$

Here and further, under velocity we understand differentiation with respect to the monotonically increasing parameter q_0 and formula (1.1) we get :

$$\dot{q} = 1 + \lambda\sin^2\varphi. \qquad (2.5)$$

Note that the stability process occurs in the plane of the ring and in view of the thinness of walls the circumferential stress σ varies in thickness by the linear law. Following this assumption, we accept [3]

$$\sigma = -\frac{qR}{2h} + \frac{3z}{2h^3}M \text{ or } \dot{\sigma} = -\frac{\dot{q}R}{2h} + \frac{3z}{2h^3}\dot{M}. \qquad (2.6)$$

For finding the stationary value of functionals (2.2) and (2.4) we apply the Rayleigh-Rits method. To this end as approximating functions we assume

$$w = w_0(q) + w_1(q)\cos 2\varphi\,, \quad M = m(q)\cos 2\varphi \qquad (2.7)$$

or in velocities

$$\dot{w} = \dot{w}_0 + \dot{w}_1\cos 2\varphi, \dot{M} = \dot{m}\cos 2\varphi. \qquad (2.8)$$

The further course of the calculations is that the expressions (2.1), (2.5)-(2.8) are substituted in expressions (2.2) and (2.4). Thus, we get analytic formulas for the functional, the functionals K_{yn} and $K_{H/yn}$ are found as the functions w_0, w_1, m and their derivatives with respect to q_0. Then, the solution can be carried out from the stationary condition $\delta K = 0$ with an additional requirement $\frac{dq_0}{dw} = 0$. The functional (2.2) and (2.4) vary in $\dot{w}_0, \dot{w}_1$ and $\dot{m}$, i.e. having equated

$$\frac{\partial K}{\partial\dot{w}_0} = 0, \qquad \frac{\partial K}{\partial\dot{w}_1} = 0, \qquad \frac{\partial K}{\partial\dot{m}} = 0,$$

as a result we get a system of equations. After a series of elementary calculations we arrive at the first order equations with respect to w_1 for the cases of elasticity and nonlinear elasticity. Introducing the following dimensionless variables

$$\xi = \frac{h}{R}, \qquad a = \frac{w_1}{h}, \qquad \gamma = \frac{E}{\sigma_0}, \qquad \tau = \frac{q_0}{E}$$

and passing to dimensionless differentiation according by the rule

$$d/dq_0 = E^{-1}d/d\tau,$$

we finally get the equation:

$$\frac{d\tau}{da} = \left[4\pi\xi^3 - \frac{3\pi}{4}(2+\lambda)\tau\right] \times \left[\frac{3\pi}{4}(2+\lambda)a\right]^{-1} \tag{2.9}$$

at elasticity ,

$$\frac{d\tau}{da} = \left[4\pi\xi^3 - \frac{3\pi}{4}(2+\lambda)\tau - \xi^{-n}\tau^{n+1}\gamma^n \sum_{p=0}^{n} \varphi_{p+2}a^p\right] \times$$

$$\times \left[\frac{3\pi}{4}(2+\lambda)a - \xi^{-n}\tau^n\gamma^n \sum_{p=0}^{n} \left(\varphi_{p+1}a^p - \varphi_{p+2}a^{p+1}\right)\right]^{-1}, \tag{2.10}$$

at nonlinear elasticity.

In (2.10) we introduced the denotation

$$\varphi_{p+i} = \frac{C_n^p(n+1)(-1)^{n-p}3^{p+i}\left[1-(-1)^{p+i+1}\right](2+\lambda)^{p+i-1}}{2^{n+p+i+1}(p+i+1)}K_{p+i},$$

where

$$K_{p+i} = \int_0^{2\pi} \left(1+\lambda\sin^2\varphi\right)^{n-p+(2-i)} \cos^{p+i} 2\varphi d\varphi,$$

$$C_n^p = \frac{n!}{p!(n-p)!} \quad (i = 1\,,\,2).$$

Equations (2.9) and (2.10) should be complemented with the initial condition

$$a(0) = w_1^0/h = a_0,$$

where w_1^0 is the given amplitude of the initial imperfection.

For $\lambda = 0$ we get known formulas for critical hydrostatic load [4].

3 Numerical analysis and conclusions

Having accepted $\xi = 10^{-1},\ \gamma = 300$, by the Runge-Kutta numerical method we solved the Cauchy problem for equations (2.9) and (2.10) for $a_0 = 10^{-1}$ and determined the values of critical forces with additional requirement $d\tau/\,da = 0$.

In Table 1 the dependences of τ_{kp} on λ are given for the case of elasticity and nonlinear elasticity for $n = 2, 4, 6$.

Based on the numerical experiments we can make the following conclusions:

1 The difference of numerical values for critical forces at elasticity is much more-greater, for example for $\lambda = 0,2$ and this is approximately 81,8%;
2 As the parameter λ increases, the critical force decreases in all cases, because parameter is λ is included in the formula (2.9) and (2.10) in the same way:

Table 3.9. Dependence of τ_{kp} on λ for $n = 2, 4, 6$.

λ	τ_{cr} (el. case)	$\tau_{cr}(n = 2)$	$\tau_{cr}(n = 4)$	$\tau_{cr}(n = 6)$
0	$0,24 \cdot 10^{-2}$	$0,44 \cdot 10^{-3}$	$0,35 \cdot 10^{-3}$	$0,32 \cdot 10^{-3}$
0,2	$0,22 \cdot 10^{-2}$	$0,4 \cdot 10^{-3}$	$0,32 \cdot 10^{-3}$	$0,29 \cdot 10^{-3}$
0,4	$0,20 \cdot 10^{-2}$	$0,37 \cdot 10^{-3}$	$0,29 \cdot 10^{-3}$	$0,27 \cdot 10^{-3}$
0,6	$0,18 \cdot 10^{-2}$	$0,34 \cdot 10^{-3}$	$0,27 \cdot 10^{-3}$	$0,26 \cdot 10^{-3}$
0,8	$0,17 \cdot 10^{-2}$	$0,32 \cdot 10^{-3}$	$0,25 \cdot 10^{-3}$	$0,24 \cdot 10^{-3}$
1	$0,16 \cdot 10^{-2}$	$0,29 \cdot 10^{-3}$	$0,24 \cdot 10^{-3}$	$0,23 \cdot 10^{-3}$

3) critical force essentially depends on the parameter λ. So that for $n = 2$ consideration of non-uniformity of external pressure reduces the critical force up to 34%;

4) with an increase in the nonlinearity, (n) the critical force decreases:

5) difference of numerical variables, for critical forces for $n = 4$ and $n = 6$is far less this for $n = 2$ and $n = 4$ that is quite understandable in describing the behavior of the diagram (1.1) for large values of nonlinearity indices.

The practical value of the results obtained, is that when simulating constructions, the value of pressure may be increased (decreased).

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