Transactions of NAS of Azerbaijan, Issue Mechanics, **37** (7), 19–26 (2017).
Series of Physical-Technical and Mathematical Sciences.

Determination of hydraulic characteristics of a pipeline for a non-stationary flow of viscous liquids

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Received: 04.04.2017 / Revised: 05.07.2017 / Accepted: 22.09.2017

Abstract. *The process of nonstationary flow of an incompressible viscous fluid through a pipeline, described by a one-dimensional equation of parabolic type, is considered. First, by integrating and replacing the variable, and then using the time discretization, the original equation is transformed to a semi-discrete equation with the corresponding conditions. Within the framework of the model obtained, an inverse problem is posed in the variational formulation for determining the hydraulic characteristics of a pipeline with an unknown boundary condition on the pipeline wall. For the numerical solution of the variational problem, a computational.*

Keywords. pipeline transport · hydraulic characteristics of the pipeline · combined inverse problem · local regularization method · difference method.

Mathematics Subject Classification (2010): 76M30

1 Introduction

In modern technology, pipelines are used to move a variety of fluids, with the smallest quantities used in laboratory equipment and instrumentation, to trunk lines. Typically, when designing pipelines, flow fluids are set which provide the main characteristics of the pipeline's performance in accordance with its purpose, and the position of the pipeline's starting and ending points. One of the main tasks is to determine the hydraulic characteristics of the pipeline, i.e. Determination of the pressure drop, flow rate for the passage of the specified flow rate of the liquid through this pipeline. Darcy-Weisbach [2], [6], [11]

$$\Delta P = \lambda \frac{\rho \overline{u}^2}{2d} l, \tag{1.1}$$

where ΔP - is the pressure drop across the pipeline the length of l, d - is the diameter of the pipeline, λ - is the coefficient of hydraulic resistance, ρ - is the density of the fluid, $\overline{u}$ - the average velocity along the pipeline section.

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Formula (1.1), as well as an explicit expression for the coefficient of hydraulic resistance for a laminar regime, can be obtained from the exact solution of the stationary flow equation for homogeneous incompressible liquids through a pipeline with the corresponding rheological laws. The so-called "sticking condition" is used as the boundary condition on the pipeline wall. Thus a known parabolic velocity profile arises in stationary flows of viscous fluids under the action of a pressure drop. However, in recent years, numerous experimental and numerical studies have shown that there is a slip condition on the solid wall of the pipeline [8], [13], [7]. Three models of interaction of liquids with a solid wall are considered in the literature, which correspond to the following boundary conditions: adhesion, slippage according to Navier's law and slippage with limited voltage [9], [3], [12]. In this regard, it is necessary to note a very important circumstance regarding the boundary condition on the pipeline wall. The fact is that the velocity of the fluid flow on the pipeline wall is not available to direct measurement and cannot be controlled. Consequently, an accurate representation of the condition on the pipeline wall is practically not possible.

Therefore, for the practice of pipeline transport, research is important to determine the hydraulic characteristics of the pipeline for non-stationary flows of transported viscous liquids with an unknown boundary condition on the pipeline wall.

In this paper, the problem of determining the hydraulic characteristics of a pipeline is represented as a combined inverse problem for the equation of the nonstationary flow of an incompressible viscous fluid in a pipeline in a variational setting.

2 Formulation of the problem

Let there be a horizontally located simple pipeline with rigid walls, length of l, radius of R, and a viscous incompressible fluid is pumped through it. It is assumed that the Oz axis is directed along the axis of the pipeline and the flow is directed along the axis of the tube so that only one of the three velocity components (u_r, u_φ, u_z) remains, $u_z \neq 0$, $u_r = 0$ and $u_\varphi = 0$. Assuming the flow of the fluid to be axisymmetric, a complete system of differential equations describing a given flow can be represented in the form [6]:

$$\frac{\partial u_z}{\partial t} + u_z \frac{\partial u_z}{\partial z} = \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) - \frac{1}{\rho}\frac{\partial P}{\partial z}, 0 < r < R, \quad 0 < t \le T,$$

$$\frac{\partial u_z}{\partial z} = 0, \quad \frac{\partial u_z}{\partial \varphi} = 0, \quad \frac{1}{\rho}\frac{\partial P}{\partial r} = 0, \quad \frac{1}{\rho}\frac{\partial P}{\partial \varphi} = 0, \tag{2.1}$$

where P- is the pressure, ν– is the kinematic viscosity of the liquid. From the second and third equations of system (2.1) it follows that u_z represents a function only r and t, and of the last two, independence of the pressure P of r and φ, i.e.

$$u_z = u_z(r,t), P = P(z,t).$$

Then from the system (2.1) we arrive at the following form of the equation of the nonstationary flow of a viscous incompressible fluid through a pipeline

$$\frac{\partial u_z}{\partial t} - \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) = -\frac{1}{\rho}\frac{\partial P}{\partial z}$$

Let us pay attention to the following singularity of the last equation: the left-hand side of it does not depend on z, and the right-hand side does not depend on r. This is possible only if $\frac{\partial P}{\partial z}$ is a function of time. Assuming

$$u(r,t) = u_z(r,t), \ -\frac{\partial P}{\partial z} = \frac{\Delta P\,(t)}{l},$$

the mathematical model of the flow of a viscous incompressible fluid in a pipeline can be written in the form

$$\frac{\partial u}{\partial t}-\frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)=\frac{1}{\rho}\frac{\Delta P\left(t\right)}{l},0<r<R,\quad 0<t\leq t_1. \qquad (2.2)$$

The initial and boundary conditions for equation (2.2) can be represented in the form

$$u\left.\right|_{t=0}=\varphi(r), \qquad (2.3)$$

$$\frac{\partial u}{\partial r}\left.\right|_{r=0}=0, \qquad (2.4)$$

$$u\left.\right|_{r=R}=\eta(t). \qquad (2.5)$$

The boundary condition (2.4) is equivalent to the boundedness condition of the solution of equation (2.2) while $r=0$.

Obviously, knowing the laws of changing the pressure drop $\Delta P(t)$ and the velocity of the fluid flow on the pipeline wall $\eta(t)$ in time, while solving problem (2.2) - (2.5), one can find the law of variation of the volumetric flow of liquid in time through the pipeline

$$Q(t)=\int_0^R 2\pi r u\,dr. \qquad (2.6)$$

Equation (2.2), conditions (2.3) - (2.5), and also ratio (2.6) can be represented in dimensionless form. Let us introduce the following dimensionless variables

$$\overline{r}=\frac{r}{R},\ \overline{t}=\frac{t}{t^*},\ \overline{u}=\frac{u}{u^*},\ \ \overline{Q}=\frac{Q}{Q^*},\quad \overline{\Delta P}=\frac{\Delta P}{P^*},\ \bar{\varphi}=\frac{\varphi}{u^*},\ \overline{\eta}=\frac{\eta}{u^*}$$

where $u^*,\ t^*,\ P^*,\ Q^*$ are the characteristics dimensional quantities.
Omitting the dashes over the dimensionless variables of equation (2.2) and condition (2.3) - (2.5), we write in the form

$$\frac{\partial u}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{R^2P^*}{\nu\rho u^* l}\Delta P(t),0<r<1,\quad 0<t\leq T, \qquad (2.7)$$

$$u(r,0)=\varphi(r), \qquad (2.8)$$

$$\frac{\partial u}{\partial r}\left.\right|_{r=0}=0, \qquad (2.9)$$

$$u\left.\right|_{r=1}=\eta(t). \qquad (2.10)$$

In this case ratio (2.6) takes the form

$$Q=\int\limits_0^1 r u\,dr. \qquad (2.11)$$

We multiply both sides of equation (2.7) by r and integrate the result on the interval $[0,r]$ with respect to the variable r. Integrating by parts and taking into account condition (2.9), we obtain

$$\frac{\partial}{\partial t}\int_0^r u\xi d\xi=r\frac{\partial u}{\partial r}+\frac{r^2R^2P^*}{\nu\rho u^* l}\Delta P(t)$$

Denoting

$$\int_0^r u\xi d\xi=w(r,t), \qquad (2.12)$$

The last integral relation is written in the form

$$\frac{\partial w}{\partial t}=\frac{\partial^2 w}{\partial r^2}-\frac{1}{r}\frac{\partial w}{\partial r}+Dr^2\Delta P(t),0<r<1,\quad 0<t\leq T. \tag{2.13}$$

In this case, the initial and boundary conditions for equation (2.13) take the form

$$w\left|_{t=0}\right.=\psi(r), \tag{2.14}$$

$$w\left|_{r=0}\right.=0, \tag{2.15}$$

$$\frac{\partial w}{\partial r}\left|_{r=1}\right.=\eta(t), \tag{2.16}$$

and the integral ratio (2.11) is written in the form

$$w\left|_{r=1}\right.=Q(t), \tag{2.17}$$

where $\psi(r)=\int\limits_0^r \xi\varphi(\xi)d\xi,\quad D=\frac{r^2R^2P^*}{2\nu\rho u^*l}$.

Equation (2.13) is discretized by time t. To this end, we introduce a uniform difference grid in the $[0\leq t\leq T]$ area with respect to the variable t

$$\overline{\omega_\tau}=\left\{t_j=j\Delta t,\ \ j=\overline{0,m}\right\}$$

in $\Delta t=\frac{T}{m}$ increment. The derivative $\frac{\partial w}{\partial t}$ in equation (2.13), while $t_j,\ j=\overline{1,m}$, is approximated by the difference "backward"

$$\frac{\partial w}{\partial t}\Big|_{(r,t_j)}\approx\frac{w\left(r,t_j\right)-w\left(r,t_{j-1}\right)}{\Delta t}$$

Introducing the notation $w^j(r)\approx w\left(r,t_j\right)$, we can write the equation (2.13), conditions (2.14) - (2.16), and the additional ratio (2.17) in the form

$$\frac{w^j\left(r\right)-w^{j-1}\left(r\right)}{\Delta t}=\frac{d^2w^j}{dr^2}-\frac{1}{r}\frac{dw^j}{dr}+Dr^2\Delta P^j,0<r<1, \tag{2.18}$$

$$w^0\left(r\right)=\psi(r), \tag{2.19}$$

$$w^j\left|_{r=0}\right.=0, \tag{2.20}$$

$$\frac{dw^j}{dr}\left|_{r=1}\right.=\eta^j, \tag{2.21}$$

$$w^j\left|_{r=1}\right.=Q^j, \tag{2.22}$$

$$j=1,\ 2,\ ...,m,$$

where $Q^j=Q\left(t_j\right)$, $\Delta P^j\approx\Delta P\left(t_j\right),\quad \eta^j\approx\eta(t_j)$.

Now, on the basis of the obtained model (2.18) - (2.21), we set the following problem of determining the hydraulic characteristics of the pipeline:

Let the law of the change in the volumetric flow rate of the fluid $Q^j,j=\overline{1,m}$ be known and it is required to find a law of change in the pressure drop $\Delta P^j,j=\overline{1,m}$, that would ensure that a given flow rate of liquid is passed through the pipeline with unknown $\eta^j,j=\overline{1,m}$.

It should be noted that in [4] the problem of determining the hydraulic characteristics of a pipeline for a nonstationary flow of nonlinear viscous liquids was investigated with specifying the sticking condition on the pipeline wall.

This problem belongs to the class of combined inverse problems connected with the reconstruction of the right-hand sides of partial differential equations and boundary conditions [1], [10], [5].

3 Solution method

We formulate the inverse problem for the semi-discrete equation (2.18) as a variational problem with the use of local regularization [1]. To this end, in accordance with (2.22), we introduce a smoothing functional in the form

$$J(\Delta P^j, \eta^j) = \left[w^j\left|_{r=1}\right. - Q^j\right]^2 + \gamma_1 \left(\Delta P^j\right)^2 + \gamma_2 \left(\eta^j\right)^2, \tag{3.1}$$

where γ_1, γ_2- are the regularization parameters.

Thus, at each time layer $j = 1,\ 2, ., m$, the minimum of the smoothing functional (3.1) is sought to determine the pressure drop along the length of the pipeline ΔP^j and the boundary condition on the pipeline wall η^j under conditions (2.18) - (2.21).

The solution of problem (2.18) - (2.21) on each time layer $j = 1,\ 2,\ . . ., m$ can be represented in the form

$$w^j\left(r\right) = \theta^j(r) + \Delta P^j \phi\left(r\right) + \eta^j \lambda(r), \tag{3.2}$$

where $\theta^j(r),\ \phi(r),\ \lambda(r)$ are unknown functions. Substituting (3.2) into equation (2.18), we have

$$\frac{\theta^j\left(r\right) + \Delta P^j \phi\left(r\right) + \eta^j \lambda(r) - w^{j-1}\left(r\right)}{\Delta t} = \frac{d^2\theta^j}{dr^2} - \frac{1}{r}\frac{d\theta^j}{dr} +$$
$$+\Delta P^j \frac{d^2\phi}{dr^2} - \Delta P^j \frac{1}{r}\frac{d\phi}{dr} + Dr^2 \Delta P^j + \eta^j \frac{d^2\lambda}{dr^2} - \eta^j \frac{1}{r}\frac{d\lambda}{dr}.$$

Hence we obtain the following boundary-value problems with respect to unknown functions $\theta^j(r),\quad \phi(r),\quad \lambda(r)$:

$$\begin{gathered}\frac{\theta^j\left(r\right) - w^{j-1}\left(r\right)}{\Delta t} = \frac{d^2\theta^j}{dr^2} - \frac{1}{r}\frac{d\theta^j}{dr},\\ \theta^j\left|_{r=0}\right. = 0,\\ \frac{d\theta^j}{dr}\left|_{r=1}\right. = 0.\end{gathered} \tag{3.3}$$

$$\begin{gathered}\frac{\phi\left(r\right)}{\Delta t} = \frac{d^2\phi}{dr^2} - \frac{1}{r}\frac{d\phi}{dr} + Dr^2,\\ \phi\left|_{r=0}\right. = 0,\\ \frac{d\phi}{dr}\left|_{r=1}\right. = 0.\end{gathered} \tag{3.4}$$

$$\begin{gathered}\frac{\lambda\left(r\right)}{\Delta t} = \frac{d^2\lambda}{dr^2} - \frac{1}{r}\frac{d\lambda}{dr},\\ \lambda\left|_{r=0}\right. = 0,\\ \frac{d\lambda}{dr}\left|_{r=1}\right. = 1.\\ j = 1, 2, ..., m.\end{gathered} \tag{3.5}$$

Substituting (3.2) into the functional (3.1), we have

$$J(\Delta P^j, \eta^j) = \left[\theta^j(r)\left|_{r=1}\right. + \Delta P^j \phi\left(r\right)\left|_{r=1}\right. +\right.$$
$$\left.+\eta^j \lambda(r)\left|_{r=1}\right. - Q^j\right]^2 + \gamma_1 \left(\Delta P^j\right)^2 + \gamma_2 \left(\eta^j\right)^2$$

The minimum of this functional is achieved when

$$\left[\theta^j(r)\,|_{r=1} + \Delta P^j \phi\,(r)\,|_{r=1} + \eta^j \lambda(r)\,|_{r=1} - Q^j\right]\ \phi\,(r)\,|_{r=1} + \gamma_1 \Delta P^j = 0,$$

$$\left[\theta^j(r)\,|_{r=1} + \Delta P^j \phi\,(r)\,|_{r=1} + \eta^j \lambda(r)\,|_{r=1} - Q^j\right]\ \lambda(r)\,|_{r=1} + \gamma_2 \eta^j = 0.$$

From the resulting system of equations, we can determine ΔP^j and η^j

$$\Delta P^j = \frac{\gamma_2\left(Q^j - \theta^j(r)\,|_{r=1}\right)\phi\,(r)\,|_{r=1}}{\gamma_2(\phi\,(r)\,|_{r=1})^2 + \gamma_1(\lambda\,(r)\,|_{r=1})^2 + \gamma_1\gamma_2}, \tag{3.6}$$

$$\eta^j = \frac{\gamma_1\left(Q^j - \theta^j(r)\,|_{r=1}\right)\lambda\,(r)\,|_{r=1}}{\gamma_2(\phi\,(r)\,|_{r=1})^2 + \gamma_1(\lambda\,(r)\,|_{r=1})^2 + \gamma_1\gamma_2}. \tag{3.7}$$

Thus, the proposed method of local regularization for the solution of the inverse problem (2.18) - (2.21), (3.1) by defining ΔP^j and η^j, $j = 1, 2, ..., m$ based on the solution of the direct problems (3.3), (3.4), (3.5), calculation of ΔP^j and η^j according to (3.6), (3.7) and using the representation (3.2) for the solution of problem (2.18) - (2.21).

For the numerical solution of problems (3.3), (3.4) and (3.5), the finite difference method can be used. We introduce a uniform difference grid in the $[0 \le r \le 1]$ area with respect to the variable r

$$\overline{\omega_h} = \left\{r_i = i\Delta r,\ \ i = \overline{0, n}\right\}$$

with $\Delta r = \frac{1}{n}$ step. Applying the integral method, the discrete analogs of problems (3.3), (3.4) and (3.5) on the difference grid $\overline{\omega_h}$ can be represented as

$$\begin{gathered}
\frac{\theta_i^j - w_i^{j-1}}{\Delta t} = \frac{\theta_{i+1}^j - 2\theta_i^j + \theta_{i-1}^j}{\Delta r^2} - \frac{1}{r_i}\frac{\theta_i^j - \theta_{i-1}^j}{\Delta r}, \quad i = 1, 2, 3, ..., n-1, \\
\theta_0^j = 0, \\
\frac{\theta_n^j - \theta_{n-1}^j}{\Delta r} = 0,
\end{gathered} \tag{3.8}$$

$$\begin{gathered}
\frac{\phi_i}{\Delta t} = \frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{\Delta r^2} - \frac{1}{r_i}\frac{\phi_i - \phi_{i-1}}{\Delta r} + Dr_i^2,\ \ i = 1, 2, 3, ..., n-1, \\
\phi_0 = 0, \\
\frac{\phi_n - \phi_{n-1}}{\Delta r} = 0,
\end{gathered} \tag{3.9}$$

$$\begin{gathered}
\frac{\lambda_i}{\Delta t} = \frac{\lambda_{i+1} - 2\lambda_i + \lambda_{i-1}}{\Delta r^2} - \frac{1}{r_i}\frac{\lambda_i - \lambda_{i-1}}{\Delta r}, \quad i = 1, 2, 3, ..., n-1, \\
\lambda_0 = 0, \\
\frac{\lambda_n - \lambda_{n-1}}{\Delta r} = 1, \\
j = 1, 2, ..., m,
\end{gathered} \tag{3.10}$$

where $w_i^j \approx w^j(r_i), \quad \theta_i^{j-1} \approx \theta^j(r_i), \ \ \lambda_i \approx \lambda(r_i), \ \ \phi_i \approx \phi(r_i)\,.$

The resulting difference problems (3.8), (3.9), and (3.10) are a linear system of algebraic equations with a tridiagonal matrix in which the approximate values of the unknown functions $\theta^j(r), \phi(r), \lambda(r)$ in the inner nodes of the difference grid, i.e. $\theta_i^j, \phi_i, \ \ \lambda_i, i = \overline{1, n-1}$. The solutions of the difference problems (3.8), (3.9) and (3.10) can be found by the usual method of sweeping.

4 Results of numerical calculations

On the basis of the proposed numerical algorithm, numerical experiments were carried out for model problems. The numerical experiment is carried out according to the following scheme: for given functions $\psi(r)$, $\Delta P(t)$ and $\eta(t)$, the direct problem (2.13) - (2.16) is solved. To this end, we construct a discrete analogue of the problem (2.13) - (2.16) on the difference grid $\overline{\omega_\tau} \times \overline{\omega_h}$

$$\frac{w_i^j - w_i^{j-1}}{\Delta t} = \frac{w_{i+1}^j - 2w_i^j + w_{i-1}^j}{\Delta r^2} - \frac{1}{r_i}\frac{w_i^j - w_{i-1}^j}{\Delta r} + Dr_i^2 \Delta P^j,$$

$$i = 1,2,3,...,n-1,\ j = 1,2,3,...,m,$$

$$w_i^0 = \psi_i, i = \overline{0,n},$$

$$w_0^j = 0,$$

$$\frac{w_n^j - w_{n-1}^j}{\Delta r} = 0$$

and the resulting system of difference equations is solved by the sweep method. The dependence found $Q^j = w_n^j$, $j = 1,2,...,m$ is taken as the exact data for recovery of ΔP^j and η^j, $j = 1,2,...,m$.

The results of the numerical experiment carried out for the case of $\mu = 10^{-3}$ Pa's; $\rho = 1000kg/m^3$; $R = 0.6$ m; $\psi(r) = 0$; $\Delta P(t) = 1 - 0.2\sin 3t$; $\eta(t) = 0.02$; $\Delta t = 0.002$; $\Delta r = 0.05$; $\gamma_1 = 0.08$; $\gamma_2 = 0.008$ are presented in the table; ΔP^t, η^t are the exact values of the functions $\Delta P(t)$ and $\eta(t)$, $\Delta\tilde{P}$, $\tilde{\eta}$ are the calculated values of $\Delta P(t)$ and $\eta(t)$ respectively.

An analysis of the results of a numerical experiment indicates that the proposed computational algorithm allows us to restore the required functions with a sufficiently high accuracy.

Table.

t	ΔP^t	$\Delta\tilde{P}$	η^t	$\tilde{\eta}$
0.5	0.8005	0.8005	0.02	0.1997
1.0	0.9718	0.9718	0.02	0.0210
1.5	1.1955	1.1955	0.02	0.0202
2.0	1.0559	1.0559	0.02	0.0203
2.5	0.8124	0.8124	0.02	0.0200
3.0	0.9176	0.9176	0.02	0.0210
3.5	1.1759	1.1759	0.02	0.0209
4.0	1.1073	1.1073	0.02	0.0203
4.5	0.8392	0.8392	0.02	0.0204
5.0	0.8699	0.8699	0.02	0.0210
5.5	1.1424	1.1424	0.02	0.0208
6.0	1.1502	1.1502	0.02	0.0202
6.5	0.8789	0.8789	0.02	0.0207
7.0	0.8327	0.8327	0.02	0.0205
7.5	1.0974	1.0974	0.02	0.0207
8.0	1.1811	1.1811	0.02	0.0209
8.5	0.9282	0.9282	0.02	0.0212
9.0	0.8087	0.8087	0.02	0.0199
9.5	1.0447	1.0447	0.02	0.0205
10	1.1976	1.1976	0.02	0.0209

5 Conclusion

The inverse problem connected with the determination of the hydraulic characteristics of the pipeline for a non-stationary flow of a viscous fluid with an unknown boundary condition on the pipeline wall is considered. The computational algorithm of the problem is based on the use of the variational formulation of the inverse problem with local regularization. Unlike the global regularization method, where the solution of the inverse problem is determined at all times simultaneously, the proposed approach takes into account the specifics of the inverse problem and the solution is determined successively at individual instants of time. An analysis of the results of a numerical experiment indicates that the proposed computational algorithm ensures the stability of the solution to the errors of the input data.

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