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Physical nonlinear elastic deformations of smooth ring

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Abstract. *Smooth rings as flexible elements of constructions are widely used in practice. In their analysis and statically unsolved systems encountered in joint operation with other structural elements, problems on definition of displacements for composing discontinuities of deformations arise.*

Keywords. In the mentioned cases · the effective method for definition of displacements of a ring is a method based on application of an elastic line to differential equation.

Mathematics Subject Classification (2010): 74D05 ·

1 Introduction

Assume that the loads extended in radial and peripheral directions and with intensity q_1 and q_2 act on an annular ring of average radius r and with the cross-section sizes of $b \cdot h$ (Fig.1). Lets separate an elementary part $cd = r \cdot d\varphi$ by radially cutting (Fig. 1) and show external and internal forces (N is a normal force, Q_z is a cutting force, M_y is bending moment) acting on it (Fig 2., a).

Compose balance equations of the separated element

$$\sum R = 0;\ \ dQ_z \cdot \cos\frac{d\varphi}{2} - N \cdot \sin\frac{d\varphi}{2} \cdot 2 + q_1 \cdot r d\varphi = 0; \tag{1.1}$$

$$\sum T = 0;\ \ -dN \cdot \cos\frac{d\varphi}{2} - Q_z \cdot \sin\frac{d\varphi}{2} \cdot 2 + q_2 \cdot r d\varphi = 0; \tag{1.2}$$

$$\sum M_{0_1} = 0;\ dM_y - Q_z \cdot r \cdot \frac{d\varphi}{2} \cdot 2 = 0. \tag{1.3}$$

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By composing equations above the second order quantities (their product) were small, they are neglected. In equations (1.1) and (1.2), taking into account substitutions $\cos\frac{d\varphi}{2} \approx$, $\sin\frac{d\varphi}{2} \approx \frac{d\varphi}{2}$ and simplyfing equation (1.3) we reduce to the following form:

$$\frac{dQ_z}{d\varphi} - N = -q_1 \cdot r; \ \ \frac{dN}{d\varphi} - Q_z = -q_2 \cdot r;$$

$$\frac{1}{r}\frac{dM_y}{d\varphi} - Q_z = 0. \tag{1.4}$$

Inputing N from the first equation, cutting force Q_z from the third equation of (1.4) in the second equation, we obtain a differential equation expressed bu the bending moment M_y:

$$\frac{1}{r^2}\frac{d^3M_y}{d\varphi^3} + \frac{1}{r^2}\frac{dM_y}{d\varphi} + \frac{dq_1}{d\varphi} + q_2 = 0. \tag{1.5}$$

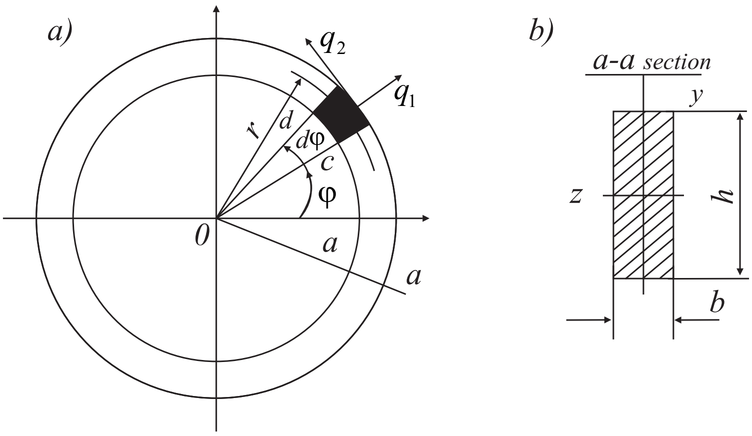

Fig.1.

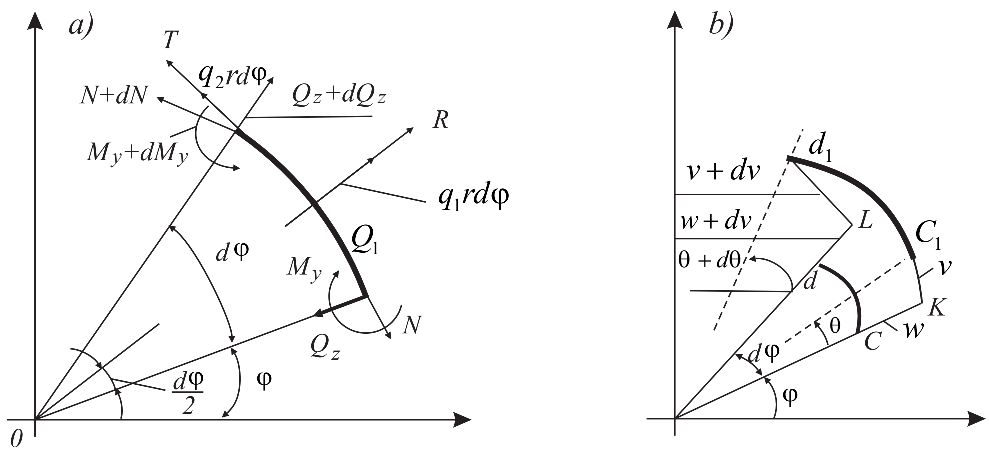

Fig. 2.

2 Problem formulation and discussion.

External and internal force factors of the smooth ring act on the same surface. Therefore, their replacements occur at this surface.

Assume that deformations of the ring in peripherals direction are negligible.

Let's now consider displacements and deformations of the elementary part cd of the ring for that comparision of the states of the element cd before and after deformation (Fig.2,b). Replace radial displacement of the point c on the surface of the ring by w tangential displacement by v angular displacement (turning angle) passing from this point by θ. Then displacement of the cut d get gains increments and become $w + dw$,$v + dv$ and $\theta + d\theta$, respectively (Fig. 2 b). Relationship between the displacements w, v and θ are expressed as in following expressions [1]:

$$\theta = \frac{v}{r} - \frac{1}{r}\frac{dw}{d\varphi}; \;\; w = -\frac{dv}{d\varphi}. \tag{2.1}$$

The change in the curvature of infinitesimal elements cd equals the derivative of turning angle θ with respect to the arc c_1d_1:

$$\chi = \frac{1}{r}\frac{d\theta}{d\varphi} \tag{2.2}$$

Inputing the first expression of (2.1) in (2.2), we get:

$$\chi = \frac{1}{r^2}\frac{dv}{d\varphi} - \frac{1}{r}\frac{d^2w}{d\varphi^2} \tag{2.3}$$

Taking into account the second expression of (2.1), expression (2.3) takes transforms into:

$$\chi = \frac{1}{r^2}\left(\frac{d^2w}{d\varphi^2} + w\right) \tag{2.4}$$

ρ is the curvature radius of ring's neutral layer. Then $\chi = \frac{1}{p}$ is the relative circular linear deformation of the ring located at the distance z from the circular neutral layer

$$\varepsilon_\varphi = \frac{z}{\rho} = z\chi \tag{2.5}$$

or taking into consideration (2.4),

$$\varepsilon_\varphi = \frac{z}{\rho} = -\frac{z}{r^2}\left(\frac{d^2w}{d\varphi^2} + w\right) \tag{2.6}$$

Assuming that the ring's material is nonlinear elastic, we take the relationship between deformations and stresses as following [2]:

$$\sigma_\varphi = E_0\varepsilon_\varphi - E_1\varepsilon_\varphi^3 \tag{2.7}$$

Here σ_φ is the circular normal stress of the ring, ε_φ is circular relative deformation, $*$ are elasticity constants of the material. Having written expression (2.6) in (2.7), we get:

$$\varepsilon_\varphi = E_0\frac{z}{r^2}\left(\frac{d^2w}{d\varphi^2} + w\right) + E_1\frac{z^3}{r^6}\left(\frac{d^2w}{d\varphi^2} + w\right)^3 \tag{2.8}$$

Compose the bending force in the cross section of the ring (2.9):

$$M_y = \int_A \sigma_\varphi z \cdot dA \tag{2.9}$$

Having written expression (2.8) in (2.9), we get:

$$M_y = -E_0 j_{y(3)} \cdot \frac{1}{r^2}(w'' + w) + E_1 j_{y(5)} \cdot \frac{1}{r^6}(w'' + w)^3 \tag{2.10}$$

Here $J_{y_{(3)}} = \frac{bh^3}{12}$, $J_{y_{(5)}} = \frac{bh^5}{80}$ are geometrical features of the cross section of the ring. Substituting (2.6) in (2.10), we get an equation of radial curvatures in nonlinear elastic deformations:

$$\frac{E_0 J_{y(3)}}{r^4}\left(\frac{d^5 w}{d\varphi^5} + 2\frac{d^3 w}{d\varphi^3} + \frac{dw}{d\varphi}\right) - \frac{E_1 j_{y(5)}}{r^8}\left[\frac{d^3(\phi^3)}{d\varphi^3} + \frac{d(\phi^3)}{d\varphi}\right] = \frac{dq_1}{d\varphi} + q_2 \tag{2.11}$$

Here we accept the denotation

$$\phi = \frac{d^3 w}{d\varphi^3} + w. \tag{2.12}$$

In order to solve the principle nonlinear differential equation of the problem we use the small parameters method [3]. For that we divide all terms of equation (2.12) by the expression $\frac{E_0 J_{y(3)}}{r^4}$, introduce the small parameter into the expression obtained in front of the square brackets. Therefore,

$$v = \frac{E_1 \varepsilon_s^2}{E_0} \tag{2.13}$$

we give to this equation the following form:

$$\frac{d^5 w}{d\varphi^5} + 2\frac{d^3 w}{d\varphi^3} + \frac{dw}{d\varphi} = \frac{r^4}{E_0 J_{y(3)}}\left(\frac{dq_1}{d\varphi^3} + q_2\right) +$$

$$+v\frac{J_{y_{(5)}}}{\varepsilon_s^2 J_{y_{(3)}} r^4}\left[\frac{d^3(\phi^3)}{d\varphi^3} + \frac{d(\phi^3)}{d\varphi}\right] \tag{2.14}$$

Accept the curvature function w in the form of series with respect to a small parameter:

$$w = \sum_{m=0}^{n} v^m w_m(\varphi) \tag{2.15}$$

Having substituted expression (2.16)- in nonlinear differential equation (2.17), equating separately the expressions of v different forces of the small parameter v to zero, we get the following system of recurrent linear differential equations (the first four equations of the system are presented):

$$\frac{d^5 w_0}{d\varphi^5} + 2\frac{d^5 w_0}{d\varphi^3} + \frac{dw_0}{d\varphi} = \frac{r^4}{\varepsilon_s^2 J_{y(3)}}\left(\frac{dq_1}{d\varphi^3} + q_2\right);$$

$$\frac{d^5 w_1}{d\varphi^5} + 2\frac{d^3 w_1}{d\varphi^3} + \frac{dw_1}{d\varphi} = \frac{J_{y_{(5)}}}{r^4 \varepsilon_s^2 J_{y(3)}}\left[\left(\phi_0^3\right)' + \left(\phi_0^3\right)'''\right];$$

$$\frac{d^5 w_2}{d\varphi^5} + 2\frac{d^3 w_2}{d\varphi^3} + \frac{dw_2}{d\varphi} = \frac{3J_{y_{(5)}}}{r^4 \varepsilon_s^2 J_{y(3)}}\left[\left(\phi_0^3 \phi_1\right)' + \left(\phi_0^3 \phi_1\right)'''\right];$$

$$\frac{d^5 w_3}{d\varphi^5} + 2\frac{d^3 w_3}{d\varphi^3} + \frac{dw_3}{d\varphi} = \frac{3J_{y_{(5)}}}{r^4 \varepsilon_s^2 J_{y(3)}}\left[\left(\phi_0^3 \phi_1^2 + \phi_0^3 \phi_2\right)' + \left(\phi_0 \phi_1^2 + \phi_0^3 \phi_2\right)'''\right]; \tag{2.16}$$

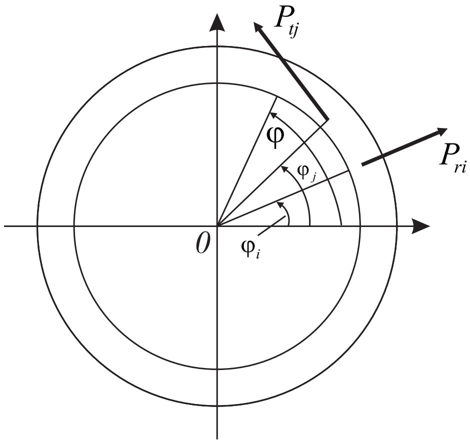

Fig. 3.

Here $\phi_m(\varphi) = w_m'' + w_m \ \ (m = 0, 1, 2, 3)$.

The solution of the last system of linear equations is made easier within appropriate boundary conditions (depending on loading and strengthening conditions of the ring). The expressions in the ring). the expressions in the right hand side of found in the form of series.

Assume, that at any ϕ_i the closed ring is being cut and P_{ri} is a radialacting force, while at the ϕ_j the concentrated tangential forces P_{tj} are applied (Fig. 3) . By expanding these in series, we find intensities of the forces $q_1 = q_r$ and $q_1 = q_t$ wirte then in the first equation of (2.16) and reduce it to the form:

$$\frac{d^5 w_0}{d\varphi^5} + 2\frac{d^3 w_3}{d\varphi^3} + \frac{dw_0}{d\varphi} = \frac{r^4}{\varepsilon_s^2 J_{y(3)}} \left[\sum_{i=1}^{n} \sum_{k=1}^{\infty} \left(-\frac{P_{r^i}}{\pi r} k \sin k(\varphi - \varphi_i) \right) + \sum_{i=1}^{n} \sum_{k=1}^{\infty} \frac{P_{r^i}}{\pi r} \cos k(\varphi - \varphi_i) \right]. \quad (2.17)$$

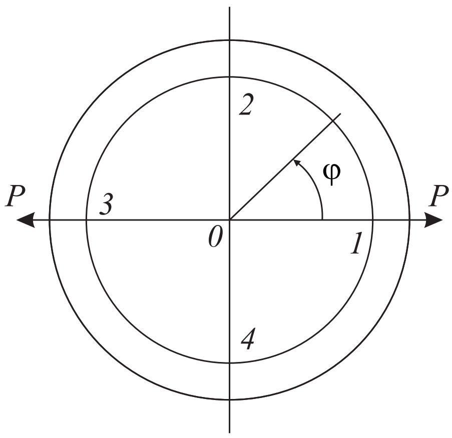

Fig. 4.

The solution of equation (2.18) is the sum of the general solution of the homogeneous equation and special solution corresponding to the function in the right hand side of the equation:

$$w_0 = c_{1(0)} \cos\varphi + c_{2(0)} \sin\varphi + \sum_{i=1}^{n} \frac{P_{r^i} r^3}{\pi \varepsilon_s^2 J_{y(3)}} \sum_{k=2}^{\infty} \frac{\cos k(\varphi - \varphi_i)}{(k^2-1)^2}$$

$$+\sum_{j=1}^{m} \frac{P_{tj} r^3}{\pi \varepsilon_s^2 J_{y(3)}} \sum_{k=2}^{\infty} \frac{\sin k(\varphi - \varphi_i)}{k\,(k^2-1)^2}. \quad (2.18)$$

Here $c_{1_{(0)}}$, $c_{2_{(0)}}$, are integral constants and may be evaluated from strengthening conditions of the ring. Other equations of the system (2.16) should be solved in this way. Here we should take into account that as the physical nonlinearity of the ring's material is a small elastic nonlinearity, and by solving the second and other differential equations of system (2.18) it suffices to retain from the series with the solution of previous equations only the term.

For example let's for the case when an annular ring is stretched by two identical forces P along horizontal diameter (Fig. 4), define radial displacement δ_{13} of applications points of the forces. The sizes of the ring $r = 5sm$, $b = 2sm$, $h = 0,5sm$, $r_0 = 5,25sm$ elastic constants of the material $E_1 = 2 \cdot 10^7 N/sm^2$,$E_0 = 1,2 \cdot 10^{10} N/sm^2$, $\varepsilon_s = 0,002$.

Calculate geometrical characteristics of the ring:

$$J_{y(3)} = \frac{bh^3}{12} = \frac{2 \cdot 0,5}{12} = 0,02085cm^4,$$

$$J_{y(5)} = \frac{bh^5}{80} = \frac{2 \cdot 0,5}{80} = 0,03125cm^6.$$

The small parameter $v = \frac{E_1 \varepsilon_s^2}{E_0} = \frac{2 \cdot 10^7 \cdot 0,02^2}{1,2 \cdot 10^{10}} = 0,24$ By calculations it was determined that radial displacement of cuts 1 and 3 of the ring

$$\delta_{13} = \delta_{13}^{(0)} + \nu \delta_{13}^{(1)} + \nu^2 \delta_{13}^{(2)} (a)$$

Here $\delta_{13}^{(0)}, \delta_{13}^{(1)}, \delta_{13}^{(2)}$ - are the values of the system (2.17) obtained from the solutions of the first, second and third equations for $P = 4000N$:

$$\delta_{13}^{(0)} = 0,164sm, \quad \delta_{13}^{(1)} = 0,04584sm, \quad \delta_{13}^{(2)} = 0,0163sm$$

Substituting the last values and $v = 0,24(a)$, we get:

$$\delta_{13} = 0,164 + 0,24 \cdot 0,1305 + 0,24^2 \cdot 0,00863 = 0,164 + 0,313 + 0,000497 = 0,1759sm$$

3 Conclusions.

It is seen that taking into account physical features of the ring's material with accepted values of force, displacements of the points 1 and 3 becomes more by 22%. At the same time, the difference between the third and second approximations in displacement δ_{13} is 1,6% and this is enables the satisfactions by the third approximation and that the convergence of series used in the paper is satisfactory.

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