Transactions of NAS of Azerbaijan, Issue Mechanics, **36** (7), 60–67 (2016).
Series of Physical-Technical and Mathematical Sciences.

Solution of contact problem for a plane weakened by a variable width slot

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Received: 02.09.2016 / Revised: 11.10.2016 / Accepted: 25.11.2016

Abstract. *By the methods of elasticity theory we construct a mathematical model of partial closure in isotropic medium of a variable width slot. It is assumed that the interaction of surfaces of the slot under the action of applied volumetric and surface loads may lead to origination of the contact zones of their surfaces. We study the case of origination of several contact areas of the slot faces. Herewith it is assumed that at some part of the contact area there arises stick of faces, in the remaining part there may happen slippage. The problem on equilibrium of a slot with partially contacting faces is reduced to the problem of linear conjugation of analytic functions. Definition of the unknown parameters characterizing the partial closure of a variable width slot is reduced to the solution of the system of singular integral equations. The contact stresses, the sizes of the contact zones were determined.*

Keywords. isotropic medium · variable width slot · body-forces · contact zones · contact stresses.

Mathematics Subject Classification (2010): 74A45, 74H10

1 Introduction

Consider the fracture of an isotropic medium weakened by a rectilinear variable width slot $h(x)$ whose surface is under the action of gas. The quasistatic deformation process is investigated. Recently, there have been published a number of papers devoted to investigation of bodies with cracks with regard to existence of cohesive forces between the faces and possibility of their contact [1-15].

As is known, to get the solution of a fracture mechanics problem with regard to contact of faces is considerably difficult. This is connected with increase of the amount of unknown parameters of the problem such as contact stresses, contact boundaries, etc. At the same time, these problems, with regard to partial contact of crack faces are of great interest by investigating the fracture of composite materials, rocks, stick-slip effect and so on. To present

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day, the problems of partial contact of faces of variable width slots have not been studied enough.

We can say that accounting of variability of the slot width at the contact of its faces has not been investigated. In the present paper we give general statement of the problem in which the variability of the slot width and friction and action of body forces are taken into account. The contact stresses are determined by quadrature that is convenient for practical application in engineering calculations.

2 Formulation of the problem.

Assume that in the isotropic medium occupying the plane xOy these is a variable width slot $h(x)$ comparable with elastic deformations. The slot's length is accepted as $2l = b - a$ (Fig. 1). It is assumed that the body forces $F = X + iY$ $(X(x, y), Y(x, y)$ are the given functions, $i^2 = -1)$ act on the particles of the medium. As $x \to \infty,\ y \to \infty$ the displacement vector and stress tensor components tend to zero.

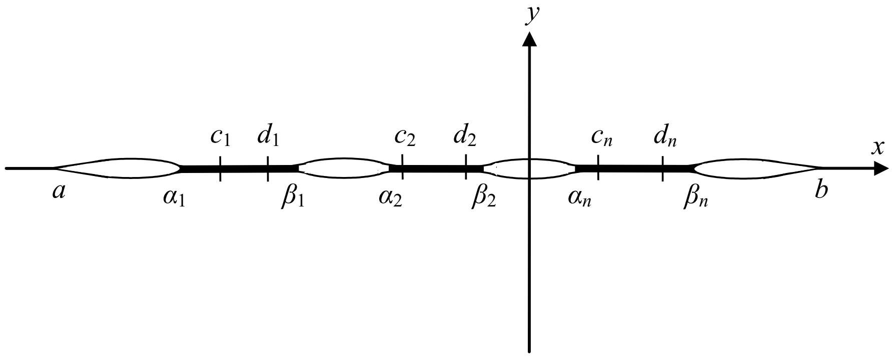

Fig. 1. Computational diagram of contact problem for a plane with a variable width slot.

In the process loading of the body at some ratio of physical and geometrical parameters of medium and acting volumetric and surface loads, there appear the contractive stresses zones in which the slot's faces may get in contact, and this will reduce to appearance of contact stresses on the given area of the slot faces. We assume that in the deformation process the slot faces get in contact on the areas (α_k, β_k) (k=1,2,…,n). It is accepted that each contact area consists of areas of stick of faces (c_k, d_k) and two areas (α_k, c_k), (d_k, β_k) on which there may happen slippage.

Denote by L_1 the totality of stick areas, by L_2 the totality of slippage areas, by L_3 the totality of areas on which the gas pressure $p(x)$ acts.

In the process of loading of the body, in the areas where the slot's faces get in contact, there arise normal $p_y(x)$ and tangent $p_{xy}(x)$ stresses whose values are not known in advance and to be defined. The boundary conditions on the slot's faces for the problem under consideration with stresses vanishing at infinity, have the form:

$$\sigma_y - i\tau_{xy} = p_y - ip_{xy} \text{ on } L_1, \sigma_y - i\tau_{xy} = (1 - if(x))\, p_y \text{ on } L_2 \tag{2.1}$$

$$\sigma_y - i\tau_{xy} = -p(x) \text{ on } L_3,$$

$$\frac{\partial}{\partial x}\left(v^+ - v^-\right) = -h'(x) \text{ on } L_1 + L_2, \tag{2.2}$$

$$\frac{\partial}{\partial x}\left(u^{+}-u^{-}\right)=0 \, \text{on} \, L_{1}. \tag{2.3}$$

Here it is accepted that on the slippage areas there hold the dry friction forces (the friction law is accepted in the Amonton-Coulomb form); $f(x)$ is the friction factor; $(u^{+}-u^{-})$ is a tangential, $(\upsilon^{+}-\upsilon^{-})$ is a normal component of the opening of the slot's faces.

A model of a contact with friction and stick was first considered by L.A. Galin [16, 17].

The contact zone sizes are not known in advance and to be determined.

3 The method of the boundary value problem solution.

We represent the stress state in the plane with a slot in the form of the sums

$$\sigma_{x}=\sigma_{x}^{0}+\sigma_{x}^{1}, \sigma_{y}=\sigma_{y}^{0}+\sigma_{y}^{1}, \tau_{xy}=\tau_{xy}^{0}+\tau_{xy}^{1}, \tag{3.1}$$

where σ_{x}^{0}, σ_{y}^{0}, τ_{xy}^{0} is any particular solution of the equations of plane theory of elasticity in the presence of body-forces; σ_{x}^{1}, σ_{y}^{1}, τ_{xy}^{1} is the solution of the equations of plane theory of elasticity in the absence of body-forces.

For the stresses σ_{x}^{0}, σ_{y}^{0}, τ_{xy}^{0} we have the relations

$$\sigma_{x}^{0}+\sigma_{y}^{0}=-\frac{2}{1+\kappa}\frac{\partial Q}{\partial z},\ z=x+iy, \tag{3.2}$$

$$\sigma_{y}^{0}-\sigma_{x}^{0}+2i\tau_{xy}^{0}=\frac{1}{1+\kappa}\frac{\partial}{\partial z}\ \left(\kappa\bar{Q}-\bar{F}_{1}\right)$$

that contain the two functions $Q\left(z,\bar{z}\right)$ and $F_{1}\left(z,\bar{z}\right)$ that represent any particular solutions of the equations

$$\frac{\partial^{2}Q}{\partial z\partial\bar{z}}=F\left(z,\bar{z}\right),\frac{\partial^{2}F_{1}}{\partial z^{2}}=\overline{F\left(z,\bar{z}\right)}. \tag{3.3}$$

Here κ is the Muskhelishvili constant, $\kappa=3-4\nu$ for plane strain; $\kappa=(3-\nu)/(1+\nu)$ for plane stress (ν is the Poisson's ratio of the material).

Allowing for formulas (3.1) we write boundary conditions (2.1) in the form

$$\sigma_{y}^{1}-i\tau_{xy}^{1}=p_{y}-ip_{xy}-f^{0} \text{ on } L_{1}, \sigma_{y}^{1}-i\tau_{xy}^{1}=\left(1-if\right)p_{y}-f^{0} \text{ on } L_{2}, \tag{3.4}$$

$$\sigma_{y}^{1}-i\tau_{xy}^{0}=-p\left(x\right)-f^{0} \, \text{on} \, L_{3},$$

where $f^{0}=\left(\sigma_{y}^{0}-i\tau_{xy}^{0}\right)=-\frac{1}{1+\kappa}Re\frac{\partial Q}{\partial z}+\frac{1}{2(1+\kappa)}\frac{\partial}{\partial z}\left(\kappa\frac{\partial\bar{Q}}{\partial z}-\frac{\partial\bar{F}_{1}}{\partial z}\right)$ for $y=0$.

We express the components stress tensor σ_{x}^{1}, σ_{y}^{1}, τ_{xy}^{1} and displacement vector u_{1}, υ_{1} by two complex variable piecewise-analytic functions $\Phi\left(z\right)$ and $\Omega\left(z\right)$

$$\sigma_{y}^{1}-i\tau_{xy}^{1}=\Phi\left(z\right)+\Omega\left(\bar{z}\right)+\left(z-\bar{z}\right)\overline{\Phi'\left(z\right)}, \tag{3.5}$$

$$2\mu\frac{\partial}{\partial x}\left(u_{1}+i\upsilon_{1}\right)=\kappa\Phi(z)-\Omega(\bar{z})-(z-\bar{z})\overline{\Phi'(z)},$$

where μ is the shear modulus of the material. Following [18] Muskhelishvili N.I., based on boundary conditions (3.4) we arrive at the linear conjugation problem with discontinuous coefficients

$$\left[\Phi\left(t\right)+\Omega\left(t\right)\right]^{+}+\left[\Phi\left(t\right)+\Omega\left(t\right)\right]^{-}=2f_{0}\left(t\right), \tag{3.6}$$

$$[\Phi(t) - \Omega(t)]^{+} - [\Phi(t) - \Omega(t)]^{-} = 0,$$

where $f_0(t) = \begin{cases} p_y - ip_{xy} - f^0 & \text{on } L_1 \\ (1 - if)p_y - f^0 & \text{on } L_2 \\ -p(x) - f^0 & \text{on } L_3 \end{cases}$.

We write the solution of boundary value problem (3.6) in the form

$$\Phi(z) = \Omega(z) = \frac{1}{2\pi i X(z)} \int_a^b \frac{X(t) f_0(t)}{t - z} dt, \tag{3.7}$$

$$X(z) = \sqrt{(z - a_1)(z - b_1)}.$$

As $z \to \infty$ $X(z) \to z + O(1/z)$. The root under the integral sign represents the value of the branch of the corresponding function distinguished by the reduced condition on the upper face of the slot.

Relations (3.6) and (3.7) contain unknown contact stresses $p_y(x)$, $p_{xy}(x)$. Let us construct integral equations for determining the unknown functions $p_y(x)$, $p_{xy}(x)$. The relations (2.2), (2.3) are the conditions that determine the listed unknown functions. Using the second formula in relations (3.5), and boundary values of the functions $\Phi(z)$, $\Omega(z)$ on the segment $y = 0$, $a \le x \le b$ we get

$$\Phi^{+}(x) - \Phi^{-}(x) = \frac{2\mu}{1+\kappa} \left[\frac{\partial}{\partial x} \left(u^{+} - u^{-} \right) + i \frac{\partial}{\partial x} \left(v^{+} - v^{-} \right) \right]. \tag{3.8}$$

Using the Sokhotsky-Plemelj formula [18] and taking into account formula (3.7), we find

$$\Phi^{+}(x) - \Phi^{-}(x) = -\frac{i}{\pi X^{+}(x)} \int_a^b \frac{X^{+}(t) f_0(t)}{t - x} dt. \tag{3.9}$$

Taking into account relations (2.2), (2.3), (3.8), (3.9), after some transformations we get the system of integral equations with respect to unknown functions $p_y(x)$, $p_{xy}(x)$:

$$-\frac{1}{\pi X_1^{+}(x)} \left[\int_{L_1+L_2} \frac{X_1^{+}(t) p_y(t) dt}{t - x} - \int_a^b \frac{X_1^{+}(t) \sigma_y^0(t) dt}{t - x} - \right. \tag{3.10}$$

$$\left. - \int_{L_3} \frac{X_1^{+}(t) p(t) dt}{t - x} \right] = -\frac{2\mu}{1+\kappa} h'(x).$$

$$\int_{L_1} \frac{X_1^{+}(t) p_{xy}(t) dt}{t - x} + \int_{L_2} \frac{X_1^{+}(t) f p_y(t) dt}{t - x} - \int_a^b \frac{X_1^{+}(t) \tau_{xy}^0(t) dt}{t - x} = 0. \tag{3.11}$$

where $X_1^{+}(t) = \sqrt{(t - a_1)(b_1 - t)}$.

4 Calculation method and analysis of results.

As might be expected, the stated problem decays into two independent problems: for a opening mode slot (see (3.10)), and sliding mode (the transverse shear slot) (see (3.11)).

The solution of integral equation (3.10) may be obtained by solving the appropriate Riemann problem [18, 19]. We represent integral equation (3.10) in the form

$$\int_{L_1+L_2} \frac{p_y^{*}(\tau)}{\tau - t} d\tau = f_{*}(t),$$

where

$$p_y^*(t) = p_y(t)X_1(t),$$

$$f_*(t) = \frac{2\mu h'(t)X_1(t)}{1+\kappa} + \int_a^b \frac{X_1^+(\tau)\sigma_y^0(\tau)d\tau}{\tau - t} + \int_{L_3} \frac{X_1^+(\tau)p(\tau)d\tau}{\tau - t}.$$

Introduce the piecewise-analytic function $F_*(z)$ given by the Cauchy integral whose density is the sought-for solution of the integral equation

$$F_*(z) = \frac{1}{2\pi i}\int_{L_1+L_2} \frac{p_y^*(\tau)}{\tau - z}d\tau.$$

The analytic function $F_*(z)$ represents the solution of the problem of linear conjugation of boundary values

$$F_*^+(z) + F^-(\tau) = \frac{f_*(\tau)}{\pi i}. \tag{4.1}$$

The solution of boundary value problem (4.1) in the class of everywhere bounded functions has the form

$$F_*(z) = \frac{X_i(z)}{2\pi i}\int_{L_1+L_2} \frac{f_*^1(\tau)}{X_2^+(\tau)(\tau - z)}d\tau,$$

where

$$X_2(z) = \prod_{k=1}^{n} \sqrt{(z-\alpha_k)(z-\beta_k)}, f_*^1(\tau) = \frac{f_*(\tau)}{\pi i},$$

$$X_2^+(\tau) = \prod_{k=1}^{n} \sqrt{(\tau-\alpha_k)(\tau-\beta_k)}.$$

Allowing for Sokhotsky-Plemelj formulas, we get the solution of integral equation (3.10)

$$p_y^*(t) = F_*^+(t) - F_*^-(t),$$

$$F_*^+(t) = X_2^+(t)\left(\frac{1}{2}\frac{f_*^1(t)}{X_2^+(t)} + \frac{1}{2\pi i}\int_{L_1+L_2} \frac{f_*^1(\tau)}{X_2^+(\tau)(\tau - t)}d\tau\right),$$

$$F_*^-(t) = X_2^-(t)\left(-\frac{1}{2}\frac{f_*^1(t)}{X_2^+(t)} + \frac{1}{2\pi i}\int_{L_1+L_2} \frac{f_*^1(\tau)}{X_2^+(\tau)(\tau - t)}d\tau\right).$$

Taking into account $X_2^-(t)/X_2^+(t) = -1$, we have

$$p_y^*(t) = \frac{X_2^+(t)}{\pi i}\int_{L_1+L_2} \frac{f_*^1(\tau)}{X_2^+(\tau)(\tau - t)}d\tau,$$

from which it follows

$$p_y(t) = X_1^+(t)\frac{X_2^+(t)}{\pi i}\int_{L_1+L_2} \frac{f_*^1(\tau)}{X_2^+(\tau)(\tau - t)}d\tau. \tag{4.2}$$

For determining the parameters α_k and β_k, we have the equations

$$\int_{L_1+L_2} \frac{f_*(t)}{X_2^+(t)}t^{k-1}dt = 0 k = 1, 2, \ldots, n. \tag{4.3}$$

We get the missing n equations for determining the coordinates of the ends of the contact areas of the slot faces from the conditions

$$\upsilon^{+}(\alpha_k) - \upsilon^{-}(\alpha_k) = -h(\alpha_k) \;\; k = 1, 2, \ldots, n.$$

We have

$$\upsilon^{+}(x) - \upsilon^{-}(x) = \frac{1}{4\pi i\gamma}\int_{\alpha_1}^{x} G(t)\,dt,$$

where

$$\gamma = \frac{\mu}{\pi(1+\kappa)}, G(t) = \left[\Phi + \bar{\Phi}\right]^{+} - \left[\Phi + \bar{\Phi}\right]^{-}.$$

Using the previous formulas, we find the sought-for equations

$$\int_{a_1}^{\alpha_1} G(t)\,dt = -4\pi i\gamma h(\alpha_1), \tag{4.4}$$

$$\int_{\beta_k}^{\alpha_{k+1}} G(t)\,dt = -4\pi i\gamma\left[h(\alpha_{k+1}) - h(\beta_k)\right] \;\; k = 1, 2, \ldots, n-1.$$

In the same way, by solving singular integral equation (3.11), we get

$$p_{xy}(x) = \frac{X_1^{+}(x)X_3^{+}(x)}{\pi^2}\int_{L_1}\frac{f_{xy}(\tau)}{X_3^{+}(\tau)(\tau - x)}d\tau, \tag{4.5}$$

where

$$X_3^{+}(x) = \prod_{k=1}^{m}\sqrt{(x - c_k)(x - d_k)},$$

$$f_{xy}(x) = -\int_{L_2}\frac{fp_y(t)\,dt}{X_1^{+}(t)(t-x)} + \int_{a_1}^{b_1}\frac{\tau_{xy}^{0}(t)\,dt}{X_1^{+}(t)(t-x)} - \int_{L_4}\frac{q_{xy}(t)\,dt}{X_1^{+}(t)(t-x)}.$$

For determining the unknowns c_k and d_k, we have

$$\int_{L_1}\frac{f_{xy}(t)}{X_3^{+}(t)}t^{k-1}dt = 0 \;\; k = 1, 2, \ldots, m. \tag{4.6}$$

We get the missing m equations for finding the coordinates of the ends of the contact areas, from the conditions

$$u^{+}(c_k) - u^{-}(c_k) = \int_{a_1}^{c_k}\frac{\partial}{\partial t}\left(u^{+} - u^{-}\right)dt = 0 \quad k = 1, 2, \ldots, m.$$

We have

$$\int_{a_1}^{c_1}\left[\Phi^{+} - \Phi^{-}\right]dt = -2\pi i\gamma h(c_1), \tag{4.7}$$

$$\int_{d_k}^{c_{k+1}}\left[\Phi^{+} - \Phi^{-}\right]dt = -2\pi i\gamma\left[h(c_{k+1}) - h(d_k)\right] \quad k = 1, 2, \ldots, m-1.$$

For determining the stick parts we have the complete system of equations.

The joint solution of equations (4.2)-(4.4), (4.5)-(4.7) permits to determine the contact stresses $p_y(x)$, $p_{xy}(x)$ and the sizes of contact zones.

For simplifying calculations, the functions $X(x,y)$ and $Y(x,y)$ were expanded into Taylor series in the vicinity of the origin of coordinates, and this expansion was limited by several first members. As a result of integration of equations (3.3), we have

$$Q(z,\bar{z}) = \int^{z} dz\int^{\bar{z}} F(z,\bar{z})\,d\bar{z}, F_1(z,\bar{z}) = \int^{z} dz\int^{z}\overline{F(z,\bar{z})}dz. \tag{4.8}$$

Using the functions $Q(z,\bar{z})$ and $F_1(z,\bar{z})$, according to (3.2) we find the function $f^0(x)$.

The results of calculations of absolute values of contact stresses p_y/F_0 of the slot along the contact zone (α_1,β_1) for different values of relative slot size $l_* = (b-a)/R$ are depicted in Fig. 2.

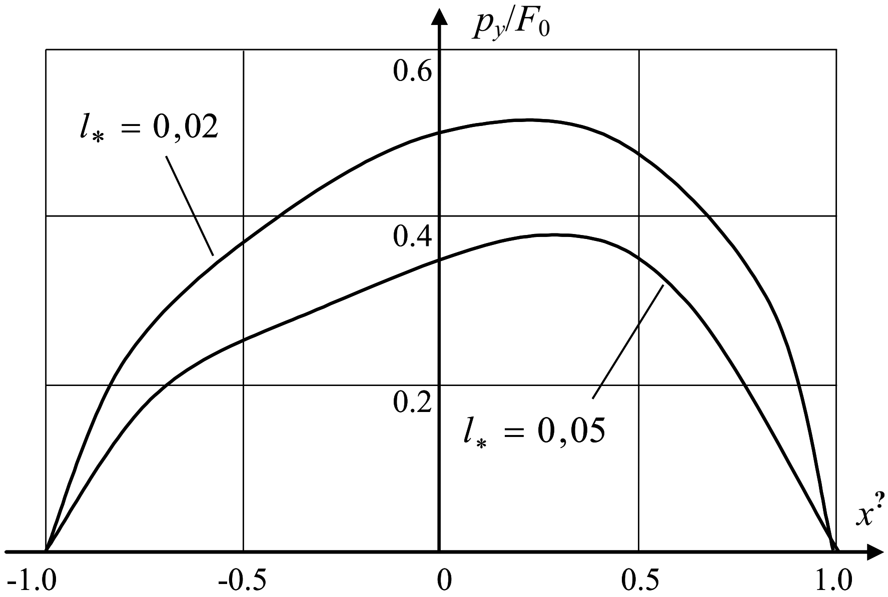

Fig. 2. Distributions of the normal contact stresses p_y/F_0 for different relative slot sizes

Here F_0 are body forces per unit area (MPa), R is a typical linear size of the plane. In calculations the dimensionless coordinate x' connected with x by the following relation

$$x = \frac{\alpha_1+\beta_1}{2} + \frac{\beta_1-\alpha_1}{2}x'$$

was used.

The largest values of contact stresses are attained in the middle part of the contact zone where the slot faces interlock.

The character of change of tangential contact stresses $p_{xy}(x)$ along the contact zone is similar to change of normal contact stresses $p_y(x)$, but the absolute value of tangential stresses are significantly less.

5 Conclusions.

The effective calculation scheme of a variable width slot partially closed by body forces under the action of internal pressure is suggested. Analysis of the model of partial closure of a variable width slot in the isotropic medium in the existence of body forces is reduced to parametrical investigation of the system of singular integral equations at different geometrical and physical parameters of the medium. The contact stresses $p_y(x)$, $p_{xy}(x)$ and the sizes of contact zones are immediately determined from the solutions of the obtained systems. The obtained relations permit to find the solution of the inverse problem, i.e. to determine the parameters of body forces and the stress state of the isotropic medium at which the given contact domain of variable width slot's faces holds.

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