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Studying the problem of torsion of a spherical shell with variable shear module

Natik K. Akhmedov · Natavan S. Gasanova

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Abstract. *In the paper the problem of torsion of a radially-inhomogeneous isotropic sphere of small thickness is studied by the method of homogeneous solutions. Some special cases of radius dependence elastic characteristics (linear dependence, quadratic dependence, inverse proportionality of the square of distance) are considered. Homogeneous solutions are constructed and based on the carried out asymptotic analysis the character of stress-strain state is clarified.*

Keywords. method of homogeneous solutions · boundary layer · characteristic equation · asymptotic solution.

Mathematics Subject Classification (2010): 74D05

1 Introduction

Studies of inhomogeneous, mainly of sandwich thin-shelled structures play a significant role in theory of shells. Variety of inhomogeneous constructions and complexity of phenomena arising in deformation of an inhomogeneous shell give rise to a number of applied theories. Applicability fields of applied theories of inhomogeneous shells have not been studied enough. Existence of different applied theories for inhomogeneous shells requires their critical analysis based on three-dimensional equations of elasticity theory. Analysis of inhomogeneous shells from point of view of three-dimensional theory of elasticity for creating new refined applied theories, is also urgent.

H.K. Akhmedov
Azerbaijan State University of Economics (UNEC)
Department of Mathematics, Baku, Azerbaijan
Istiqlaliyyat str. 6, Azerbaijan
E-mail: anatiq@gmail.com

N.S. Gasanova
Ganja State University, Ganja, Azerbaijan
E-mail: anatiq@gmail.com

2 Problem statement

Let as consider a problem of torsion of a small thickness isotropic hollow sphere. In the spherical system of coordinates we denote the domain occupied by the sphere as : $\Gamma = \{r \in [r_1; r_2],\ \theta \in [\theta_1; \theta_2], \varphi \in [0; 2\pi]\}$. It is supposed that the shell contains none of the poles 0 and π .We will assume $G = G(r)$(shear modulus) an arbitrary positive continuous function of variable r whose values may change within one order.

When there are no mass forces in spherical system of coordinates, the equilibrium equations have the from [5]:

$$\frac{\partial \sigma_{r\varphi}}{\partial r} + \frac{1}{r}\ \frac{\partial \sigma_{\varphi\theta}}{\partial \theta} + \frac{3\sigma_{r\varphi} + 2\sigma_{\varphi\theta} ctg\theta}{r} = 0, \tag{2.1}$$

where $\sigma_{r\varphi}, \sigma_{\varphi\theta}$ are the stress tensor components that are determined by the displacement vector component $u_\varphi = u_\varphi(r, \theta)$ in the following way [5]:

$$\sigma_{r\varphi} = G\left(\frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r}\right),\ \sigma_{\varphi\theta} = \frac{G}{r}\left(\frac{\partial u_\varphi}{\partial \theta} - u_\varphi ctg\theta\right). \tag{2.2}$$

Substituting (2.2) in (2.1) , we get equilibrium equations in displacements

$$\frac{\partial}{\partial r}\left[G(r)\left(\frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r}\right)\right] + \frac{3G(r)}{r}\left(\frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r}\right)$$

$$+\frac{G(r)}{r^2}\left(\frac{\partial^2 u_\varphi}{\partial \theta^2} + \frac{\partial u_\varphi}{\partial \theta} ctg\theta - \frac{\cos 2\theta}{\sin^2\theta} u_\varphi\right) = 0. \tag{2.3}$$

Assume that face surfaces of the sphere are free from stresses

$$\sigma_{r\varphi} = 0 \text{ for } r = r_s, \tag{2.4}$$

and on conical surfaces (end-wall) are given the boundary conditions

$$\sigma_{\varphi\theta} = f_s(r) \text{ for } \theta = \theta_s, \tag{2.5}$$

where $f_s(r)\ \ (s = 1, 2)$are rather smooth functions satisfying the equilibrium conditions.

3 Problem solution

We shall look for the solution of (2.3) in the form :

$$u_\varphi(r, \theta) = \upsilon(r) \cdot m(\theta) \tag{3.1}$$

where $m(\theta)$ is the solution of the Legendre equation [3]:

$$m''(\theta) + ctg\theta \cdot m'(\theta) + \left(z^2 - \frac{1}{4} - \frac{1}{\sin^2\theta}\right) m(\theta) = 0. \tag{3.2}$$

Substituting (3.1) in (2.3), (2.4) allowing for (3.2) we get the following boundary value problem:

$$\left[G(r)\left(\upsilon'(r) - \frac{\upsilon(r)}{r}\right)\right]' + \frac{3G(r)}{r}\left(\upsilon'(r) - \frac{\upsilon(r)}{r}\right) + \frac{G(r)}{r^2}\left(\frac{9}{4} - z^2\right)\upsilon(r) = 0, \tag{3.3}$$

$$G(r)\left(\upsilon'(r) - \frac{\upsilon(r)}{r}\right) = 0 \qquad \text{for} \quad r = r_s\,. \tag{3.4}$$

Represent (3.3) (3.4) in the form

$$A\upsilon = \lambda\upsilon, \tag{3.5}$$

where

$$A\upsilon = \left\{ -\frac{r^2}{G(r)} \left[G(r) \left(\upsilon'(r) - \frac{\upsilon(r)}{r} \right) \right]' - 3r \left(\upsilon'(r) - \frac{\upsilon(r)}{r} \right); \right.$$

$$\left. G(r) \left(\upsilon'(r) - \frac{\upsilon(r)}{r} \right) = 0 \text{ for } r = r_s \right\};$$

$$\lambda = \frac{9}{4} - z^2.$$

Let us introduce the Hilbert space $L_2(r_1, r_2)$ with a scalar product

$$(\upsilon, g) = \int_{r_1}^{r_2} G(r)\,\upsilon(r)\,g(r)\,dr.$$

Lemma: *$A : L_2 \to L_2$ is a symmetric operator.*

Proof. For any function $\upsilon(r) \in D_A,\ g(r) \in D_A$ we have:

$$(A\upsilon, g) = \int_{r_1}^{r_2} g \cdot A\upsilon \cdot G(r)\,dr$$

$$= \int\limits_{r_1}^{r_2} g(r) \left[-\frac{r^2}{G(r)} \left(G(r) \left(\upsilon'(r) - \frac{\upsilon(r)}{r} \right) \right)' - 3r \left(\upsilon'(r) - \frac{\upsilon(r)}{r} \right) \right] G(r)\,dr. \tag{3.6}$$

After integration by parts and taking into account boundary conditions (3.4), from (3.6) we get:

$$(A\upsilon, g) = (\upsilon, Ag).$$

From (3.6) we have:

$$(A\upsilon, \upsilon) = \int_{r_1}^{r_2} G(r) \cdot \upsilon \cdot A\upsilon \cdot dr = \int_{r_1}^{r_2} G(r) \left(\frac{d\upsilon}{dr} - \frac{\upsilon(r)}{r} \right)^2 r^2 dr \geq 0,$$

i.e. $A : L_2 \to L_2$ is non-negative .

Nonzero eigen values of the operator A $\lambda_k > 0,\ \lambda_k \to \infty$ as $k \to \infty$ and the set of eigen functions $\{\upsilon_k\}$ form an orthogonal basis in space L_2:

$$(\upsilon_k, \upsilon_t) = \alpha_k \cdot \delta_{kt} \tag{3.7}$$

where $\alpha_k = \int_{r_1}^{r_2} G(r) \cdot \upsilon_k^2(r)\,dr$.

Let us consider some special cases of dependence of elastic characteristics on r [1,6].

Linear dependence. Assume that the shear modulus is given in the form:

$$G(r) = G_0 r, \tag{3.8}$$

where G_0 is a constant.

Allowing for (3.8), from (3.3), (3.4) we have:

$$r^2\upsilon''(r) + 3r\upsilon'(r) - \left(\frac{3}{4} + z^2 \right) \upsilon(r) = 0, \tag{3.9}$$

$$G_0 r\left(\upsilon'(r)-\frac{\upsilon(r)}{r}\right)=0 \quad \text{for} \quad r=r_s\,. \tag{3.10}$$

General solution of (3.9) has the form:

$$\upsilon(r)=D_1 r^{p-1}+D_2 r^{-(p+1)} \tag{3.11}$$

where D_1, D_2 -are arbitrary constants; $p=\sqrt{z^2+\frac{7}{4}}$.

By (3.11) satisfying boundary conditions (3.10), with respect to D_1, D_2 we get a homogeneous linear system of algebraic equations. From the condition of existence of nontrivial solutions of this system, we get the characteristic equation:

$$\varDelta_1(z,\varepsilon)=\left(\frac{9}{4}-z^2\right) sh\left(2\varepsilon\sqrt{z^2+\frac{7}{4}}\right)=0, \tag{3.12}$$

where $\varepsilon=\frac{1}{2}\ln\left(\frac{r_2}{r_1}\right)$ is a small parameter that characterizes the thickness of the spherical shell.

Let us perform analysis of the roots of equation (3.12) The function $\varDelta_1(z,\varepsilon)$ has the following two groups of zeros:

1^0) $z_0^{\pm}=\pm\frac{3}{2}$,

2^0) Denumerable set of zeros

$$z_n^{\pm}=\pm i\sqrt{\frac{7}{4}+\frac{\pi^2 n^2}{4\varepsilon^2}}\ ,(n=1,2,...)\,. \tag{3.13}$$

Displacements and stresses corresponding to $z_0^{\pm}=\pm\frac{3}{2}$ are given by the following formulas:

$$u_\varphi^{(1)}(r,\theta)=C_0 r\left(\frac{1}{2}\sin\theta\cdot\ln\left(ctg^2\left(\frac{\theta}{2}\right)\right)+ctg\theta\right), \tag{3.14}$$

$$\sigma_{r\varphi}^{(1)}=0,\ \sigma_{\varphi\theta}^{(1)}=\frac{-2G_0 r}{\sin^2\theta}C_0.$$

Displacements and stress corresponding to zeros of (3.13), are of the form:

$$u_\varphi^{(2)}(r,\theta)=\sum_{n=1}^{\infty}\frac{1}{r}\left[\frac{\pi n}{\varepsilon}\cos\left(\frac{\pi n}{2\varepsilon}\cdot\ln\left(\frac{r_2}{r}\right)\right)-4\sin\left(\frac{\pi n}{2\varepsilon}\cdot\ln\left(\frac{r_2}{r}\right)\right)\right]m_n(\theta),$$

$$\sigma_{r\varphi}^{(2)}(r,\theta)=\sum_{n=1}^{\infty}\frac{G_0}{r}\left(8+\frac{\pi^2 n^2}{2\varepsilon^2}\right)\cdot\sin\left(\frac{\pi n}{2\varepsilon}\cdot\ln\left(\frac{r_2}{r}\right)\right)m_n(\theta), \tag{3.15}$$

$$\sigma_{\theta\varphi}^{(2)}=\sum_{n=1}^{\infty}\frac{G_0}{r}\left[\frac{\pi n}{\varepsilon}\cos\left(\frac{\pi n}{2\varepsilon}\cdot\ln\left(\frac{r_2}{r}\right)\right)-4\sin\left(\frac{\pi n}{2\varepsilon}\cdot\ln\left(\frac{r_2}{r}\right)\right)\right]$$
$$\times\left(m_n'(\theta)-m_n(\theta)\,ctg\theta\right),$$

where $m_n(\theta)=A_n P'_{z_n-\frac{1}{2}}(\cos\theta)+B_n Q'_{z_n-\frac{1}{2}}(\cos\theta)\,;\ P'_{z_n-\frac{1}{2}}(\cos\theta)\,,Q'_{z_n-\frac{1}{2}}(\cos\theta)$ are Legendre's first and second kind adjoint functions, respectively; A_n, B_n are arbitrary constants.

Indicate the character of constructed solutions. We represent the displacements and stresses in the form:

$$u_{\varphi}(r,\theta)=C_0 r\left(\frac{1}{2}\sin\theta\cdot\ln\left(ctg^2\left(\frac{\theta}{2}\right)\right)+ctg\theta\right)+\sum_{k=1}^{\infty}\upsilon_k(r)\,m'_k(\theta), \qquad (3.16)$$

$$\sigma_{\theta\varphi}=\frac{-2G_0 r}{\sin^2\theta}C_0+G_0\sum_{k=1}^{\infty}\upsilon_k(r)\left(m'_k(\theta)-m_k(\theta)\,ctg\theta\right). \qquad (3.17)$$

For torques M_{kp} of stress acting in the section $\theta=const$, we have [2;4]:

$$M_{kp}=2\pi\sin^2\theta\int_{r_1}^{r_2}\sigma_{\theta\varphi}r^2dr. \qquad (3.18)$$

Substitute (3.17) in (3.18):

$$M_{kp}=-\pi G_0\left(r_2^4-r_1^4\right)C_0$$

$$+2\pi G_0\cdot\sin^2\theta\cdot\sum_{k=1}^{\infty}\left(\int_{r_1}^{r_2}r^2\upsilon_k(r)\,dr\right)\cdot\left(m'_k(\theta)-m_k(\theta)\,ctg\theta\right). \qquad (3.19)$$

Multiplying the both hand sides of (3.9) by r^2 and integrating the obtained one in $[r_1,r_2]$ allowing for (3.10), we have:

$$\int_{r_1}^{r_2}r^2\upsilon_k(r)\,dr=0 \qquad (3.20)$$

After substitution of (3.20) in (3.19), we get:

$$M_{kp.}=-\pi G_0\left(r_2^4-r_1^4\right)C_0. \qquad (3.21)$$

The constant C_0 is proportional to the torque $M_{kp.}$ of stresses acting in the section $\theta=const$.

Substituting (3.17) in (2.5) and scalarly multiplying by $\upsilon_n(r)\,(n=1,2,...)$, allowing for (3.17) we have:

$$m'_n(\theta_s)-m_n(\theta_s)\,ctg\theta_s=\frac{\int_{r_1}^{r_2}rf_s(r)\,\upsilon_n(r)\,dr}{G\int_{r_1}^{r_2}r\upsilon_n^2(r)\,dr}$$

$$-\frac{2M_{kp.}\cdot\int_{r_1}^{r_2}r^2\upsilon_n(r)\,dr}{\pi G_0\left(r_2^4-r_1^4\right)\sin^2\theta_s\int_{r_1}^{r_2}r\upsilon_n^2(r)\,dr},\ (s=1,2) \qquad (3.22)$$

The unknown constants A_n,B_n are determined from system (3.22).

(3.14) determine inner stress-strain state of the shell.

For the second group of roots, the principal term of the asymptotic solution of equation (3.2) has the form:

$$m_n(\theta)=\begin{cases}\left(\frac{\pi n}{2}\right)^{-\frac{1}{2}}\frac{1}{\sqrt{\sin\theta}}\exp\left[-\frac{\pi n}{2\varepsilon}(\theta-\theta_1)\right](1+O(\varepsilon)),\text{in vicinity of}\,\theta=\theta_1,\\ \left(\frac{\pi n}{2}\right)^{-\frac{1}{2}}\frac{1}{\sqrt{\sin\theta}}\exp\left[\frac{\pi n}{2\varepsilon}(\theta-\theta_2)\right](1+O(\varepsilon)),\text{in vicinity of}\,\theta=\theta_2.\end{cases} \qquad (3.23)$$

The stress state corresponding to the second group of roots is of boundary layer character [2;4].

Quadratic dependence. Assume that the shear modulus is given in the form:

$$G(r) = G_0 r^2.$$

From (3.3), (3.4) we have:

$$r^2 \upsilon''(r) + 4r\upsilon'(r) - \left(\frac{7}{4} + z^2\right)\upsilon(r) = 0, \tag{3.24}$$

$$G_0 r^2 \left(\upsilon'(r) - \frac{\upsilon(r)}{r}\right) = 0 \quad \text{for} \quad r = r_s. \tag{3.25}$$

General solution of (3.24) has form:

$$\upsilon(r) = D_3 r^{-\left(\frac{3}{2}+t\right)} + D_4 r^{t-\frac{3}{2}}, \tag{3.26}$$

where D_3, D_4 are arbitrary constants; $t = \sqrt{z^2+4}$.

By means of (3.26) satisfying boundary conditions(3.25) , we get the characteristic equation:

$$\Delta_2(z,\varepsilon) = \left(\frac{9}{4} - z^2\right) sh\left(2\varepsilon\sqrt{z^2+4}\right) = 0. \tag{3.27}$$

Equation (3.27) has two group of roots:

1^0) $z_0^{\pm} = \pm\frac{3}{2}$,

2^0) Denumerable set of roots $z_n = \pm i\sqrt{4 + \frac{\pi^2 n^2}{4\varepsilon^2}}$.

Displacements and stresses corresponding to the root $z_0^{\pm} = \pm\frac{3}{2}$ are of the form:

$$u_\varphi^{(1)}(r,\theta) = D_0 r\left(\frac{1}{2}\sin\theta \cdot \ln\left(ctg^2\left(\frac{\theta}{2}\right)\right) + ctg\theta\right), \tag{3.28}$$

$$\sigma_{r\varphi}^{(1)} = 0,\ \sigma_{\theta\varphi}^{(1)} = \frac{-2G_0 r^2}{\sin^2\theta} D_0.$$

The constants D_0, is proportional to the torque $M_{kp.}$ of stresses acting in the section $\theta = const$:

$$M_{kp.} = \frac{-4\pi G_0\left(r_2^5 - r_1^5\right)}{5} D_0.$$

(3.28) determines the inner stress strain state of the shell.

Displacements and stresses corresponding to the second group of roots are of the form:

$$u_\varphi^{(2)}(r,\theta) = \sum_{n=1}^{\infty} r^{-\frac{3}{2}}\left[\frac{-\pi n}{\varepsilon}\cos\left(\frac{\pi n}{2\varepsilon}\cdot\ln\left(\frac{r_2}{r}\right)\right) + 5\sin\left(\frac{\pi n}{2\varepsilon}\cdot\ln\left(\frac{r_2}{r}\right)\right)\right] m_n(\theta),$$

$$\sigma_{r\varphi}^{(2)} = \sum_{n=1}^{\infty} G_0 r^{-\frac{1}{2}}\left(\frac{25}{2} + \frac{\pi^2 n^2}{2\varepsilon^2}\right)\cdot\sin\left(\frac{\pi n}{2\varepsilon}\cdot\ln\left(\frac{r}{r_2}\right)\right) m_n(\theta), \tag{3.29}$$

$$\sigma_{\theta\varphi}^{(2)} = \sum_{n=1}^{\infty} G_0 r^{-\frac{1}{2}}\left[\frac{-\pi n}{\varepsilon}\cos\left(\frac{\pi n}{2\varepsilon}\cdot\ln\left(\frac{r_2}{r}\right)\right)\right.$$

$$\left. +5\sin\left(\frac{\pi n}{2\varepsilon}\cdot\ln\left(\frac{r_2}{r}\right)\right)\right]\cdot\left(m_n'(\theta) - m_n(\theta)\, ctg\theta\right).$$

For the second group of roots the principal term of asymptotic solution (3.2) has the form (3.23). Solutions (3.29) are of boundary layer character [2;4].

Inverse proportionality to square of distance. Suppose that the shear modulus is given in the form:

$$G(r) = \frac{G_0}{r^2}.$$

From (3.3)-(3.4) we get:

$$r^2 \upsilon''(r) + \left(\frac{9}{4} - z^2\right) \upsilon(r) = 0, \tag{3.30}$$

$$\frac{G_0}{r^2}\left(\upsilon'(r) - \frac{\upsilon(r)}{r}\right) = 0 \quad for \quad r = r_s\,. \tag{3.31}$$

Solution of (3.30) has the form:

$$\upsilon(r) = D_5 r^{\frac{1}{2}-p} + D_6 r^{\frac{1}{2}+p}, \tag{3.32}$$

where $p = \sqrt{z^2 - 2}$; D_5, D_6 are arbitrary constants.

By means of (3.32) satisfying (3.31) , we get the characteristical equation:

$$\varDelta_3(z, \varepsilon) = \left(\frac{9}{4} - z^2\right) sh\left(2\varepsilon\sqrt{z^2 - 2}\right) = 0. \tag{3.33}$$

Equation (3.33) has two groups of zeros:

1^0) $z_0^{\pm} = \pm\frac{3}{2}$,

2^0) Denumerable set of roots $z_n = \pm i\sqrt{\frac{\pi^2 n^2}{4\varepsilon^2} - 2}$.

Displacements and stresses corresponding to the root $z_0^{\pm} = \pm\frac{3}{2}$, are given by the following formula:

$$u_\varphi^{(1)}(r, \theta) = E_0 r\left(\frac{1}{2}\sin\theta \cdot \ln\left(ctg^2\frac{\theta}{2}\right) + ctg\theta\right),$$

$$\sigma_{r\varphi}^{(1)} = 0,\ \sigma_{\varphi\theta}^{(1)} = \frac{-2G_0}{r^2\sin^2\theta}E_0. \tag{3.34}$$

Displacements and stresses corresponding to the second group of zeros have the form:

$$u_\varphi^{(2)}(r, \theta) = \sum_{n=1}^{\infty} r^{\frac{1}{2}}\left[\frac{\pi n}{\varepsilon}\cos\left(\frac{\pi n}{2\varepsilon}\cdot\ln\left(\frac{r_2}{r}\right)\right) - \sin\left(\frac{\pi n}{2\varepsilon}\cdot\ln\left(\frac{r_2}{r}\right)\right)\right] m_n(\theta),$$

$$\sigma_{r\varphi}^{(2)} = \sum_{n=1}^{\infty} G_0 r^{-\frac{5}{2}}\left(\frac{1}{2} + \frac{\pi^2 n^2}{2\varepsilon^2}\right)\cdot\sin\left(\frac{\pi n}{2\varepsilon}\cdot\ln\left(\frac{r_2}{r}\right)\right) m_n(\theta), \tag{3.35}$$

$$\sigma_{\theta\varphi}^{(2)} = \sum_{n=1}^{\infty} G_0 r^{-\frac{5}{2}}\left[\frac{\pi n}{\varepsilon}\cos\left(\frac{\pi n}{2\varepsilon}\cdot\ln\left(\frac{r_2}{r}\right)\right) - \sin\left(\frac{\pi n}{2\varepsilon}\cdot\ln\left(\frac{r_2}{r}\right)\right)\right]$$
$$\times\left(m_n'(\theta) - m_n(\theta)\,ctg\theta\right).$$

(3.34) determine the inner stress –stain state of the shell. The constant D_0 is proportional to the torque $M_{kp.}$ of stresses acting in the section $\theta = const$:

$$M_{kp.} = -4\pi G_0 (r_2 - r_1) E_0.$$

(3.35) is of boundary layer character [2;4].

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