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Stability of inhomogeneous nano-micro beams of Euler-Bernoulli based on nonlocal elasticity theory of Eringen K.A.

Fahreddin K. Isayev · **Vugar G. Rajabov** ·
Orhan Y. Efendiyev

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Abstract. *In this article, the bending and stability analysis of nonhomogeneous nano- and micro elements have been made by using the nonlocal elasticity theory. Beam has been chosen as the structural model type. Euler-Bernoulli beam theories have been used for beam theories. The motion equations of Euler-Bernoulli beam theories were expressed by using nonlocal elasticity theory, which was proposed by Eringen. According to different boundary conditions, the bending and stability equations of micro-nano beam have been generated. Then, in order to observe the effect of non-local behaviour, analysis have been made over carbon nanotube and microtubules and results have been compared with the classical theory.*

Keywords. nonlocal elasticity theory · nano-micro elements · nonhomogeneous · Euler-Bernoulli beam theories · stility · critical force.

Mathematics Subject Classification (2010): 74A60

1 Introduction

Various issues of stability and strength of one and multi-core designs elements from homogeneous materials have been studied enough in the scientific literature. In these studies, the classic ratios of theory of elasticity are mainly used [1 - 3].

In recent years, new composite-synthetic materials are used extensively in the technique. Therefore, these processes are put in front of designers, researchers have increased require-

Fahreddin K. Isayev
Qafqaz University, Baku, Azerbaijan,
E-mail: fisayev@qu.edu.az

Vugar G. Rajabov
Qafqaz University, Baku, Azerbaijan,
E-mail: fisayev@qu.edu.az

Orhan Y. Efendiyev
Azerbaijan University of Architecture and Construction, Baku, Azerbaijan
E-mail: o.efendiyev@mail.ru

ments for assessing the strength, stability and vibration as well as in various working conditions and loading conditions, a number of issues that require the new challenges of the stress-strain state and the determination of the critical parameters. In many cases, the layered structure elements are made from a variety of non-uniformly elastic materials. The cause of non-uniformity can be manufactured by design technology, thermal processing of materials, heterogeneous composition, etc. Consideration of these factors in solving problems of stability and vibration of structures is very important. Therefore, it is required to use a more refined hypothesis or theory in solving many problems of stability and vibration of structural elements from heterogeneous composite materials. One such theory is the theory of nonlocal elasticity theory proposed by A.K. Eringen [13 10]. In the work [12] some of the challenges and bending strength of inhomogeneous nano-micro elements were considered. In this work, we study the problem of stability of inhomogeneous cores based on nonlocal Eringen theory [13].

2 Formulation of the problem

It is known that the Cauchy equation of motion of homogeneous elastic bodies on the basis of nonlocal theory of elasticity consists of the following equations [13]:

$$\tau_{kl,l} + \rho\left(f_i - \frac{\partial^2 u_i}{\partial t^2}\right) = 0. \tag{2.1}$$

Here, physical correlations are as follows:

$$\tau_{kl}(x) = \int_v \varepsilon_{klmn}(x - x')\varepsilon_{mn} dv(x'), \tag{2.2}$$

where τ_{kl} - components of the stress tensor, - density of body weight, f - density of mass force, u - component of the displacement vector, v - volume of body, t-time, $_{kl}$ -components of the strain tensor, and are determined by the following formulas:

$$\varepsilon_{kl} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right). \tag{2.3}$$

As seen, $_{klmn}$ are $x - x$ vector functions, and stresses at the points x-depend on the deformations and displacements at points x'. Connection between the components of the stress and strain at the points x' is determined on the basis of the generalized Hooke's law [13]:

$$\tau(x') = \lambda\varepsilon_{mn}(x')\delta_{ke} + 2\mu\varepsilon_{ke}(x'), \tag{2.4}$$

$$\varepsilon_{kl}(x') = \frac{1}{2}\left(\frac{\partial u_k(x')}{\partial x'_l} + \frac{\partial u_l(x')}{\partial x'_k}\right).$$

Equations of state of nonlocal elasticity theory proposed by K.A. Eringen are as follows [13 - 10]:

$$[1 - (l_0 a)^2\nabla^2]\upsilon_{kl} = \tau_{kl}, \tag{2.5}$$

$$[1 - (l_0 a)^2\nabla^2]\tau_{kl} = \lambda\varepsilon_{kl}\delta_{kl} + 2\mu\varepsilon_{kl}.$$

Here, l_0 characteristic internal length, a - material constant. From equations (5) can be obtained equation for the beam:

$$[1 - (l_0 a)^2\frac{\partial^2}{\partial x^2}]\sigma_{xx} = E\varepsilon_{xx}, \tag{2.6}$$

$$[1-(l_0 a)^2\frac{\partial^2}{\partial x^2}]\tau_{xx}=2G\varepsilon_{xz}.$$

Assume that the beam material is inhomogeneous i.e.,$E=E(z)$ (the elastic modulus of the beam material is a continuous function of the coordinate of the thickness). If we consider the theory of the Euler-Bernoulli beams, then we can write:

$$\sigma_{xx}=E\left(\frac{\partial u}{\partial x}-z\frac{\partial^2 w}{\partial x^2}\right). \tag{2.7}$$

Here, u- moving by axis direction, w - the deflection of the axis of the beam.

In this case, the components of forces and moments are calculated according to the formulas:

$$P=\int_A\sigma_{xx}dA,\ \ N=\int_A\tau_{xx}dA,\quad M=\int_A\sigma_{xx}zdA, \tag{2.8}$$

where, S- cross-sectional area of the beam. Taking into account equation (7) from equation (8) the following for the time is obtained:

$$M=KI\frac{d^2w}{dx^2}. \tag{2.9}$$

Here, KI- generalized stiffness of the beam under consideration. If the inhomogeneity has the form: $E=E_0\left(1+\gamma_l\frac{z^2}{h^2}+\gamma_2\frac{z^4}{h^4}\right)$ then we get:

$$KI=E_0I\left\{1+\gamma_1\frac{3}{20}+\gamma_2\frac{3}{112}\right\}. \tag{2.10}$$

Where, E_0I bending stiffness of a homogeneous beam. After some transformations from (6) we can get:

$$\left[1-(e_0a)^2\frac{\partial^2}{\partial x^2}\right]M=-KI\frac{d^2w}{dx^2}. \tag{2.11}$$

The equations of motion of the beam under consideration are as follows:

$$\frac{\partial P}{\partial x}+f=m_0\frac{\partial^2 U}{\partial t^2} \tag{2.12}$$

$$\frac{\partial^2 M}{\partial x^2}+q-\frac{\partial}{\partial x}\left(P\frac{\partial w}{\partial x}\right)=m_0\frac{\partial^2 w}{\partial t^2}-m_2\frac{\partial^2 w}{\partial x^2\partial t^2}. \tag{2.13}$$

Where, P- axial compressive force, q- evenly distributed force,

$$m_0=\int_S\rho ds=\rho s;\quad m_2=\int_S z^2ds=\rho s\frac{h^2}{12}. \tag{2.14}$$

Here, the following expressions can be obtained for the time and force:

$$P=Ks\frac{\partial U}{\partial x}+\mu\frac{\partial}{\partial x}\left(m_0\frac{\partial^2 U}{\partial t^2}-f\right) \tag{2.15}$$

$$M=-KI\frac{\partial^2 w}{\partial t^2}+\mu\left[\frac{\partial}{\partial x}\left(P\frac{\partial w}{\partial x}\right)-q-m_0\frac{\partial^2 w}{\partial t^2}-m_2\frac{\partial^4 w}{\partial x^2\partial t^2}\right]. \tag{2.16}$$

Substituting the expression for the time (16) into the equation (13) we obtain the following equation of motion under consideration of an inhomogeneous beam:

$$\frac{\partial^2}{\partial x^2}\left(-KI\frac{\partial^2 w}{\partial x^2}\right)+\mu\frac{\partial^2}{\partial x^2}\left[\frac{\partial}{\partial x}\left(P\frac{\partial w}{\partial x}\right)-q+m_0\frac{\partial^2 w}{\partial t^2}-m_2\frac{\partial^4 w}{\partial x^2\partial t^2}\right]+$$

$$+q-\frac{\partial}{\partial x}\left(P\frac{\partial w}{\partial x}\right)=m_0\frac{\partial^2 w}{\partial t^2}-m_2\frac{\partial^4 w}{\partial x^2\partial t^2}. \quad (2.17)$$

By adding to this equation, the boundary conditions, we obtain the general statement of the problem considered.

3 Solution of stability problem of a compressed beam.

In general, the solution of equation (17) is associated with great mathematical difficulties. Therefore, we consider the case when the beam has only effective compressive load (i.. q=0). In this case, the equation (17) simplifies and turns in the form of:

$$\frac{d^2}{dx^2}\left(-KI\frac{d^2 w}{dx^2}\right)+\frac{d}{dx}\left(P\frac{dw}{dx}\right)+\mu\frac{d^2}{dx^2}\left[\frac{d}{dx}\left(P\frac{dw}{dx}\right)\right]=0. \quad (3.1)$$

Double integrating this equation, we obtain

$$EI\frac{d^2 w}{dx^2}-\mu P\frac{d^2 w}{dx^2}+Nw=k_1x+k_2. \quad (3.2)$$

Where, k_1,k_2- constant integration. If we consider the homogeneous equation, we get:

$$\frac{d^2 w}{dx^2}+\lambda^2 w=0. \quad (3.3)$$

Here:

$$\lambda^2=\frac{P}{KI-\mu P}\quad \text{or}\quad P=\frac{\lambda^2}{1+\mu\lambda^2}KI. \quad (3.4)$$

The general solution of equation (19) is obtained in the form of:

$$w=c_1\sin\lambda x+c_2\cos\lambda x+\frac{1}{\lambda^2}\left(k_1x+k_2\right). \quad (3.5)$$

Where integration constants are determined from the boundary conditions of the problem. Consider the case where the ends of the beam are rigidly fixed. In this case the boundary conditions are:

$$w=0;\quad \text{and}\quad \frac{dw}{dx}=0\text{if}x=0;\quad \text{a.} \quad (3.6)$$

Taking into account (22) from (23) the following transcendental equation is obtained:

$$\lambda a\sin\lambda a+2\cos\lambda a-2=0; \quad (3.7)$$

As seen from the solution, the equation (24) is a method of Newton that can be shown as $\lambda=2\Pi$. Then we get the formula for the critical load:

$$P=KI\frac{4\left(\frac{n\pi}{a}\right)^2}{1+4\mu\left(\frac{n\pi}{a}\right)^2}. \quad (3.8)$$

Hence, from the minimum value we obtain the formula of the critical load:

$$P_{kr} = \frac{KL}{a^2}\left(\frac{4\pi^2}{1+4\mu\pi^2}\right). \tag{3.9}$$

It should be noted that when =0,from (26) is obtained solution of a similar problem on the basis of the classical theory of elasticity. Taking into account (10) from (26) we find:

$$P_{kr} = \left(1+\gamma_1\frac{3}{20}+\gamma_2\frac{3}{112}\right)P_{kr}R. \tag{3.10}$$

Where $P_{krR} = \frac{El}{a^2}\left(\frac{4\pi^2}{1+4\mu\pi^2}\right)$- Reddy critical load for the considered homogeneous beam of Euler-Bernoulli [13]. In numerical calculations for specific parameters, the following values are taken:
$p = 2300kg/m^3$; $E_0 = 1000GPA$; $\nu = 0.19$; $G = 420GPA$; $d = 1.0 \cdot 10^{-9}m$; $I = 4.91 \cdot 10^{38}m^4$;
$A = 7.85 \cdot 10^{19}m^2; l_0 = 1.5 \cdot 10^9m$
The results of numerical calculations are presented in Table 1.

Table 1.

a/l_i		10	20	30	40	50
$l_0 = 0.5$	$\gamma_1 = \gamma_2 = 0$	0.9102	0.9758	0.9853	0.9938	0.9561
	$\gamma_1 = 0.5\ \gamma_2 = 0$	0.9785	1.049	1.0635	1.0683	1.0708
	$\gamma_1 = 0.5\ \gamma_2 = 1$	0.9788	1.052	1.0638	1.0686	1.0714
$l_0 = 1$	$\gamma_1 = \gamma_2 = 0$	0.7172	0.9102	0.9583	0.9758	0.9844
	$\gamma_1 = 0.5\ \gamma_2 = 0$	0.771	0.9785	1.030	1.0489	1.058
	$\gamma_1 = 0.5\ \gamma_2 = 1$	0.774	0.9788	1.033	1.0492	1.062

4 Conclusions.

The article gives a general statement of the problem of stability of nano-micro elements such as non-uniform beam of the Euler-Bernoulli equation using state of nonlocal theory of elasticity Eringen K.A. Solutions stability problem of considered beams under axial compression are obtained. When fixing the hard edges of the beam, the formula for the determination of critical load is found.

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