Transactions of NAS of Azerbaijan, Issue Mechanics, **35** (7), 22-31 (2016).
Series of Physical-Technical and Mathematical Sciences.

Pulsating flow of two-phase viscous bubbly fluid in an elastic semi-infinite cylindrical tapering tube

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Received: 16.06.2015 / Accepted: 11.09.2015

Abstract. *The problem of pulsating flow of a viscous two-phase bubbly fluid in an elastic semi-infinite cylindrical tube in view of narrowing effect was considered. A linear one-dimensional equations were used. Pulsating pressure was set at the end of the tube in order to describe the pressure, density,fluid flow and displacement. The task was created to resolve the Sturm-Lyuville's singular boundary value problem, which, in turn, is reduced to an equivalent integral equation of Volterra type, which is solved by successive approximations method. Under the condition of integrability of the potential is proved to converge to the exact solution. For the numerical implementation a flexible tubing with constant cross section and flowing glycerol containing small additions of air bubbles was considered and their influence on wave characteristics was numerically found.*

Keywords. pulsating flow · bubbly fluid · elastic tube · viscosity · gas

Mathematics Subject Classification (2010): 76T10

1 Introduction

Due to the wide spread in technology and living organisms transport phenomena, the problem of wave propagation in fluid filled deformable tube is a very topical and is of interest in several aspects. In the theoretical aspect - this is the problem of mathematical physics, and as the applied aspect of problem - a necessary stage in the calculation of the system, subject to dynamic stress. To date, the totality of these tasks, the investigation of which laid down in the fundamental works of [2, 4, 15] is a widely developed field of hydrodynamics [7, 10, 13]. However, a number of features associated with simultaneous consideration of two-phase fluid, alongside with viscosity and tube tapering remain understudied.

Therefore, the investigation of regularities of wave dynamics of a bubble viscous fluids flowing in deformable tubes, caused by importance of the application of research results to problems of the aircraft hydraulic systems, oil and gas industry, chemical engineering, hemodynamics, etc. [3,6,11].

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Basic relations

Set of equations describing the propagation of waves in two-phase bubble viscous fluid contained in an elastic tube of variable circular cross section.

A mathematical model of fluid. Two-phase medium comprising a mixture of a liquid with fine gas bubbles, are a very important example of relaxing media. Experimental and theoretical studies have shown that, solving the problem of transport of two-phase flows, it is necessary to keep in mind that such environments differ from other two-phase environments by the fact that the heat capacity of the carrier phase is much greater than the heat capacity of the dispersed phase due to predominant mass content of carrying phase in unit volume. In this regard, the liquid can be considered as a thermostat having a constant temperature [9]. Following [6], we put the following assumptions, which are the basis of the theory used here to describe the flow of bubble mixtures by the methods of continuum mechanics, which simplify the formulation and solution of the problem without distorting the essence of the phenomenon:

-in each elementary macrovolume of bubbles are present in the form of spherical inclusions of the same radius r_0, and the volume concentration of bubbles is low (the mixture is monodisperse), and the value of r_0 is much smaller than the characteristic dimensions of the problem;

-direct interactions and collisions of bubbles with each other can be neglected;

-the merge processes (coagulation), crushing and formation of new vesicles are absent;

-speed of the bubbles and the carrier phase are the same;

-bubbles have neutral buoyancy, i.e. do not settle and do not float;

-the viscosity of the carrier phase is much greater than the viscosity of gas bubbles (for example, the viscosity of water is 10 times greater than the viscosity of air) and therefore the viscosity of the mixture practically does not depend on volume content of bubbles.

As part of the assumptions we write the momentum equation:

$$\rho_0 \frac{\partial u}{\partial t} + \frac{\partial p}{\partial} = 0 \tag{1.1}$$

and the rheological equation of state of the mixture [9]:

$$p = a^2 \rho + \frac{\xi}{\rho_0} \frac{\partial \rho}{\partial t}. \tag{1.2}$$

For a one-dimensional approximation the continuity equation for a tube with variable cross section can be obtained based on the following physical considerations. Select in the space filled with the mixture elementary volume $S(x)dx$, where $S(x) = \pi R^2(x)$ - cross-sectional area of the tube. Calculate the difference between the rate of the liquid flowing through a time dt through the opposite plane at a distance dx:

$$\{[Su + \frac{\partial}{\partial x}(Su)dx] - Su\}dt = \frac{\partial}{\partial x}(Su)dxdt.$$

On the other hand, the extra flow rate is due to deformation of the pipe walls and shown as:

$$L\frac{\partial w}{\partial t}dxdt,$$

where $L(x) = 2\pi R(x)$ – the length of its circumference. For a compressible medium it is necessary to take into account the charge associated with the reduction of its density:

$$-S\frac{1}{\rho_0}\frac{\partial \rho}{\partial t}dxdt.$$

Thus, the continuity equation is finally written in the form:

$$S\frac{1}{\rho_0}\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(Su) + L\frac{\partial w}{\partial t} = 0. \tag{1.3}$$

In equations (1.1) - (1.3) $u(x,t)$ - the flow rate of the mixture, $p(x,t)$- hydrodynamic pressure, $\rho(x,t)$ - density of mixture; $w(x,t)$ - the radial throw of the wall,

$$a^2 = \frac{1}{\alpha_{20}(1-\alpha_{20})}\left(\frac{\rho_{10}}{\rho_{10}-\rho_{10}}\right)\frac{p_0}{\rho_{10}} \tag{1.4}$$

is the square of the equilibrium sound velocity,

$$\rho_0 = \alpha_{10}\rho_{10} + \alpha_{20}\rho_{20} \qquad (\alpha_{10} + \alpha_{20} = 1) \tag{1.5}$$

$$\xi = \frac{4}{3}\frac{\mu(1-\alpha_{20})}{\alpha_{20}} \tag{1.6}$$

is the bulk viscosity, where μ - dynamic viscosity of the carrier phase. Here α_{20} - volume fraction of bubbles,ρ_{10}, ρ_{20}- the densities of the carrier and the dispersed phase, p_0 – configurable static pressure. An 0 index means the value in the equilibrium state. It should be noted that in the linear setting the equilibrium α_{20} are using instead of the current bulk concentration of α_2, and this approach assumes a priori of the presence of bubbles ($\alpha_{20} \neq 0$). If the volume fraction of bubbles is sufficiently small, ($\alpha_{20} << 1$), the medium can be thought as homogeneous. The peculiarity of such liquid with $\rho_{20} << \rho_{10}$ is that:

$$\rho_0 = \alpha_{10}\rho_{10} + \alpha_{20}\rho_{20} \approx \alpha_{10}\rho_{10} \approx \rho_{10}. \tag{1.7}$$

This allows with a sufficient degree of accuracy to rewrite formulas (1.4) and (1.6) as follows:

$$a^2 \approx \frac{p_0}{\alpha_{20}\rho_{10}},\ \xi \approx \frac{4}{3}\frac{\mu}{\alpha_{20}}. \tag{1.8}$$

In this case, as it follows from the first formula (1.8), compression of the mixture occurs due to the gas component.

The equation of tube motion. Now, to closure equations (1.1) - (1.3) we write the equation of motion of the tube, believing it is linear elastic, that the ratio of wall thickness h to the radius and that the tube is rigidly attached to the environment, causing it cannot move along its axis. Under these conditions, it is sufficient to use the following equation [14] :

$$p = \frac{hE}{(1-\nu^2)R^2(x)}w + \rho_* h\frac{\partial^2 w}{\partial t^2}, \tag{1.9}$$

where ρ_* - the density of wall, E - Young's modulus, ν - Poisson's ratio. The value $(1-\nu^2)^{-1}$ in the latter equation is necessary to account for ties, which prevents axial displacement. The second term in (1.9) expresses the inertia of the tube wall. This effect is usually considered negligible in the present case it can be ignored [7]. So we write down

$$w = \frac{(1-\nu^2)R^2(x)}{hE}p. \tag{1.10}$$

So, equations (1.1) – (1.3) and (1.10) represent a closed system of hydroelasticity, which can be used to describe the evolution of small perturbations in a tube of variable cross section containing a gas-liquid medium.

Resolving equation. Now, without loss of generality, the function $R(x)$ suppose by equality $R(x) = R_\infty g(x)$ and we assume that the function $g(x)$ twice differentiable. Let's also assume that at infinity the tube has a constant cross-section with a radius of R_∞. Hence we conclude that:

$$\lim_{x\to\infty} g(x) = 1. \tag{1.11}$$

At the same time believe that

$$\lim_{x\to\infty} g'(x) = 0,\ \lim_{x\to\infty} g''(x) = 0. \tag{1.12}$$

Here and below, primes means differentiation with respect to x. An example of such function is [10]:

$$g(x) = 1 + e^{-\alpha x},\ \ \alpha > 0, \tag{1.13}$$

which characterizes the tapering tube along its length. Now equation (1.1) - (1.3) can be represented as follows:

$$\frac{1}{\rho_0}\frac{\partial \rho}{\partial t} + 2\frac{g\prime(x)}{g(x)}u + \frac{\partial u}{\partial x} + \frac{2}{R_\infty g(x)}\frac{\partial w}{\partial t}$$

$$\frac{\partial u}{\partial t}+\frac{1}{\rho_0}\frac{\partial p}{\partial x}=0$$
$$p=a^2\rho+\frac{\xi}{\rho_0}\frac{\partial \rho}{\partial t} \tag{1.14}$$
$$w=\frac{(1-\nu^2)R^2g^2(x)}{hE}p.$$

Characteristic for this consideration is the appropriateness of reduction of system (1.14) to one equation relative to the function $\rho(x,t)$. To this end, we proceed as follows: by means of differentiation from the first two equations of (1.14) eliminate the function $u(x,t)$. The result:

$$\frac{2}{R_\infty g(x)}\frac{\partial^2 w}{\partial t^2}+\frac{1}{\rho_0}\frac{g\prime(x)}{g(x)}+\frac{1}{\rho_0}\frac{\partial^2\rho}{\partial t^2}-\frac{1}{\rho_0}\frac{\partial^2\rho}{\partial x^2}=0.$$

Given the following two equations, after simple transformations, introducing for brevity the notation recording

$$c_0^2\frac{Eh}{2\rho_0(1-\nu^2)R_\infty},$$

finally we can write:

$$\left\{1+g(x)\frac{a^2}{c_0^2}\right\}\frac{\partial^2\rho}{\partial t^2}+\frac{\xi}{\rho_0c_0^2}\frac{\partial^3\rho}{\partial t^3}-2\frac{g\prime(x)}{g(x)}\left\{a^2\frac{\partial\rho}{\partial x}+\frac{\xi}{\rho_0}\frac{\partial^2\rho}{\partial x\partial t}\right\}=0. \tag{1.15}$$

Hence, in particular, for $g(x)=1$ case is implemented tube of constant circular cross-section.

2 A solution of the problem

With equation (1.15), we reduce it to solving of ordinary differential equation.

Boundary problem. For the description of complex pulses specific to wave motion, harmonic analysis is used, i.e. pulses of complex shapes decomposed on the sinusoidal components that make up the Fourier series. Because of linearity and homogeneity of the original equations is traced through each harmonic frequency $n\omega$, where n is a natural number and for determining the pulse shape at any point in the system are summed components, corresponding to the given point. Hence we may conclude that in the mathematical aspect of fundamental importance is the consideration of the part of sinusoidal vibrations with one frequency ω. Therefore, using the method of separation of variables, the solution of equation (1.15) will be sought in the class of functions:

$$\rho(x,t)=\varphi(x)\exp(i\omega t), \tag{2.1}$$

where $\varphi(x)$ - required, generally speaking, complex function, and $i=\sqrt{-(1)}$- the imaginary unit. As a result, substituting (2.1) in equation (1.5), we have:

$$\varphi''2\frac{g\prime(x)}{g(x)}\varphi'+G(x)\varphi=0. \tag{2.2}$$

It is assumed here:

$$G(x)=\left\{\omega^2\left[1+g(x)\frac{a^2}{c_0^2}\right]+i\omega^3g(x)\ \frac{\xi}{\rho_0c_0^2}\right\}\left(a^2+i\omega\frac{\xi}{\rho_0}\right)^{-1}. \tag{2.3}$$

Using the change of Liouville [5]:

$$y=\varphi(x)\exp\frac{1}{2}\int 2\frac{g\prime(x)}{g(x)}dx=\varphi(x)\exp\ln g(x), \tag{2.4}$$

we write the reduced form of the wave equation

$$y''+I(x)y=0 \tag{2.5}$$

at the invariant

$$I(x) = G(x) - \frac{1}{4}\left\{2\frac{g\prime(x)}{g(x)}\right\}^2 - \frac{1}{2}\left\{2\frac{g\prime(x)}{g(x)}\right\}',$$

which, after simplification, becomes

$$I(x) = G(x) - \frac{g\prime(x)}{g(x)}. \tag{2.6}$$

For further considerations from (2.6), following (1.11) and (1.12), set the limit using the equation:

$$\lim_{x\to\infty} I(x) = \frac{m_1 + im_2}{a^2 + im_3} = \delta^2, \tag{2.7}$$

in which

$$m_1 = \omega^2\left(1 + \frac{a^2}{c_0^2}\right),\ m_2 = \omega^3\frac{\xi}{\rho_0 c_0^2}\ , m_3 = \omega\frac{\xi}{\rho_0}.$$

Transforming equation (2.5) using the substitution [1]

$$q(x) = 1 - \frac{I(x)}{\delta^2}, \tag{2.8}$$

as a differential equation of the problem we get:

$$y'' + \delta^2 y = \delta^2 q(x)y. \tag{2.9}$$

According to the rule of square root of complex number, following the dispersion equation (2.7) and introducing additional notation

$$k_1 = \frac{m_1 a^2 + m_2 m_3}{a^4 + m_3^2}, \quad k_2 = \frac{m_1 m_3 - m_2 a^2}{a^4 + m_3^2},$$

define the value of δ

$$\delta = \pm(\delta_0 - i\delta_1)$$

where

$$\delta_0 = \sqrt{\frac{2 + k_1}{2}}, \qquad \delta_1 = \sqrt{\frac{r - k_1}{2}} \qquad r = \sqrt{k_1^2 + k_2^2}.$$

Next, we use root for which

$$\mathrm{Jm}\delta < 0 \tag{2.10}$$

i.e.

$$\delta = \delta_0 - i\delta_2$$

and on the potential q(x) we impose the integrability condition

$$\int_0^\infty |q(x)|dx < +\infty. \tag{2.11}$$

It is easy to show that the constructed according to the formula (2.8) the function $q(x)$ in (1.13) meet the conditions (2.11). To build the solution, equation (2.9) should be supplemented with the following boundary conditions:

$$y(0) = y_0, y \to 0\ \ y \to 0, \quad x \to \infty. \tag{2.12}$$

Note that the second condition (2.12) ensures the boundedness of the desired solution. Move compute the value of y_0 depends on the mode of operation of the system. A typical case is a situation in which at the end of the tube set to a pulsating pressure

$$p(0,t) = \check{p}\exp(i\omega t), \tag{2.13}$$

where $\check{p}$ determined by experiment.

From (2.4) and (2.1):

$$\rho = \frac{y(x)}{\exp \ln g(x)} \exp(i\omega t).$$

Taking into account this equality, from (1.2) we obtain:

$$p = \frac{y(x)}{\exp \ln(x)} (a^2 + im_3) \exp(i\omega t). \tag{2.14}$$

Side-by-side comparison of (2.13) and (2.14) at $x = 0$ can be defined

$$y_0 = \frac{\breve{p}}{a^2 + im_3} \exp \ln(0). \tag{2.15}$$

Thus, the solution of the problem was reduced to a singular boundary value problem of the Sturm-Liouville problem (2.9) and (2.12) under the condition (2.11), when y(0) is determined by formula (2.15).

The equivalent integral equation. It is usually advisable solution of the stated above boundary value problem is reduced to solution of integral equations. Homogeneous equation:

$$y'' + \delta^2 y = 0 \tag{2.16}$$

has a fundamental system of solutions

$$y_1(x) = e^{i\delta x} \ \ and \ \ y_2(x) = e^{-i\delta x}.$$

Considering (2.9) as an inhomogeneous equation with known right-hand side

$$\delta^2 q(x) y(x)$$

and, applying the method of variation of arbitrary constants, after the procedure, the solution of problem (2.9) and (2.12) under the condition (2.10) is reduced to the equivalent integral equation

$$y(x, -\delta) = Ce^{-i\delta x} + \delta \int_x^\infty \sin \delta(\tau - x) q(\tau) y(\tau, -\delta) d\tau, \tag{2.17}$$

where C is a constant of integration, which we define thus, to satisfy the first boundary condition (2.12). It will write the expression:

$$C = \frac{y_0}{f(0, -\delta)}.$$

The value of y we can define by the equality of the form

$$y(x, -\delta) = y_0 \frac{f(x, -\delta)}{f(0, -\delta)}.$$

Here a new function $f(x, -\delta)$ is determined by solving the integral equation

$$f(x, -\delta) = e^{-i\delta x} + \delta \int_x^\infty \sin \delta(\tau - x) q(\tau) y(\tau, -\delta) d\tau, \qquad (2.18)\,,$$

which is the equation of Volterra type and can be solved by successive approximations. By definition [13] find the solution of (2.18) as follows:

$$f_0(x, -\delta) = \exp(-i\delta x)$$

....

$$f_{n+1}(x, -\delta) = \exp(-i\delta x) + \delta \int_x^\infty \sin \delta(\tau - x) q(\tau) f_n(\tau, -\delta) d\tau$$

$$|f_0(x, -\delta)| \le \exp(\mathrm{Im}\delta)x.$$

Furthermore, in view of (2.11), by induction we will prove the estimate

$$|f_n(x,-\delta) - f_{n-1}(x,-\nu)| \le \frac{B_\delta^n(x)}{n!}\ e^{(\mathrm{Im}\delta)x}, \tag{2.19}$$

in which

$$B_\delta(x) = |\delta| \int\limits_x^\infty |q(\tau)| d\tau.$$

Thus, on the basis of (2.11)

$$B_\delta(x) = |\delta| \int\limits_x^\infty |q(\tau)| d\tau \le |\delta| \int\limits_0^\infty |q(x)| dx = B_\delta(0) < +\infty$$

and, consequently,

$$|f_n(x,-\delta) - f_{n-1}(x,-\delta)| \le \frac{B_\delta^n(x)}{n!}\ e^{(\mathrm{Im}\delta)x}. \tag{2.20}$$

Further limiting only the most essential calculations and taking into account the assessment

$$|\sin\delta(x-\tau)| \le \exp(-\mathrm{Jm}\delta)(\tau - x) \qquad\qquad (\tau \ge x)$$

for $n = 1$ we have:

$$|f_1(x,-\delta) - f_0(x,-\delta)| = |\delta| \int\limits_x^\infty \left|\sin(\tau - x) q(\tau) e^{-i\delta\tau}\right|$$

$$\le |\delta| \int\limits_x^\infty e^{-\mathrm{Jm}\delta(\tau-x)})|q(\tau)| e^{\mathrm{Jm}\delta\tau} d\tau = B_\delta(x) e^{(\mathrm{Jm}\delta)x}.$$

Let (2.19) holds for $n = m$. We prove its validity for $n = m+1$:

$$|f_{m+1}(x,-\delta) - f_m(x,-\delta)| \le |\delta| \int\limits_x^\infty |\sin\delta(\tau - x)||f_m(\tau,-\delta) f_{m-1}(\tau,-\delta)||q(\tau)| d\tau$$

$$\le \frac{e^{(\mathrm{Jm}\delta)x}}{m!} |\delta| \int\limits_x^\infty B_\delta^m(\tau)|q(\tau)| d\tau = \frac{B_\delta^{m+1}(\tau)(x)}{(m+1)!} e^{(\mathrm{Jm}\delta)x}.$$

Noticing that

$$|f_0(x,-\delta)| \le \exp(\mathrm{Jm}\delta)x \le 1$$

by (2.20) we conclude that:

$$f_0(x,-\delta) + \sum_{n=1}^{\infty} \{f_n(x,-\delta) - f_{n-1}(x,-\delta)\} \tag{2.21}$$

dominated in the interval $[0, +\infty)$ converging positive number series:

$$\sum_{n=1}^{\infty} \frac{B_\delta^n(0)}{n!}$$

and, therefore, on the basis of *Weierstrass* [11] it converges uniformly for $x \in [0, \infty)$ and its amount is only limited and the solution of equation (2.18). Note that from the structure of the series (2.21) it follows that the ranks obtained his memberwise differentiation on x, also converges uniformly. Now by a direct calculation it is easy to establish that this solution is also a solution of equation (2.9).

For practical purposes it is convenient to a different view of (2.21). Believing

$$f_n(x,-\delta) - f_{n-1}(x,-\delta) = \delta^n \psi_n(x,-\delta) \tag{2.22}$$

note, that

$$f(x,-\delta)=\sum_{n=1}^{\infty}\delta^n\psi_n(x,-\delta), \tag{2.23}$$

where

$$\psi_0(x,-\delta)=\exp(-i\delta x), \tag{2.24}$$

and for $n\geq 1$

$$\psi_n(x,-\delta)=\int_x^{\infty}\sin\delta(\tau-x)\psi_{n-1}(\tau-\delta)d\tau. \tag{2.25}$$

Thus, the series (2.23) combined with (2.24) and (2.25) gives a meaningful representation of the solution.

Now, given a formula for a function $y(x,-\delta)$, and the ratio (2.1), (2.4), equation (1.14) and the expression (2.15), after simple transformations we come to the relations

$$\rho=\breve{p}\frac{\exp\ln g(0)}{\exp\ln g(x)}(a^2+im_3)^{-1}\frac{f(x,-\delta)}{f(0,-\delta)}\exp(i\omega t), \tag{2.26}$$

$$p=\breve{p}\frac{\exp\ln g(0)}{\exp\ln g(x)}\frac{f(x,-\delta)}{f(0,-\delta)}\exp((i\omega t), \tag{2.27}$$

$$w=\breve{p}\frac{(1-\nu^2)R_\infty^2}{hE}\frac{\exp\ln g(0)}{\exp\ln g(x)}\frac{f(x,-\delta)}{f(0,-\delta)}\exp(i\omega t). \tag{2.28}$$

To determine the flow velocity will hold the following reasoning. As above, separating the variables, we write

$$u(x,t)=U(x)\exp(i\omega t),$$

and by $\varphi(x)$ denote the function

$$\phi(x)=\frac{\exp\ln g(0)}{\exp\ln g(x)}\frac{f(x,-\delta)}{f(0,-\delta)}.$$

Then from equation (1.1) we can to write an expression for the velocity distribution. It has the form:

$$u=\breve{p}\frac{1}{\omega\rho_0}\phi'(x)\exp(i\omega t). \tag{2.29}$$

Special case. Leaving aside the factor of contraction, we will focus on the consideration of issues of interest to hydrodynamics. This schematization aims to gain a clear dependence, allowing to estimate influence of the concentration of bubbles on wave characteristics. In this special case we have the obvious equality:

$$g(x)\equiv 1\ \ (R=R_\infty),\ \ q(x)=0,$$

from which follows:

$$f(x,-\delta)=e^{-i\delta x},\ \ f(0,-\delta)=1,\ \phi(x)=e^{-i\delta x}.$$

Now the solution of (2.26) - (2.29) is simplified and, keeping the same notation can be written as follows:

$$\rho=\breve{p}(a^2+im_3)^{-1}\exp[i(\omega t-\delta x)],$$

$$p=\breve{p}\exp[i(\omega t-\delta x)]$$

$$w=\breve{p}\frac{R^2}{hE}(1-\nu^2)\exp[i(\omega t-\delta x)]$$

$$u=-\breve{p}\frac{\delta}{\rho_0\omega}\exp[i(\omega t-\delta x)].$$

Hence, in accordance with *Euler's* formula for the amplitudes of these functions we got:

$$|\rho|=\frac{\breve{p}e^{-\delta_1 x}}{\sqrt{a^4+m_3^2}}$$

$$|p| = \check{p}e^{-\delta_1 x} \tag{2.30}$$

$$|w| = \check{p}e^{-\delta_1 x}\frac{R^2(1-\nu^2)}{hE}$$

$$|u| = \check{p}\left(\frac{e^{-\delta_1 x}}{\rho_0\omega}\right)\sqrt{\delta_0^2+\delta_1}.$$

The formulas (2.30) can form the basis of the calculation of the unknown amplitudes of the wave velocity $c = \omega/\delta_0$ and damping of δ_1, depending on α_{20}.

A numerical example. For numerical implementation we define the parameters of the corresponding data for the rubber tube with the following characteristics: $E = 4.105H/m^2$; $\nu = 0,5$; $R = 0,002m$; $\omega = 10^{-1}$ with; $\check{p} = 14 \cdot 10N/m^2$ with respect to the option of filling with a mixture of glycerol containing small additions of air $\alpha_{20} = \{10^{-2} \sim 10^{-1}\}$. Next, we define $\rho_0 = 13 \cdot 102kg/m^2$, $\mu = 1,4kg/m \cdot s\ and\ p_0 = 105N/m^2$.

Table

$\lvert\rho\rvert/\rho_0$	$c = \omega/\delta_0$	$\lvert u\rvert/h\omega$	α_{20}
0,00001	45,59608	11,80938	0,01
0,00003	40,45567	1330992	0,02
0,00004	36,73592	14,65763	0,03
0,00006	33,88371	15,89146	0,04
0,00007	31,60698	17,03616	0,05
0,00008	29,73505	18,10865	0,06
0,00012	8,16064	20,08252	0,07
0,00011	26,81245	21,0	0,08
0,00013	25,64103	21,87904	0,09
0,00014	24,61084		0,1

The table shows the dependence of the density amplitude $|\rho|/\rho_0$, the speed of wave propagation c and the speed of the mixture $|u|/h\omega$ by volume content of bubbles. The corresponding value for the hydrodynamic pressure and the displacement is not specified, since they do not depend on the size α_{20} and $\delta_1 \approx 0$.

Conclusion

Therefore, for the chosen values of the parameters and mode of operation of the system can be concluded:

– the wave propagation speed is significantly reduced depending on the concentration of gas bubbles;

– the table shows that the amplitude of the dimensionless density increases an order of magnitude depending on α_{20};

– it is established that the viscosity only slightly changes the nature of the flow mixture.

In conclusion, we note that the change of volume content of bubbles can be increased (decrease) the velocity of the fluid and thus in a certain way to optimize the functioning of the system.

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