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Analysis of torsional vibrations of radially non-homogeneous cylinder

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Abstract. *In the paper, by the method of homogeneous solutions, we study torsional vibrations of a radially-nonhomogeneous hollow isotropic cylinder when lateral surfaces are free of stresses. Variance equation is constructed and its roots are studied. Asymptotic formulas for displacements and stresses allowing to calculate stress-strain state at different values of forcing forces frequency are obtained.*

Keywords. method of homogeneous solutions · variance equation · thickness resonance · super high-frequency vibrations · fringe effect

Mathematics Subject Classification (2010): 74H45

1 Introduction

Investigations of nonhomogeneous constructions hold a prominent place in theory of shells. Complexity of phenomena arising under deformation of nonhomogeneous constructions has lead to creation of different applied theories each of which was constructed on the basis of certain system of conjectures and whose fields of applicability were not studied enough. Existence of different applied theories for nonhomogeneous constructions requires its critical analysis based on strong mathematical approach, i.e. from the position of three-dimensional equations of elasticity theory. It is especially important when studying nonstationary and stationary vibrations in rather wide frequency range, when researching stress concentrations near boundary and local loadings and in many other cases.

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2 Problem statement

Let us consider stationary torsional vibrations of a radially-nonhomogeneous hollow isotropic cylinder. We assume that the shear modulus is a quadratic function of radius. Denote by $\Gamma = \{r \in [R_1; R_2], \ \varphi \in [0; 2\pi], \ z \in [-L; L]\}$ a domain occupied by the cylinder in cylindric system of coordinates.

Equations of motion have the form [1]:

$$\frac{\partial \sigma_{\rho\varphi}}{\partial \rho} + \frac{\partial \sigma_{\varphi\xi}}{\partial \xi} + \frac{2}{\rho}\sigma_{\rho\varphi} = \frac{gR_0^2}{G_*}\frac{\partial^2 u}{\partial t^2}. \tag{2.1}$$

Here $\rho = \frac{r}{R_0}$; $\xi = \frac{z}{R_0}$ are dimensionless coordinates; $u = \frac{u_\varphi}{R_0}$; $\sigma_{\rho\varphi} = \frac{\sigma_{r\varphi}}{G_*}$, $\sigma_{\varphi\xi} = \frac{\sigma_{\varphi z}}{G_*}$ are dimensionless quantities; $u_\varphi = u_\varphi(\rho, \xi, t)$ is a displacement vector component; g is the density of the cylinder's material; $\sigma_{r\varphi}, \sigma_{\varphi z}$ are stress tensor components; G_* is some typical parameter with dimension of shear modulus; $R_0 = \frac{R_1+R_2}{2}$ is the radius of cylinder's median surface; $\rho \in [\rho_1, \rho_2]$, $\xi \in [-l; l]$, $l = \frac{L}{R_0}$, $\rho_s = \frac{R_s}{R_0}$ $(s = 1, 2)$.

Stress tensor components are expressed by vector displacements components in the following way [1]:

$$\sigma_{\rho\varphi} = G\left(\frac{\partial u}{\partial \rho} - \frac{u}{\rho}\right), \ \sigma_{\varphi\xi} = G\frac{\partial u}{\partial \xi}, \tag{2.2}$$

where $G = G_0\rho^2$, G_0 is a constant.

Substituting (2.2) in (2.1), we get an equation of motion in displacements

$$G_0 \cdot \left(\rho^2\frac{\partial^2 u}{\partial \rho^2} + 3\rho\left(\frac{\partial u}{\partial \rho} - \frac{u}{\rho}\right) + \rho^2\frac{\partial^2 u}{\partial \xi^2}\right) = \frac{gR_0^2}{G_*}\frac{\partial^2 u}{\partial t^2}. \tag{2.3}$$

Suppose that the lateral part of the cylinder is free from stresses, i.e.

$$\sigma_{\rho\varphi} = G_0\rho^2\left(\frac{\partial u}{\partial \rho} - \frac{u}{\rho}\right)\Bigg|_{\rho=\rho_s} = 0, \tag{2.4}$$

and on the endfaces of the cylinder the following boundary conditions are fulfilled:

$$\sigma_{\varphi\xi} = G_0\rho^2 \left.\frac{\partial u}{\partial \xi}\right|_{\xi=\pm l} = f^{\pm}(\rho) \cdot e^{i\omega t}, \tag{2.5}$$

where ω is oscillations frequency.

3 Problem solution

We will look for the solution of (2.3) in the form:

$$u(\rho, \xi, t) = v(\rho)\, m(\xi)\, e^{i\omega t}, \tag{3.1}$$

where the function $m(\xi)$ is subjected to the condition

$$m''(\xi) - \mu^2 m(\xi) = 0, \tag{3.2}$$

and the parameter μ is determined after fulfilling boundary conditions on lateral surface.

After substitution of (3.1) in (2.3), (2.4) allowing for (3.2) we have:

$$v''(\rho) + \frac{3}{\rho}v'(\rho) + \left(\mu^2 + \frac{\frac{\lambda^2}{G_0} - 3}{\rho^2}\right) \cdot v(\rho) = 0, \tag{3.3}$$

$$G_0\rho^2\left(v'(\rho) - \frac{v(\rho)}{\rho}\right)\Bigg|_{\rho=\rho_s} = 0, \tag{3.4}$$

where $\lambda^2 = \frac{gR_0^2\omega^2}{G_*}$ is a dimensionless frequency parameter.

We can represent boundary value problem (3.3), (3.4) in the form

$$Av = \mu^2 v, \tag{3.5}$$

where

$$Av = \left\{ -\left[\frac{d^2 v(\rho)}{d\rho^2} + \frac{3}{\rho}\left(\frac{dv(\rho)}{d\rho} - \frac{v(\rho)}{\rho} \right) + \frac{\lambda^2}{G_0}\frac{v(\rho)}{\rho^2} \right];\ G_0\rho^2 \left(\frac{dv(\rho)}{d\rho} - \frac{v(\rho)}{\rho} \right)\Big|_{\rho=\rho_s} = 0 \right\}.$$

A is a self-adjoint operator in Hilbert space $L_2(\rho_1, \rho_2)$ with the weight ρ^3. All eigen-values μ_k^2 are real, and corresponding eigen functions are orthonormed:

$$(v_k, v_n) = \int_{\rho_1}^{\rho_2} v_k(\rho)\, v_n(\rho)\, \rho^3 d\rho = \delta_{kn}. \tag{3.6}$$

The general solution of (3.3) is of the form:

$$v(\rho) = \frac{1}{\rho}\left[C_1 J_{\sqrt{4-\frac{\lambda^2}{G_0}}}(\mu\rho) + C_2 Y_{\sqrt{4-\frac{\lambda^2}{G_0}}}(\mu\rho) \right], \tag{3.7}$$

where $J_{\sqrt{4-\frac{\lambda^2}{G_0}}}(\mu\rho)$, $Y_{\sqrt{4-\frac{\lambda^2}{G_0}}}(\mu\rho)$ are first and second order Bessel functions, respectively; $C_1,\ C_2$ are arbitrary constants.

By means of (3.7), satisfying boundary conditions (3.4), with respect to C_1 and C_2 we get homogeneous linear system of algebraic equations. From the condition of existence of nontrivial solutions of this system, we have the variance equation:

$$\Delta(\mu, \lambda, \rho_1, \rho_2) = \mu^2 \rho_1 \rho_2 L^{(1;1)}_{\sqrt{4-\frac{\lambda^2}{G_0}}}(\mu)$$

$$-2\mu\left(\rho_1 L^{(1;0)}_{\sqrt{4-\frac{\lambda^2}{G_0}}}(\mu) + \rho_2 L^{(0;1)}_{\sqrt{4-\frac{\lambda^2}{G_0}}}(\mu) \right) + 4L^{(0;0)}_{\sqrt{4-\frac{\lambda^2}{G_0}}}(\mu) = 0, \tag{3.8}$$

where $L^{(i;j)}_{\sqrt{4-\frac{\lambda^2}{G_0}}}(\mu) = J^{(i)}_{\sqrt{4-\frac{\lambda^2}{G_0}}}(\mu\rho_1)\, Y^{(j)}_{\sqrt{4-\frac{\lambda^2}{G_0}}}(\mu\rho_2) - J^{(j)}_{\sqrt{4-\frac{\lambda^2}{G_0}}}(\mu\rho_2)\, Y^{(i)}_{\sqrt{4-\frac{\lambda^2}{G_0}}}(\mu\rho_1);\ (i; j = 0; 1)$.

The left hand side of (3.8) as an entire function of the parameter μ, has a denumerable set of zeros with a concentration at infinity. The following solutions correspond to the denumerable set of zeros with a concentration at infinity. The following solutions correspond to the denumerable set of the roots of equation (3.8)

$$u = \sum_{k=1}^{\infty} \frac{1}{\rho}\left[\mu_k \rho_2 L^{(0;1)}_{\sqrt{4-\frac{\lambda^2}{G_0}}}(\mu_k\rho;\ \mu_k\rho_2) - 2L^{(0;0)}_{\sqrt{4-\frac{\lambda^2}{G_0}}}(\mu_k\rho;\ \mu_k\rho_2) \right] m_k(\xi)\, e^{i\omega t}, \tag{3.9}$$

$$\sigma_{\rho\varphi} = \sum_{k=1}^{\infty} G_0 \left[\mu_k^2 \rho_2 \rho L^{(1;1)}_{\sqrt{4-\frac{\lambda^2}{G_0}}}(\mu_k\rho;\ \mu_k\rho_2) - 2\rho_2\mu_k L^{(0;1)}_{\sqrt{4-\frac{\lambda^2}{G_0}}}(\mu_k\rho;\ \mu_k\rho_2) \right.$$

$$\left. -2\rho\mu_k L^{(1;0)}_{\sqrt{4-\frac{\lambda^2}{G_0}}}(\mu_k\rho;\ \mu_k\rho_2) + 4L^{(0;0)}_{\sqrt{4-\frac{\lambda^2}{G_0}}}(\mu_k\rho;\ \mu_k\rho_2) \right] m_k(\xi)\, e^{i\omega t}, \tag{3.10}$$

$$\sigma_{\rho\xi} = \sum_{k=1}^{\infty} G_0 \rho \left[\mu_k \rho_2 L^{(0;1)}_{\sqrt{4-\frac{\lambda^2}{G_0}}}(\mu_k\rho;\ \mu_k\rho_2) - 2L^{(0;0)}_{\sqrt{4-\frac{\lambda^2}{G_0}}}(\mu_k\rho;\ \mu_k\rho_2) \right] m_k'(\xi)\, e^{i\omega t}, \tag{3.11}$$

where $L^{(i;j)}_{\sqrt{4-\frac{\lambda^2}{G_0}}}(\mu_k\rho;\,\mu_k\rho_2) \;=\; L^{(i)}_{\sqrt{4-\frac{\lambda^2}{G_0}}}(\mu_k\rho)\,Y^{(j)}_{\sqrt{4-\frac{\lambda^2}{G_0}}}(\mu_k\rho_2) \,-\, L^{(j)}_{\sqrt{4-\frac{\lambda^2}{G_0}}}(\mu_k\rho_2)\,Y^{(i)}_{\sqrt{4-\frac{\lambda^2}{G_0}}}(\mu_k\rho)$, $(i,j=0,1)$; $m_k(\xi) = D_k e^{\mu_k\xi} + F_k e^{-\mu_k\xi}$; D_k, F_k are arbitrary constants.

Based on (3.11) from (2.5) we get:

$$\sum_{k=1}^{\infty} G_0\,\rho^2 v_k(\rho)\,m'_k(\xi)\,e^{i\omega t}\Big|_{\xi=\pm l} = f^{\pm}(\rho)\,e^{i\omega t}. \tag{3.12}$$

Multiplying (3.12) by $\rho\bar{v}_n(\rho)$, and integrating within $[\rho_1;\rho_2]$, allowing for (3.6), we have:

$$m'_n(\xi)\big|_{\xi=\pm l} = t_n^{\pm}$$

i.e.

$$\left(\mu_n e^{\mu_n\xi} D_n - \mu_n e^{-\mu_n\xi} F_n\right)\Big|_{\xi=\pm l} = t^{\pm}(\rho). \tag{3.13}$$

Here $t_n^{\pm} = \frac{1}{G_0}\int_{\rho_1}^{\rho_2} \rho f^{\pm}(\rho)\;\bar{v}_n(\rho)\,d\rho$.

From the system (3.13) we determine the unknown constants D_n and F_n:

$$D_n = \frac{t_n^{+}e^{\mu_n l} - t_n^{-}e^{-\mu_n l}}{2\mu_n sh\,(2\mu_n l)},\, F_n = \frac{t_n^{+}e^{-\mu_n l} - t_n^{-}e^{\mu_n l}}{2\mu_n sh\,(2\mu_n l)}.$$

The case $\mu = 0$ is special and corresponds to thickness resonance [3]. In the case, boundary value problem (3.3), (3.4) takes the form:

$$v''(\rho) + \frac{3}{\rho}v'(\rho) + \frac{\left(\frac{\lambda^2}{G_0} - 3\right)}{\rho^2}\cdot v(\rho) = 0 \tag{3.14}$$

$$G_0\rho^2\left(v'(\rho) - \frac{v(\rho)}{\rho}\right)\Big|_{\rho=\rho_s} = 0. \tag{3.15}$$

The solution of (3.14) has the form:

$$v(\rho) = A_1\rho^{-1-\sqrt{4-\frac{\lambda^2}{G_0}}} + A_2\rho^{-1+\sqrt{4-\frac{\lambda^2}{G_0}}}.$$

Satisfying (3.15), we get:

$$sh\left(\sqrt{4-\frac{\lambda^2}{G_0}}\,\ln\left(\frac{\rho_2}{\rho_1}\right)\right) = 0\,. \tag{3.16}$$

Equation (3.16) determines denumerable set of frequencies of thickness resonance

$$\lambda_k^2 = G_0\left(4 + \frac{\pi^2 k^2}{\ln^2\left(\frac{\rho_2}{\rho_1}\right)}\right);\ (k = 0,1,2,...)\,.$$

Assume that the cylinder has a small thickness. Study the asymptotic behavior of the problem solution.

Let us analyze the roots of the variance equation (3.8). For studying its roots we put

$$\rho_1 = 1-\varepsilon;\ \rho_2 = 1+\varepsilon, \tag{3.17}$$

where $\varepsilon = \frac{R_2 - R_1}{2R_0}$ is a small parameter characterizing the cylinder's thickness.

Substituting (3.17) in (3.8), we have:

$$D(\mu,\lambda,\varepsilon) = \mathit{\Delta}(\mu,\lambda,\rho_1,\rho_2) = 0. \tag{3.18}$$

Expand $D(\mu,\lambda,\varepsilon)$ in series with respect to ε:

$$D(\mu,\lambda,\varepsilon) = \frac{4\varepsilon}{\pi} < \mu^2 + \frac{\lambda^2}{G_0} + \varepsilon^2\left[\left(\frac{16}{3} - \frac{4}{3}\cdot\frac{\lambda^2}{G_0}\right)\mu^2 - \frac{2}{3}\mu^4 + 3\frac{\lambda^2}{G_0} - \frac{2}{3}\frac{\lambda^4}{G_0^2}\right]$$

$$+\varepsilon^4\left[\frac{2}{15}\mu^6+\left(\frac{6}{15}\frac{\lambda^2}{G_0}-2\right)\mu^4+\left(\frac{32}{3}-\frac{64}{15}\frac{\lambda^2}{G_0}+\frac{2}{5}\frac{\lambda^4}{G_0^2}\right)\mu^2+5\frac{\lambda^2}{G_0}-\frac{26}{15}\frac{\lambda^4}{G_0^2}+\frac{2}{15}\frac{\lambda^6}{G_0^3}\right]+......>=0\,. \tag{3.19}$$

For finite $\lambda\,(\lambda=O\,(1)$ as $\varepsilon\to 0)$ the function $D\,(\mu,\lambda,\varepsilon)$ has the following two groups of zeros:
a) the first group consists of two zeros with asymptotic properties $\mu_k=O\,(1)$ as $\varepsilon\to 0\;\;(k=1,2)$;
b) the second group contains a denumerable set of zeros that are of order $O\left(\varepsilon^{-1}\right)$.

For constructing the asymptotics of the zeros of the first group, we look for μ_k in the form of the following expansion:

$$\mu_k=\mu_{k0}+\varepsilon\mu_{k1}+\varepsilon^2\mu_{k2}+\dots\,. \tag{3.20}$$

Substituting (3.20) in (3.19), we have:

$$\mu_{k0}=\pm i\frac{\lambda}{\sqrt{G_0}},\;\;\mu_{k1}=0,\;\;\mu_{k2}=\mp\frac{7i\lambda}{6\sqrt{G_0}},\dots\,.$$

For constructing the asymptotics of zeros of the second group, we look for μ_k in the form:

$$\mu_k=\frac{\delta_k}{\varepsilon}+O\,(\varepsilon)\;. \tag{3.21}$$

After substitution of (3.21) in variance equation (3.18), using asymptotic expansion of the Bessel function for large μ [3], for δ_k we get the following equation:

$$\sin 2\delta_k=0. \tag{3.22}$$

(3.22) coincides with the equation that determines the indices of Saint-Venant's fringe effects in the statics of the shell [2].

Let us consider the case $\lambda\to 0$ as $\varepsilon\to 0$. Such vibrations are called super lower frequencies [2]. Assume that the principal terms of the asymptotics λ have the form:

$$\lambda=\lambda_0\varepsilon^q\quad(q>0)\;. \tag{3.23}$$

In this case the function $D\,(\mu,\lambda,\varepsilon)$ has two restricted zeros with asymptotic properties $\mu_k\to\infty$ as $\varepsilon\to 0$.

Suppose that the principal terms μ_k have the form:

$$\mu_k=\mu_{k0}\varepsilon^\beta\;\;(\beta>0)\;. \tag{3.24}$$

Substituting (3.23), (3.24) in (3.19), we get that only the case $\beta=q$ is possible. Finally we find: $\mu_{k0}=\pm\frac{i\lambda}{\sqrt{G_0}}$.

Consider the case $\lambda\to\infty$ as $\varepsilon\to 0$. Such vibrations are called superhigher frequency [2]. Here the following variations are possible: a) $\lambda\varepsilon\to 0$ as $\varepsilon\to 0$; b) $\lambda\varepsilon\to const$ as $\varepsilon\to 0$.

Define μ_k when $\lambda\varepsilon\to 0$ as $\varepsilon\to 0$. Suppose that the principal members of the asymptotics λ have the form:

$$\lambda=\lambda_0\varepsilon^{-q},\;\;\lambda_0=O\,(1)\,,\;\;0<q<1. \tag{3.25}$$

In this case equation (3.18) has only unbounded roots $\mu_k\to\infty$ as $\varepsilon\to 0$. Suppose that

$$\mu_k=\mu_{k0}\varepsilon^{-\beta},\;\;\mu_{k0}=O\,(1)\,,\;\;0<\beta<1; \tag{3.26}$$

Substituting (3.25), (3.26) in (3.19), from the condition of consistency of the constructed asymptotic process we get that only the case $q=\beta$ is possible.

Assuming (3.25) we look for μ_k in the form (3.26). After substitution of (3.25), (3.26) in (3.19), we get:

$$\mu_{k0}^2=-\frac{\lambda_0^2}{G_0}.$$

Subject to condition (3.25), we look for μ_k in the form:

$$\mu_k=\frac{\delta_k}{\varepsilon}+O\left(\varepsilon^{1-\beta}\right)\;. \tag{3.27}$$

After substitution of (3.25), (3.27) in variance equation (3.8) and transformation if by means of asymptotic expansions of the Bessel functions for large μ [3], for δ_k we get equation (3.22).

Let us study the roots of equation (3.18) when $\lambda\varepsilon \to const$ as $\varepsilon \to 0$. In this case all roots of (3.18) increase and only the case $\lambda{\sim}\mu$, i.e. $\mu\varepsilon \to const$ as $\varepsilon \to 0$ is possible.

Giving

$$\lambda = \lambda_0 \varepsilon^{-1}, \ (\lambda_0 = O(1) \quad as \quad \varepsilon \to 0) \tag{3.28}$$

we look for μ_k in the following from:

$$\mu_k = \frac{\gamma_k}{\varepsilon} + O(\varepsilon) \tag{3.29}$$

Substituting (3.28), (3.29) in (3.18) and transforming it by means of asymptotic expansion of the Bessel function for large values of the argument [3], for γ_k we have:

$$\sin 2\sqrt{\gamma_k^2 + \frac{\lambda_0^2}{G_0}} = 0\,. \tag{3.30}$$

We give asymptotic formulas for displacements and stresses. Assuming $\rho = 1 + \varepsilon\eta$ $(-1 \le \eta \le 1)$ and expanding in small parameter of ε, from (3.9)-(3.11) we get the following asymptotic formulas:

1. For the roots of (3.20)

$$u = \frac{2}{\pi}\sum_{k=1}^{\infty}\left[1 + (\eta - 2)\,\varepsilon + (3 - 2\eta)\,\varepsilon^2 + O\left(\varepsilon^2\right)\right] m_k(\xi)\, e^{i\omega t},$$

$$\sigma_{\varphi\xi} = \frac{2G_0}{\pi}\sum_{k=1}^{\infty}\left[1 + (3\eta - 2)\,\varepsilon + 3\,(1 - \eta)^2\,\varepsilon^2 + O\left(\varepsilon^2\right)\right] m_k'(\xi)\, e^{i\omega t},$$

$$\sigma_{\rho\varphi} = \frac{2G_0\varepsilon^2}{\pi}\sum_{k=1}^{\infty}\left[-\frac{\lambda^2}{G_0}\left(1 - \eta^2\right) + O(\varepsilon)\right] m_k(\xi)\, e^{i\omega t}.$$

2. For the roots of (3.24)

$$u = \sum_{k=1}^{\infty}\frac{2}{\pi}\left[1 + \varepsilon\,(\eta - 2) + O\left(\varepsilon^2\right)\right] m_k(\xi)\, e^{i\omega t},$$

$$\sigma_{\varphi\xi} = \sum_{k=1}^{\infty}\frac{2G_0}{\pi}\left[1 + \varepsilon\,(3\eta - 2) + O\left(\varepsilon^2\right)\right] m_k'(\xi)\, e^{i\omega t},$$

$$\sigma_{\rho\varphi} = \sum_{k=1}^{\infty}\frac{2G_0}{\pi}\left[\frac{\lambda_0^2}{G_0}\left(\eta^2 - 1\right)\varepsilon^{2+2\beta} + O\left(\varepsilon^{3+2\beta}\right)\right] m_k(\xi)\, e^{i\omega t}\,.$$

3. For the roots of (3.26)

$$u = \sum_{k=1}^{\infty}\frac{2}{\pi}\left[1 + (\eta - 2)\,\varepsilon + O\left(\varepsilon^{2-2\beta}\right)\right] m_k(\xi)\, e^{i\omega t},$$

$$\sigma_{\varphi\xi} = \sum_{k=1}^{\infty}\frac{2G_0}{\pi}\left[1 + (3\eta - 2)\,\varepsilon + O\left(\varepsilon^{2-2\beta}\right)\right] m_k'(\xi)\, e^{i\omega t},$$

$$\sigma_{\rho\varphi} = \sum_{k=1}^{\infty}\frac{2G_0}{\pi}\left[\frac{\lambda_0^2}{G_0}\left(\eta^2 - 1\right)\varepsilon^{2-2\beta} + O\left(\varepsilon^{3-2\beta}\right)\right] m_k(\xi)\, e^{i\omega t}$$

4. For the roots of (3.21), (3.27)

$$u = \sum_{k=1}^{\infty} \left(\delta_{0k} \cos \delta_{0k} \left(1 - \eta\right) + O\left(\varepsilon\right)\right) m_k \left(\xi\right) e^{i\omega t},$$

$$\sigma_{\rho\varphi} = \frac{G_0}{\varepsilon} \sum_{k=1}^{\infty} \left(\delta_{0k}^2 \sin \delta_{0k} \left(1 - \eta\right) + O\left(\varepsilon\right)\right) m_k \left(\xi\right) e^{i\omega t},$$

$$\sigma_{\varphi\xi} = G_0 \sum_{k=1}^{\infty} \left(\delta_{0k} \cos \delta_{0k} \left(1 - \eta\right) + O\left(\varepsilon\right)\right) m'_k \left(\xi\right) e^{i\omega t}.$$

5. For the roots of (3.29)

$$u = \sum_{k=1}^{\infty} \left[\sqrt{\gamma_k^2 + \frac{\lambda_0^2}{G_0}} \cos \left(\sqrt{\gamma_k^2 + \frac{\lambda_0^2}{G_0}} \left(1 - \eta\right)\right) + O\left(\varepsilon\right)\right] m_k \left(\xi\right) e^{i\omega t},$$

$$\sigma_{\rho\varphi} = \frac{G_0}{\varepsilon} \sum_{k=1}^{\infty} \left[\left(\gamma_k^2 + \frac{\lambda_0^2}{G_0}\right) \sin \left(\sqrt{\gamma_k^2 + \frac{\lambda_0^2}{G_0}} \left(1 - \eta\right)\right) + O\left(\varepsilon\right)\right] m_k \left(\xi\right) e^{i\omega t},$$

$$\sigma_{\varphi\xi} = G_0 \sum_{k=1}^{\infty} \left[\sqrt{\gamma_k^2 + \frac{\lambda_0^2}{G_0}} \cos \left(\sqrt{\gamma_k^2 + \frac{\lambda_0^2}{G_0}} \left(1 - \eta\right)\right) + O\left(\varepsilon\right)\right] m'_k \left(\xi\right) e^{i\omega t}.$$

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