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Generalized quasiplane strain state for linear-elastic bodies

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Abstract. *In the paper the notion of "generalized quasiplane strain state" is introduced. It is proved that under generalized quasiplane strain state the stress state in the sections perpendiculas to the axis of a finite length prismatic body is independent of mechanical characteristics of the material and is determined by boundary conditions and the shape of the cross-section contour.*

Keywords. stress, equilibrium equations, strain, plane strain, quasplane strain, prismatic body

Mathematics Subject Classification (2010): 74B10

1 Introduction

There exist strain states under which displacement vector components have the form:

$$u_1 = u_1(x_1 x_2); \quad u_2 = u_2(x_l x_2); \; u_3 = u_3(x_3), \tag{1.1}$$

where $x_1, \; x_2, \; x_3$ are the axes of Cartesian system of coordinates, the axes $x_1, \; x_2$ are in horizontal plane, while x_3 is vertically upward directed. Strain state that appears under the action of proper weight in rocks may be an example of such a strain state. We call such a strain state in finite length prismatic bodies, a generalized quasiplane strain state. Prove that under generalized quasiplane strain state the stress state in sections perpendicular to the axis of a prismatic body is independent of mechanical characteristics of the material and is determined by boundary conditions and the contour shape.

In the paper [1] strain state for finite length prismatic bodies, whose axes coincide with the axis x_3 and the there hold the following displacements

$$u_1 = u_1(x_l, x_2); \;\; u_2 = u_2(x_1, x_2): \; u_3 = ax_3 + b, \tag{1.2}$$

where a and b are constant values, is called a quasiplane strain state. It is proved that under plane strain state realized for an infinitely long prismatic body, the stress state under quasiplane strain state in cross-sections of finite length prismatic bodies is independent of mechanicsl charactestics of the material and are determined by boundary conditions and the shape of the contour of the cross-section.

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At first prove that when strains are infinitely small, in the bodies under initial stresses arising at the expense of proper weight, stresses changes that occur at the expense of external actions satisfy homogenous differential equilibrium equations. Denote the tensor components of initial stresses that occur at the expense of proper weight by σ_{ij}^0, while the tensors components of stresses arising at the expense of proper weight and action of external actions by σ_{ij}. Then the equilibrium equations before application of external forces will be:

$$\sigma_{ij,j}^0 + \rho F_i = 0 \quad ij = 1 - 3. \tag{1.3}$$

In equalities (3) by the repeated index j we perform summation from one to three, the comma means differential with respect to coordinate with the index following the comma.

The stresses arising after the action of external forces also satisfy the equilibrium equations:

$$\sigma_{ij,j} + \rho F_i = 0. \tag{1.4}$$

Because of smallnes of deformations, we can assume that the density $\rho = const$. Subtracting equation (3) from equation (4), we get:

$$\overline{\sigma}_{ij,j} = 0, \tag{1.5}$$

where $\overline{\sigma}_{ij,j} = \sigma_{ij} - \sigma_{ij}^0$ are stress changes at the expense of external actions.

Dependences of changes of small deformations on displacements changes and Hook's law in changes, will be:

$$\overline{\varepsilon}_{ij} = \frac{1}{2}\left(\overline{u}_{i,j} + \overline{u}_{j,i}\right) \tag{1.6}$$

$$\begin{aligned}
\overline{\varepsilon}_{11} &= \frac{1}{E}\left[\overline{\sigma}_{11} - v(\overline{\sigma}_{22} + \overline{\sigma}_{33})\right] \\
\overline{\varepsilon}_{22} &= \frac{1}{E}\left[\overline{\sigma}_{22} - v(\overline{\sigma}_{11} + \overline{\sigma}_{33})\right] \\
\overline{\varepsilon}_{12} &= \frac{1+v}{E}\overline{\sigma}_{12};\ \overline{\varepsilon}_{13} = \overline{\varepsilon}_{23} = 0 \\
\overline{\varepsilon}_{33} &= \frac{1}{E}\left[\overline{\sigma}_{33} - v(\overline{\sigma}_{11} + \overline{\sigma}_{22})\right].
\end{aligned} \tag{1.7}$$

System (5) - (7) is closed. In this system the number of equations and the number of unknowns equals 21, consequently, from this system we can determine all changes of the stresses $\overline{\sigma}_{ij} = \sigma_{ij} - \sigma_{ij}^0$. Whence $\sigma_{ij} = \sigma_{ij}^0 + \overline{\sigma}_{ij}$.

Now prove that under generalized quasiplane strain state, the stress state in sections perpendicular to the cylinder's axis is independent on mechanical characteristics of the material and is determined by boundary conditions the contour shape.

As is seen (1), the generalized quasiplane strain state differs from quasiplane strain state by the fact that under quasiplane deformation state u_3 linearly depends on x_3, while under generalized quasiplane deformation state there is not restriction of linearity of u_3 from x_3. We also consider that equilibrium conditions are homogeneous, i.e. volume forces are not taken into account. Allowing for (1), from geometrical relations we get:

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1};\ \ \varepsilon_{22} = \frac{\partial u_2}{\partial x_2};\ \ \ \varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) \tag{1.8}$$

$$\varepsilon_{13} = \varepsilon_{23} = 0 \quad \ \varepsilon_{33} = u_3'.$$

Take the last equation of (8) in the last equation of (7) (for simplicity in equalities (5)-(7) we will not place dashes over the quantities). Substituting the last equation of (8) in the last equation of (7), we have

$$u_3' = \frac{1}{E}\left[\sigma_{33} - v(\sigma_{11} + \sigma_{22})\right],$$

whence

$$\sigma_{33} = v(\sigma_{11} + \sigma_{22}) + Eu_3'. \tag{1.9}$$

Taking into account (9) in (7), we have:

$$\varepsilon_{11} = \frac{1+v}{E}\left[\sigma_{11} - v(\sigma_{11}+\sigma_{22})\right] - vu_3'$$

$$\varepsilon_{22} = \frac{1+v}{E}\left[\sigma_{22} - v(\sigma_{11}+\sigma_{22})\right] - vu_3'$$

$$\varepsilon_{12} = \frac{1+v}{E}\sigma_{12}. \tag{1.10}$$

Five from the six strain compatibility equations in the given case turn into identity and there remains only one equation

$$\frac{\partial \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} = 2\frac{\partial^2 e_{12}}{\partial x_1 \partial x_2}. \tag{1.11}$$

The equilibrium equations have the form:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0, \quad \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = 0.$$

We differentiate the first of these equations with respect to x1, the second with respect to x2, and putting them together, we get:

$$\frac{\partial^2 \sigma_{11}}{\partial x_1^2} + \frac{\partial^2 \sigma_{22}}{\partial x_2^2} = -2\frac{\partial^2 \sigma_{12}}{\partial x_1 \partial x_{21}}. \tag{1.12}$$

Take into account equality (10) in (11). Then

$$\frac{\partial^2}{\partial^2 x_2}\left[\sigma_{11}\right] - 4v\left(\sigma_{11}+\sigma_{12}\right) + \frac{\partial^2}{\partial x_1^2}\left[\sigma_{22} - v\left(\sigma_{11}+\sigma_{12}\right)\right] = 2\frac{\partial^2 \sigma_{12}}{\partial x_1 \partial x_2}. \tag{1.13}$$

Having substituted (12) in (13), we have:

$$\Delta\left(\sigma_{11}+\sigma_{12}\right) = 0. \tag{1.14}$$

Here Δ is Laplacian two dimensional operator.

Introducing the Airy's stress function, as in plane strain [2], from (14) we get:

$$\Delta\Delta\phi = 0. \tag{1.15}$$

Here $\phi(x_1, x_2)$ is Airy's stress function,

$$\Delta\Delta = \frac{\partial^4}{\partial^4 x_{21}} + 2\frac{\partial^4}{\partial x_1^2 \partial_2 x^2} + \frac{\partial^4}{\partial^4 x_2}.$$

It is known that for the first boundary value problem, the boundary conditions have the form: [2]

$$\sigma_{11}n_1 + \sigma_{12}n_2 = T_1 \tag{1.16}$$

$$\sigma_{21}n_1 + \sigma_{22}n_2 = T_2,$$

where $n_1,\ n_2$ are the projections of the vector of normal n to the contour of section on the coordinate axis $x_1,\ x_2,\ T_1,\ T_2$ are projections of boundary forces on coordinate axis $x_1,\ x_2$. Equations (14) - (16) don't contain mechanical characteristics of the material. Consequently, mechanical characteristics of the material will not enter the expressions of stresses determined from these equations. Thus, we proved the following theorem: Under generalized quasiplane strain state, the stress state in sections perpendicular to the axis of a finite length prismatic body is independent of mechanical characteristics of the material and is determined by boundary conditions and the shape of the cross-section contour..

This is the generalization of Morris-Lew's theorem for generalized quasiplane deformation state.

In special case, when the stresses arising at the expense of external actions in rocks are determined, the volume forces don't participate. Consequently, in this case displacement components have the form (1), therefore, stress changes in rocks at the expense of external actions are independent of mechanical characteristics of the material.

References

1. Gasanov, G.A., Gulgazli, A.S., Orujov, Yu.A.: *Strength analysis of the nearwellbore part with regard to anisotropy of rocks*. Oil and gas of Western Siberia. Proc. Of the International Conference. Tyumen, 20-22 october, 84-90 (2011).
2. Amenzade, Yu.A.: *Theory of elasticity.* M.: Vysshaya shkola. 272 p. (1976).