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# Multipoint boundary value problem for the first order elliptic equation

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**Abstract.** *The present work is devoted to the study of the solution of the boundary value problem for the Cauchy-Riemann equation on a bounded plane domain having a discontinuity inside the region under consideration. This region is divided into two parts so that the break line of the solution to this problem falls on the boundary of the broken regions.*

**Keywords.** Cauchy-Riemann equation · discontinuous solution · multipoint boundary value problem · fundamental solutions · necessary conditions · regularization · Fredholm property.

**Mathematics Subject Classification (2010):** 35F15, 35C60

## 1 Introduction

The paper [2] is dedicated to the investigation of Carleman condition influence on the Fredholm property of the boundary value problem for a first order elliptic equation. The obtained necessary conditions have led to the following results. If the boundary conditions of the given boundary-value problem satisfy Carleman condition then this problem can be reduced to a Fredholm integral equation of second kind whose kernel can contain only weak singularity. If Carleman condition is not satisfied then the posed problem is reduced to a Fredholm integral equation of first kind with non-singular kernel [3].

Some boundary value problems for the Cauchy-Riemann equation with non-local boundary conditions in several regions of plane have been investigated and solved by authors. In this paper by making use of fundamental solutions of Cauchy-Riemann equations and by presenting analytic solutions to the above-mentioned boundary value problems we try to present an analytic expression for the solution of Cauchy-Riemann equation in the first semi-quarter [1,4].

Let $D \subset R^2$ is a bounded region located in the upper half-plane. Consider the Cauchy-Riemann equation in this region, and let it have a discontinuous solution. The break line is contained within the region

$$D = \{x = (x_1, x_2) : x_1 \in (-1, 1), x_2 \in (0, 1)\}$$

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Consider the following task:

$$\frac{\partial u_k(x)}{\partial x_2}+i\frac{\partial u_k(x)}{\partial x_1}=0, x=(x_1,x_2)\in D_k\subset R^2, k=1,2, \tag{1.1}$$

$$\begin{gathered}\alpha_{k1}^{(1)}(t)u_1(-1,t)+\alpha_{k2}^{(1)}(t)u_1(-1+t,0)+\\+\alpha_{k3}^{(1)}(t)u_1(0,1-t)+\alpha_{k4}^{(1)}(t)u_1(-t,1)+\\+\alpha_{k1}^{(2)}(t)u_2(1,t)+\alpha_{k2}^{(2)}(t)u_2(1-t,0)+\alpha_{k3}^{(2)}(t)u_2(0,1-t)+\\+\alpha_{k4}^{(2)}(t)u_2(t,1)=\varphi_k(t), t\in[0,1], k=\overline{1,4},\end{gathered} \tag{1.2}$$

where $i=\sqrt{-1}$, the coefficients and the right-hand sides of the boundary condition (1.2) are given continuous functions, and conditions (1.2) are linearly independent. Given that

$$u(x)=\begin{cases}u_1(x), x\in D_1,\\u_2(x), x\in D_2,\end{cases}$$

and there is a line of discontinuity at $x_1=0$, then this line refers to the boundaries, i.e.

$$D=D_1\cup D_2; D_1=\{x=(x_1,x_2):x_1\in(-1,0), x_2\in(0,1)\},$$

$$D_2=\{x=(x_1,x_2):x_1\in(0,1), x_2\in(0,1)\}.$$

It is known that the fundamental solution of equation (1.1) has the form [6]:

$$U(x-\xi)=\frac{1}{2\pi}\frac{1}{x_2-\xi_2+i(x_1-\xi_1)} \tag{1.3}$$

## 2 The main ratio

Multiplying equation (1.1) by the fundamental solution (1.3) and integrating over the domain $D=D_1\cup D_2$, we have:

$$0=\int_D\frac{\partial u(x)}{\partial x_2}U(x-\xi)dx+i\int_D\frac{\partial u(x)}{\partial x_1}U(x-\xi)dx=\int_{D_1}\frac{\partial u_1(x)}{\partial x_2}U(x-\xi)dx+$$

$$+i\int_{D_1}\frac{\partial u_1(x)}{\partial x_1}U(x-\xi)dx+\int_{D_2}\frac{\partial u_2(x)}{\partial x_2}U(x-\xi)dx+i\int_{D_2}\frac{\partial u_2(x)}{\partial x_1}U(x-\xi)dx.$$

Using the integration formula in parts or the Ostrogradsky-Gauss formula, we obtain:

$$-\frac{i}{2\pi}\int_0^1 u_1(-1,\tau)\frac{1}{\tau-\xi_2-i(1+\xi_1)}d\tau-\frac{1}{2\pi}\int_0^1 u_1(-1+\tau,0)\frac{1}{-\xi_2-i(1-\tau+\xi_1)}d\tau-$$

$$-\frac{i}{2\pi}\int_0^1 u_1(0,1-\tau)\frac{1}{1-\tau-\xi_2-i\xi_1}d\tau-\frac{1}{2\pi}\int_0^1 u_1(-\tau,1)\frac{1}{1-\xi_2-i(\tau+\xi_1)}d\tau+$$

$$+\frac{i}{2\pi}\int_0^1 u_2(0,1-\tau)\frac{1}{1-\tau-\xi_2-i\xi_1}d\tau+\frac{1}{2\pi}\int_0^1 u_2(1-\tau,0)\frac{1}{-\xi_2+i(1-\tau-\xi_1)}d\tau+$$

$$+\frac{i}{2\pi}\int_0^1 u_2(1,\tau)\frac{1}{\tau-\xi_2+i(1-\xi_1)}d\tau+\frac{1}{2\pi}\int_0^1 u_2(\tau,1)\frac{1}{1-\xi_2+i(\tau-\xi_1)}d\tau=$$

$$=\begin{cases} u_1(\xi),\xi\in D_1,\\ \frac{1}{2}u_1(\xi),\xi\in\partial D_1,\\ u_2(\xi),\xi\in D_2,\\ \frac{1}{2}u_2(\xi),\xi\in\partial D_2. \end{cases} \tag{2.1}$$

Formula (2.1) is a basic relation consisting of two parts: the first part gives an arbitrary solution to equation (1.1), and the second part (connected with the boundary) is the necessary conditions [5].

The necessary conditions:

$$\begin{cases} u_1(-1+\tau,0)=\frac{i}{\pi}\int_0^1\frac{u_1(-1+t,0)}{t-\tau}dt+...,\\ u_1(0,1-\tau)=\frac{i}{\pi}\int_0^1\frac{u_1(0,1-t)}{t-\tau}dt+...,\\ u_1(-\tau,1)=-\frac{i}{\pi}\int_0^1\frac{u_1(-t,1)}{t-\tau}dt+...,\\ u_2(0,1-\tau)=-\frac{i}{\pi}\int_0^1\frac{u_2(0,1-t)}{t-\tau}dt+...,\\ u_2(1-\tau,0)=\frac{i}{\pi}\int_0^1\frac{u_2(1-t,0)}{t-\tau}dt+...,\\ u_2(1,\tau)=\frac{i}{\pi}\int_0^1\frac{u_2(1,t)}{t-\tau}dt+...,\\ u_2(\tau,1)=-\frac{i}{\pi}\int_0^1\frac{u_2(t,1)}{t-\tau}dt+..., \end{cases} \tag{2.2}$$

where the dots indicate the sums of non-singular terms.

Regularization. Given the boundary condition (1.2), from (2.2) we create the following linear combination:

$$-\alpha_{k1}^{(1)}(\tau)u_1(-1,\tau)+\alpha_{k2}^{(1)}(\tau)u_1(-1+\tau,0)+\alpha_{k3}^{(1)}(\tau)u_1(0,1-\tau)-\alpha_{k4}^{(1)}(\tau)u_1(-\tau,1)+$$

$$+\alpha_{k1}^{(2)}(\tau)u_2(1,\tau)+\alpha_{k2}^{(2)}(\tau)u_2(1-\tau,0)-\alpha_{k3}^{(2)}(\tau)u_2(0,1-\tau)-\alpha_{k4}^{(2)}(\tau)u_2(\tau,1)=$$

$$z=\frac{i}{\pi}\int_0^1\varphi_k(t)\frac{dt}{t-\tau}+...,k=\overline{1,4}. \tag{2.3}$$

If

$$\varphi_k(t)\in C^{(1)}(0,1),\varphi_k(0)=\varphi_k(1)=0,k=\overline{1,4}, \tag{2.4}$$

then expressions (2.3) are regular.

## 3 Conclusion

Based on the boundary conditions (1.2) and the obtained regular relations (2.3), we arrive at eight equations with eight unknowns. If there is a restriction

$$\Delta(\tau)=\begin{vmatrix}
\alpha_{11}^{(1)}(\tau)\alpha_{12}^{(1)}(\tau)\alpha_{13}^{(1)}(\tau)\alpha_{14}^{(1)}(\tau)\alpha_{11}^{(2)}(\tau)\alpha_{12}^{(2)}(\tau)\alpha_{13}^{(2)}(\tau)\alpha_{14}^{(2)}(\tau)\\
\alpha_{21}^{(1)}(\tau)\alpha_{22}^{(1)}(\tau)\alpha_{23}^{(1)}(\tau)\alpha_{24}^{(1)}(\tau)\alpha_{21}^{(2)}(\tau)\alpha_{22}^{(2)}(\tau)\alpha_{23}^{(2)}(\tau)\alpha_{24}^{(2)}(\tau)\\
\alpha_{31}^{(1)}(\tau)\alpha_{32}^{(1)}(\tau)\alpha_{33}^{(1)}(\tau)\alpha_{34}^{(1)}(\tau)\alpha_{31}^{(2)}(\tau)\alpha_{32}^{(2)}(\tau)\alpha_{33}^{(2)}(\tau)\alpha_{34}^{(2)}(\tau)\\
\alpha_{41}^{(1)}(\tau)\alpha_{42}^{(1)}(\tau)\alpha_{43}^{(1)}(\tau)\alpha_{44}^{(1)}(\tau)\alpha_{41}^{(2)}(\tau)\alpha_{42}^{(2)}(\tau)\alpha_{43}^{(2)}(\tau)\alpha_{44}^{(2)}(\tau)\\
-\alpha_{11}^{(1)}(\tau)\alpha_{12}^{(1)}(\tau)\alpha_{13}^{(1)}(\tau)-\alpha_{14}^{(1)}(\tau)\alpha_{11}^{(2)}(\tau)\alpha_{12}^{(2)}(\tau)-\alpha_{13}^{(2)}(\tau)-\alpha_{14}^{(2)}(\tau)\\
-\alpha_{21}^{(1)}(\tau)\alpha_{22}^{(1)}(\tau)\alpha_{23}^{(1)}(\tau)-\alpha_{24}^{(1)}(\tau)\alpha_{21}^{(2)}(\tau)\alpha_{22}^{(2)}(\tau)-\alpha_{23}^{(2)}(\tau)-\alpha_{24}^{(2)}(\tau)\\
-\alpha_{31}^{(1)}(\tau)\alpha_{32}^{(1)}(\tau)\alpha_{33}^{(1)}(\tau)-\alpha_{34}^{(1)}(\tau)\alpha_{31}^{(2)}(\tau)\alpha_{32}^{(2)}(\tau)-\alpha_{33}^{(2)}(\tau)-\alpha_{34}^{(2)}(\tau)\\
-\alpha_{41}^{(1)}(\tau)\alpha_{42}^{(1)}(\tau)\alpha_{43}^{(1)}(\tau)-\alpha_{44}^{(1)}(\tau)\alpha_{41}^{(2)}(\tau)\alpha_{42}^{(2)}(\tau)-\alpha_{43}^{(2)}(\tau)-\alpha_{44}^{(2)}(\tau)
\end{vmatrix}\neq 0, \tag{3.1}$$

then from system (1.2), (2.3) we obtain a normal system of Fredholm integral equations of the second kind with regular kernels.

**Theorem 3.1** *If the coefficients of the boundary condition (1.2) belong to a certain Hölder class, and the right-hand sides satisfy condition (2.4) and condition (3.1) holds, then the boundary-value problem (1.1), (1.2) is Fredholm.*

## References


1. Aliev N.A., Abbasova A.H.: The new approach to boundary problems for equation Cauchy-Riemann. *Abstracts of International Conference on Modern problems of Applied mathematics and Information technologies - Al Khorezmiy Uzbekistan, Tashkent, 18-21 September,* 28, (2009).
2. Aliev N.A., Mustafayeva Y.Y., Murtuzayeva S.M.: The influence of the Carleman condition on the Fredholm property of the boundary value problem for Cauchy-Riemann Equation. *Proceedings of the Institute of Applied Mathematics, Baku, Azerbaijan,* **1**(2), 153-162, (2012).
3. Aliyev N.A., Aliyev A.M.: Data of one nonlinear differential equation with the partial derivatives to the Cauchy-Riemann Equation. *Book of Abstracts, The Fourth Congress of The Turkic World Mathematical Society. Azerbaijan, Baku, 1-3 July,* 156,(2011).
4. Aliyev N., Fatehi M.H., Jahanshahi M.: Analytic solution for the Cauchy-Riemann equation with non-local boundary conditions in the first Semi-Quarter. *Journal of Science Tarbiat Moallem University, Iran* **9**(1), 29-40,(2010).
5. Sajjadmanesh M., Jahanshahi M., Aliyev N.: Inverse Problem of the Kind of Tikhonov-Lavrentev Including the Cauchy-Riemann Equation on a Boundary Region. *Book of Abstracts, The Fourth Congress of The Turkic World Mathematical Society. Azerbaijan, Baku, 1-3 July,* 266,(2011).
6. Vladimirov V.S.: *Equations of mathematical physics,* Nauka, Moscow, p. 524, (1981).