Tran. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.
Mechanics, 39 (7), 0-0 2019

AN INVERSE BOUNDARY VALUE PROBLEM FOR THE EQUATION OF FLEXURAL VIBRATIONS OF A BAR

Yusif M. Sevdimaliyev · Yashar T. Mehraliyev
Aysel T. Ramazanova · Kalyskan Matanova

Received: 10.09.2019 / Revised: 25.10.2019 / Accepted: 22.11.2019

Abstract. *In this work, we study one inverse boundary value problem for the equations of flexural torsional vibrations of a bar with an additional condition. Using the Fourier method, the problem reduces to solving a system of integral equations, and using method of contracting mappings, the existence and uniqueness of a solution to a system of integral equations is proved. The existence and uniqueness of the classical solution of the original problem are proved.*

Keywords. equation of vibrations of a bar · inverse problem · uniqueness of a solution

Mathematics Subject Classification (2010): 74H45

1 Introduction.

In modern technology, it is necessary to regulate vibration processes in one-dimensional distributed systems, and the relevance of these problems is increasing. For shafts, which are the basic principles of mechanical transmission, dangerous transverse vibrations are not allowed [1]. In aircraft such elements are constructed simultaneously by bending and torsional vibrations. One of the objectives of the project is to prevent the use of shaft vibrations with an adjustable speed [2,3]. For such problems, mathematical models of transverse vibrations of rods are built on the basis of a refined theory [4]. Solutions of unknown parameters in accordance with the known data of its solutions [5,6]. Such problems are called inverse problems of mathematical physics, which in many works [6] - [10], [12] , [14], [15],[17,18] were studied for partial differential equations. In problems associated with initial and boundary conditions, additional information is required. The necessary additional information is due to the presence of unknown coefficients or the right-hand sides of the equations [13].

Y. Sevdimaliyev
Baku State University

Y.Mehraliyev
Baku State University

A.Ramazanova
University Duisburg-Essen
E-mail: aysel.ramazanova@uni-due.de
K.Matanova
Kyrgyz -Turkish Manas University
E-mail: kalys.matanova@manas.edu.kg

2 Problem statement and its reduction to an equivalent task.

Consider the question of the unique solvability of the inverse boundary value problem of determining a pair of functions $\{u(x,t), a(t)\}$, that satisfy in in the domain $D_T = \{(x,t) : 0 \le x \le 1, 0 \le t \le T\}$ the equation [13]

$$u_{tt}(x,t) + u_{xxxx}(x,t) = a(t)u(x,t) + f(x,t) \tag{2.1}$$

with boundary conditions

$$u(x,0) = \varphi(x), u_t(x,0) = \psi(x)(0 \le x \le 1) \tag{2.2}$$

with periodic conditions

$$u_x(0,t) = 0, u_x(1,t) = 0, u_{xxx}(0,t) = 0, u_{xxx}(1,t) = 0, (0 \le t \le T) \tag{2.3}$$

and the additional condition

$$u(0,t) = h(t)(0 \le t \le T) \tag{2.4}$$

where $f(x,t)$, $\varphi(x)$,$\psi(x)$, $h(t)$ “given functions,$u(x,t)$ and $a(t)$“ desired functions.

Definition 2.1 *The pair $\{u(x,t), a(t)\}$ of functions $u(x,t) \in C^{2,4}(D_T)$ and $a(t) \in C[0,T]$ satisfying equation (2.1) in D_T , condition (2.2) in $[0,1]$ and conditions (2.3),(2.4) in $[0,T]$,defined as a classical solution of the inverse boundary value problem (2.1)-(2.4).*

The following lemma holds:

Lemma 2.1 *Let $\varphi(x), \psi(x) \in C[0,1], h(t) \in C^2[0,T], h(t) \ne 0$, $(0 \le t \le T)$, $f(x,t) \in C(D_T)$ and the conditions of approval be fulfilled:*

$$\varphi(0) = h(0), \psi(0) = h'(0). \tag{2.5}$$

Then the problem of finding a classical solution to problem (2.1) - (2.4)is equivalent to the problem of determining functions $u(x,t)$ and $a(t)$ with properties 1) and 2) the definition of a solution to problem (2.1) - (2.6), from (2.1) - (2.3),

$$h''(t) + u_{xxxx}(0,t) = a(t)h(t) + f(0,t)(0 \le t \le T). \tag{2.6}$$

Proof. Let $\{u(x,t), a(t)\}$ is a classical solution of problem (2.1)-(2.4). Since $h(t) \in C^2[0,T]$, differentiate (2.4) two times, we obtain:

$$u_{tt}(0,t) = h''(t), (0 \le t \le T). \tag{2.7}$$

Substituting $x = 0$ into equation (2.1), we find:

$$u_{tt}(0,t) + u_{xxxx}(x,t) = a(t)u(0,t) + b(t)u_t(0,t) + f(0,t), (0 \le t \le T). \tag{2.8}$$

From (2.8), by virtue of (2.4) and (2.7), it follows that (2.6) holds. Now, suppose that $u(x,t)$ and $u(x,t)$ are a solution to problem (2.1)-(2.3),(2.6). Then from (2.6),(2.8), we have:

$$\frac{d^2}{dt^2}(u(0,t) - h(t)) - a(t)(u(0,t) - h(t)) = 0, u_{xxx}(1,t) - u_{xxx}(0,t)) = 0 \tag{2.9}$$

By (2.2) and the compatibility condition (2.5), we get: Assuming that

$$u(0,0) - h(0) = \varphi(0) - h(0) = 0, u_t(0,0) - h'(0) =$$
$$= \psi(0) - h'(0) = 0(0 \le t \le T). \tag{2.10}$$

From (2.9) and (2.10), due to Lemma 2.1,we conclude that the condition (2.4) is satisfied. The Lemma is proved.

3 Solvability of the inverse boundary value problem

The first komponent $u(x,t)$ of the solution $\{u(x,t), a(t)\}$ of the problem (2.1)-(2.3),(2.6) will be sought in the form:

$$u(x,t) = \sum_{k=0}^{\infty} u_k(t) \cos \lambda_k x (\lambda_k = \pi k) \tag{3.1}$$

where

$$u_k(t) = m_k \int_0^1 u(x,t) \cos \lambda_k x dx (k = 0, 1, 2, ...),$$

$$m_k = \begin{cases} 1, k = 0, \\ 0, k = 1, 2, ... \end{cases}$$

Then applying the formal scheme of the Fourier method, from (2.1) and (2.2) we obtained:

$$u_k''(t) + \lambda_k^4 u_k(t) = F_k(t; u, a)(0 \le t \le T; k = 1, 2, \ldots), \tag{3.2}$$

$$u_k(0) = \varphi_k, u_k'(0) = \psi_k \tag{3.3}$$

where

$$F_k(t; u, a) = f_k(t) + a(t) u_k(t) (k = 0, 1, ...)$$

$$f_k(t) = m_k \int_0^1 f(x,t) \cos \lambda_k x dx,$$

$$\varphi_k = m_k \int_0^1 \varphi(x) \cos \lambda_k x dx, \psi_k = m_k \int_0^1 \psi(x) \cos \lambda_k x dx, \quad (k = 0, 1, \ldots),$$

Further, from (3.2),(3.3) we find:

$$u_0(t) = \varphi_0 + \psi_0 t + \int_0^t (t-\tau) F_0(\tau; u, a) d\tau \tag{3.4}$$

$$u_k(t) = \varphi_k \cos \lambda_k^2 t + \frac{1}{\lambda_k^2} \psi_k \sin \lambda_k^2 t +$$

$$+ \frac{1}{\lambda_k^2} \int_0^t F_k(\tau; u, a) \sin \lambda_k^2 (t-\tau) d\tau, (k = 1, 2, ...). \tag{3.5}$$

After expression substitution $u_k(t)(k = 0, 1, ...)$ to determine the components of the solution to problem (2.1) - (2.3), (2.6) we obtain:

$$u(x,t) = \varphi_0 + t\psi_0 + \int_0^t (t-\tau) F_0(\tau; u) d\tau + \sum_{k=1}^{\infty} \Big\{ \varphi_k \cos \lambda_k^2 t + \tag{3.6}$$

$$\frac{1}{\lambda_k^2} \psi_k \sin \lambda_k^2 t + \frac{1}{\lambda_k^2} \int_0^t F_k(\tau; u) \sin \lambda_k^2 (t-\tau) d\tau \Big\} \cos \lambda_k x, (k = 1, 2, ...)$$

Now, from (2.6), considering (3.1), we have:

$$a(t) = [h(t)]^{-1}\left\{h''(t) - f(0,t) + \right.$$
$$\left. + \sum_{k=1}^{\infty} \lambda_k^4 u_k(t)\right\}. \tag{3.7}$$

Substitute the expression (3.4) in (3.7):

$$a(t) = [h(t)]^{-1}\left\{h''(t) - f(0,t) + \sum_{k=1}^{\infty} \lambda_k^4 \left[\varphi_k \cos\lambda_k^2 t + \frac{1}{\lambda_k^2}\psi_k \sin\lambda_k^2 t + \right.\right.$$
$$\left.\left. + \frac{1}{\lambda_k^2}\int_0^t F_k(\tau; u, a)\sin\lambda_k^2(t-\tau)d\tau\right]\right\}. \tag{3.8}$$

Thus, the solution of the problem (2.1)-(2.3),(2.6) was reduced to the solution of the problem (3.6),(3.8)for the unknown functions $u(x,t)$ and $a(t)$.

Lemma 3.1 *If $\{u(x,t), a(t)\}$ any classical solution of the problem (2.1)-(2.3),(2.6) then functions $u_k(t) = m_k \int_0^1 u(x,t)\cos\lambda_k x dx$, $(k = 0, 1, 2, \ldots)$ satisfy the system (3.4), (3.5).*

Remark 3.1 From Lemma 3.1 it follows that to prove the uniqueness of the solution of the problem (2.1)-(2.3),(2.6) enough to prove the uniqueness of the solution of the problem (3.6), (3.8).

Now, we consider the following spaces:

1 We denote by $B_{2,T}^{\alpha}$ [16] ,a consisting of all functions $u(x,t)$ of the form

$$u(x,t) = \sum_{k=0}^{\infty} u_k(t)\cos\lambda_k x dx \quad (\lambda_k = \pi k),$$

considered in D_T , where each of the functions form $u_k(t)(k = 0, 1, ...)$ is continuous on $[0,T]$ and

$$J(u) \equiv \|u_0(t)\|_{C[0,T]} + \left(\sum_{k=1}^{\infty}\left(\lambda_k^{\alpha}\|u_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} < +\infty.$$

The norm in this set is defined as follows:

$$\|u(x,t)\|_{B_{2,T}^{\alpha}} = J(u)$$

2 The spaces E_T^{α} denote the space consisting of a topological product

$$B_{2,T}^{\alpha} \times C[0,T].$$

The norm of element $z(x,t) = \{u, a\}$ is determined by the formula

$$\|z\|_{E_T^{\alpha}} = \|u(x,t)\|_{B_{2,T}^{\alpha}} + \|a(t)\|_{C[0,T]}.$$

It is obvious that $B_{2,T}^{\alpha}$ and E_T^{α} are Banach spaces.

Now in the space E_T^α consider the operator

$$\Phi(u,a) = \{\Phi_1(u,a), \Phi_2(u,a)\}$$

where

$$\Phi_1(u,a) = \tilde{u}(x,t) \equiv \sum_{k=0}^{\infty} \tilde{u}_k(t) \cos \lambda_k x,$$

$$\Phi_2(u,a) = \tilde{a}(t),$$

where $\tilde{u}_0(t), \tilde{u}_k(t)(k = 1,2,...)$ and $\tilde{a}(t)$ are equal to the right hand sides of (3.4), (3.5). Now with the help of easy transformations we find:

$$\|\tilde{u}_0(t)\|_{C[0,T]} \leq |\varphi_0| + T|\psi_0| +$$

$$+T\sqrt{T}\left(\int_0^T |f_0(\tau)|^2 d\tau\right)^{\frac{1}{2}} + T^2 \|a(t)\|_{C[0,T]} \|u_0(t)\|_{C[0,T]}, \tag{3.9}$$

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k^5 \|\tilde{u}_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \leq 2\left(\sum_{k=1}^{\infty}\left(\lambda_k^5 |\varphi_k|\right)^2\right)^{\frac{1}{2}} +$$

$$+2\left(\sum_{k=1}^{\infty}\left(\lambda_k^3 |\psi_k|\right)^2\right)^{\frac{1}{2}} + 2\sqrt{T}\left(\int_0^T \sum_{k=1}^{\infty}\left(\lambda_k^3 |f_k(\tau)|\right)^2\right)^{\frac{1}{2}} +$$

$$+2T \|a(t)\|_{C[0,T]} \left(\sum_{k=1}^{\infty}\left(\lambda_k^5 \|u_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}, \tag{3.10}$$

$$\|\tilde{a}(t)\|_{C[0,T]} \leq \left\|[h(t)]^{-1}\right\|_{C[0,T]} \left\{\left\|h''(t) - f(0,t)\right\|_{C[0,T]} + \left(\sum_{k=1}^{\infty} \lambda_k^{-2}\right)^{\frac{1}{2}} \times\right.$$

$$\left[\left(\sum_{k=1}^{\infty}(\lambda_k^5 |\varphi_k|)^2\right)^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty}(\lambda_k^3 |\psi_k|)^2\right)^{\frac{1}{2}} + \sqrt{T}\left(\int_0^T \sum_{k=1}^{\infty}(\lambda_k^3 |f_k(\tau)|)^2 d\tau\right)^{\frac{1}{2}} +\right.$$

$$\left.\left. + \; T \|a(t)\|_{C[0,T]} \left(\sum_{k=1}^{\infty}(\lambda_k^5 \|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right]\right\}. \tag{3.11}$$

Suppose that the data of the problem (2.1)-(2.3), (2.6), satisfy the following conditions:

1 $\varphi(x) \in C^4[0,1], \varphi^{(5)}(x) \in L_2(0,1), \varphi'(0) = \varphi'(1) = \varphi'''(0) = \varphi'''(1) = 0;$
2 $\psi(x) \in C^2[0,1], \psi^{(3)}(x) \in L_2(0,1), \psi'(0) = \psi'(1) = 0;$
3 $f(x,t), f_x(x,t), f_{xx}(x,t) \in C(D_T), f_{xxx}(x,t) \in L_2(D_T), \; f_x(0,t) = f_x(1,t) = 0 (0 \leq t \leq T);$
4 $h(t) \in C^2[0,T], h(t) \neq 0 (0 \leq t \leq T).$

Further, from (3.9)-(3.11) we have:

$$\|\tilde{u}(x,t)\|_{B^5_{2,T}} \le A_1(T) + B_1(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^5_{2,T}} \tag{3.12}$$

$$\|\tilde{a}(t)\|_{C[0,T]} \le A_2(T) + B_2(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^5_{2,T}} \tag{3.13}$$

where

$$A_1(T) = \|\varphi(x)\|_{L_2(0,1)} + T\|\psi(x)\|_{L_2(0,1)} + T\sqrt{T}\|f(x,t)\|_{L_2(D_T)} +$$

$$+2\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)} + 2\left\|\psi^{(3)}(x)\right\|_{L_2(0,1)} + 2\sqrt{T}\,\|f_{xxx}(x,t)\|_{L_2(D_T)},$$

$$B_1(T) = T(T+2),$$

$$A_2(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\|h''(t) - f(0,t)\|_{C[0,T]} + \left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}} \times\right.$$

$$\left.\times\, 2\left[\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)} + \left\|\psi^{(3)}(x)\right\|_{L_2(0,1)} + \sqrt{T}\,\|f_{xxx}(x,t)\|_{L_2(D_T)}\right]\right\},$$

$$B_2(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}} T.$$

From inequalities (3.12),(3.13) we conclude:

$$\|\tilde{u}(x,t)\|_{B^5_{2,T}} + \|\tilde{a}(t)\|_{C[0,T]} \le A(T) + B(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^5_{2,T}}\,. \tag{3.14}$$

where

$$A(T) = A_1(T) + A_2(T), B(T) = B_1(T) + B_2(T).$$

So, we can prove the following theorem:

Theorem 3.1 *Let conditions 1-4 be fulfilled and*

$$B(t)(A(T)+2)^2 < 1 \tag{3.15}$$

Then the problem (2.1)-(2.3),(2.6) has a unique solution in the sphere $K = K_R(\|z\|_{E^3_T} \le R = A(T)+2)$ *of the space* E^5_T.

Proof. In the space E^5_T consider the equation

$$z = \Phi z, \tag{3.16}$$

where $z = \{u, a\}$, the components $\Phi_i(u,a)(i=1,2)$, of the operator $\Phi(u,a)$ are determined by the right hand sides of equations (3.6)-(3.8). Consider the operator $\Phi(u,a)$ in the sphere $K = K_R$ from E^5_T. Similar to (3.14) we obtained that for any $z, z_1, z_2 \in K_R$ the following estimate are valid:

$$\|\Phi z\|_{E^3_T} \le A(T) + B(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^5_{2,T}} \le$$

$$\le A(T) + B(T)\,(A(T)+2)^2\,, \tag{3.17}$$

$$\|\Phi z_1 - \Phi z_s\|_{E^5_T} \le$$

$$\le B(T)R(\|a_1(t) - a_2(t)\|_{C[0,T]} + \|u_1(x,t) - u_2(x,t)\|_{B^5_{2,T}} \tag{3.18}$$

Then from (3.17) and (3.18), with considering (3.15), it follows that operator Φ acts in the sphere $K = K_R$ and it is contraction mapping. Therefore, in the sphere $K = K_R$ the operator Φ has a unique fixed point $\{u, a\}$, that is a solution of equation (3.6),(3.8). The function $u(x,t)$, as the element of the space $B^5_{2,T}$, has continuous derivatives $u_x(x,t)$, $u_{xx}(x,t)$, $u_{xxx}(x,t)$, $u_{xxxx}(x,t)$ in D_T.

From (3.15) we get:

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k \left\|u_k''(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \leq$$

$$\left[\left(\sum_{k=1}^{\infty}\left(\lambda_k^5 \left\|u_k(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} + \sqrt{3}\left\|\left\|f_x(x,t)+a(t)u_x(x,t)\right\|_{C[0,T]}\right\|_{L_2(0,1)}\right] (i=1,2)..$$

Then it follows that $u_{tt}(x,t)$ is continuous in D_T. It is easy to verify that (2.1)-(2.3),(2.6) are satisfied in the ordinary sense. By virtue of the lemma 3.1, it is unique in the sphere $K = K_R$. Theorem is proved.

Using the Theorem 1 the following Lemma is proved.

Theorem 3.2 *Let all the conditions of the theorem 1 be satisfied,*

$$\varphi(0) = h(0), \psi(0) = h'(0).$$

Then, problem (2.1)-(2.4) has in the sphere $K = K_R(\|z\|_{E^5_{T,T}} \leq R = A(T) + 2)$ *from* E^5_T *unique classical solution.*

References

1. Samara A.A. On some problems of the theory of differential equations *Differential equations,* **16**(11),(1980) 1925–1935.
2. Cannon J.R. The solution of the heat equation subject to the specification of energy *Quart. Appl. Math.,* **5**(21), (1963),155–160.
3. Ionkin N.I. The solution of a boundary value problem of the theory of heat conduction with a nonclassical boundary condition *Differential Equations,* **13**(2),(1977), 294–304.
4. Nakhushev A.M. About one approximate method for solving boundary value problems for differential equations and its approximation to the dynamics of soil moisture and groundwater *Differential Equations,* **18**(1), (1982),72–81.
5. Kapustin N.Yu., Moiseev E.I On spectral problems with a spectral parameter in the boundary condition *Differential Equations.* **33**(1), (1997),115–119.
6. Kozhanov A.I., Pulkina L.S. On the solvability of boundary value problems with a non-local boundary condition of integral type for multidimensional hyperbolic equations. *Differential equations.* **42**(9), (2006),1166 - 1179.
7. Gordeziani D.G., Avalishvili G.A. Solutions of non-local problems for one-dimensional environment fluctuations. *Math modeling.* **12**(1), (2000),94 – 103.
8. M. S. Hussein, D. Lesnic, M. I. Ismailov; An inverse problem of finding the time-dependent diffusion coefficient from an integral condition, *Mathematical Methods in the Applied Sciences* **39**(2) (2016), 963-98.
9. T. Megraliyev, F. Kh. Alizade, An inverse boundary value problem for a fourth-order Boussinesq equation with non-self-adjoint boundary and with additional integral conditions, *Vestnik TVGU. Series: Applied Mathematics,* **2**,(2017), 17-36
10. Prilepko A. I., Kostin A. B. On some inverse problems for parabolic equations with final and integral observation. *Mathematical collection.* **183**(4), (1992), 49 – 68.

11. Prilepko A.I., Tkachenko D.S. Properties of solutions of a parabolic equation and the uniqueness of the solution of the inverse problem of a source with an integral redefinition. *Journal of computational mathematics and mathematical physics.* **43**(4), (2003), 562-570.
12. Kamynin V. L. On the inverse problem of determining the right-hand side in a parabolic equation with the condition of an integral redefinition. *Mathematical notes,* **77**, (2005) 522–534.
13. Efimov A.I., Kashkarov P.K. *Mechanics and electromagnetism.*, Moscow University Publishing House, (2010)
14. Mehraliyev Y.T and Kanca F. An Inverse Boundary Value Problem for a Second Order Elliptic Equation in a Rectangle. *Mathematical Modelling and Analysis* (2014). **19**(2), 241-256.
15. Budak B.M., Samara A.A., Tikhonov A.N. *Collection of problems in mathematical physics.* Moscow: Science, (1972), 668.
16. Khudaverdiev K.I, Veliyev A.A. *Investigation of a one-dimensional mixed problem for a class of third-order pseudohyperbolic equations with a nonlinear operator right-hand side,* Baku: Chashyoglu, (2010), 168.
17. Ramazanova Aysel T.,Guliyev H.F., Arnd Roesch. An inverse problem for determining right hand side of equations for hyperbolic equation of fourth order *Advances in Differential Equations and Control Processes*, **20**(2), (2019), 143-161,
18. Ramazanova Aysel T. On determining Initial conditions of Equations Flexural Torsional Vibrations of a bar *European Journal of Pure and Applied Mathematics*, **12**(1), (2019) 25–38.