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Ground consolidation under the fractal filtration law

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Abstract. *The paper evaluates the influence of the fractal structure of the ground on the fluids flow. Currently, there is no analytical solution to such problems, and the available ones are obtained only for the simplest types of deformation. Based on the results of the research, a two-dimensional equation for the consolidation of a two-phase, fully water-saturated ground (in the absence of a gas phase) is obtained, when the fractality of the pore space geometry is taken into account only in the Darcy law.*

Keywords. consolidation · fractals · filtration · ground · pressure · stress · fluid flow

Mathematics Subject Classification (2010): 76B07

1 Introduction

The theory of consolidation of water-saturated grounds, reflecting the process of convergence of ground particles and the reduction of pore volume, accompanied by the water displacement, attracts growing attention of researchers due to the increasing volume of construction ground and underground work in complex grounds. As a problem of unsteady filtration in a deformable medium, taking into account the variable porosity and ground permeability, its multiphase nature and the complex rheology of the phase components, it is sometimes not amenable to an analytical solution, and in this case, appropriate numerical methods are used to analyze the situation.

Recent studies allow us to view the fractal nature [1, 2, 4] of the structure of the ground mass. The assessment of the influence of the fractality of the ground structure on the occurring processes associated with the solution of the corresponding problems. Currently, there are few such solutions, and they are obtained only for the simplest types of deformation, such as for a cylindrical or spherically symmetric type.

The construction of the theory of consolidation of the ground mass of the fractal structure is urgently required. However, it is natural that the resulting generalized system of equations will be many times more mathematically complex both in terms of its solution and analysis. Therefore, a step-by-step way of complicating the existing theories of ground consolidation, taking into account the fractal geometry of its structure, is preferable [9, 10].

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2 Formulation of the problem and solution method

In the present paper, a two-dimensional equation for the consolidation of a two-phase, fully water-saturated ground (in the absence of a gas phase) is obtained, when the fractality of the pore space geometry is taken into account only in the Darcy law. In its output, we will adhere to the notation adopted in [5, 6].

The filtration law in the ground with fractal geometry is assumed according to [3] in the form:

$$\begin{aligned} u_x - \frac{n}{m} v_x &= -k_x \left(\frac{\partial}{\partial x} \left(D^{\alpha}_{a_+} H \right) (x) - i_0 \right); \\ v_z - \frac{n}{m} v_z &= -k_z \left(\frac{\partial}{\partial z} \left(D^{\beta}_{0_+} H \right) (z) - i_0 \right); \end{aligned} \tag{2.1}$$

where,

$$\begin{aligned} \left(D^{\alpha}_{a_+} H \right) (x) &= \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial x} \int_a^x \frac{H(\xi)\, d\xi}{(x-\xi)^a}, \\ \left(D^{\beta}_{0_+} H \right) (z) &= \frac{1}{\Gamma(1-\beta)} \frac{\partial}{\partial z} \int_a^z \frac{H(\xi)\, d\xi}{(z-\xi)^a} \end{aligned} \tag{2.2}$$

respectively, the left-hand fractional derivatives of the orders $\alpha; 0 < \alpha < 1$ and $\beta; 0 < \beta < 1$.

Here $u_x,\ u_y$ are projections of the average fictitious velocity of water movement in the pores; ν_x, ν_y are projections of the average velocity of the ground skeleton; n is the porosity of the ground; v - the volume of pores in the unit volume of the ground; m is the volume of solid particles in the unit of the ground, and $m + n = 1$; k_x and k_y are the filtration coefficients in the direction of the x and y axes; i_0 is the initial pressure gradient; H is the pressure function associated with the pressure p and the height z of the considered point above the comparison plane addiction

$$H = \frac{p}{\gamma} + z, \tag{2.3}$$

where γ is the water density.

In addition, in (2.2), the last α and β - are the parameters of the fractality of the filtration component; $\Gamma(1-\alpha)$ is the Euler function.

The equations of continuity of the fluid and solid components of the ground are taken in the classical form (which is our simplification at the first stage of the study):

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} + \frac{\partial n}{\partial t} = 0; \ \frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z} + \frac{\partial m}{\partial t} = 0 \tag{2.4}$$

From the system of equations (2.1) – (2.4) following the sequence of actions given in [6], we come to the following consolidation equation for a ground with a fractal structure.

$$\begin{aligned} \frac{\partial e}{\partial t} = (1+e) \Bigg[& \frac{\partial}{\partial x} k_x \left(\frac{\partial}{\partial x} \left(D^{\alpha}_{a_+} H \right) (x) - i_0 \right) + \\ & + \frac{\partial}{\partial z} k_z \left(\frac{\partial}{\partial z} \left(D^{\beta}_{0_+} H \right) (z) - i_0 \right) \Bigg], \end{aligned} \tag{2.5}$$

where $e = \frac{n}{m} = \frac{n}{1-n}$ is the porosity coefficient that determines the ratio of the pore volume to the volume of the ground skeleton (particles).

We simplify this equation by assuming the absence of the initial pressure gradient $i_0 = 0$, the constancy and uniformity of the filtration coefficients in different directions, $k_x = k_z = k$. Then we get

$$\frac{\partial e}{\partial t} = (1+e)\,k\left(\frac{\partial^2}{\partial x^2}\left(D^{\alpha}_{a_+}H\right)(x) + \frac{\partial^2}{\partial z^2}\left(D^{\beta}_{0_+}H\right)(z)\right) \tag{2.6}$$

The resulting equation relates the change in the porosity of the ground to the process of unsteady water filtration.

To solve applied problems, it is necessary to move to the tense state of the consolidating ground [7, 8]. In this regard, there is a need to attract additional physical representations of the consolidation process. Such an essential physical assumption is the principle of hydraulic capacity of N.M. Gersevanov, which consists in the fact that the change in the ground porosity coefficient (compaction) is determined only by the change in the sum of the main stresses $\sigma = \sigma_x + \sigma_y + \sigma_z$, i.e. $e = f(\sigma)$. Then equation (2.6) can be written as

$$\frac{\partial \theta}{\partial t} = \frac{(1+e)}{\frac{de}{d\theta}}\left(\frac{\partial^2}{\partial x^2}\left(D^{\alpha}_{a_+}H\right)(x) + \frac{\partial^2}{\partial z^2}\left(D^{\beta}_{0_+}H\right)(z)\right) \tag{2.7}$$

This equation is called the basic equation of two-dimensional consolidation under the fractal filtration law, because it allows to take into account any computational model of the stress-strain state of the ground, i.e., any dependence of $de/d\theta$. In particular, for the linear relationship between stresses and deformations, for example, the linear relationship of the porosity coefficient with the sum of the principal stresses in the form of a straightened section of the compression curve, when

$$e = -\frac{a\sigma}{1+\xi} + b, \tag{2.8}$$

where a (compaction coefficient) and b are empirically determined constants, ξ is the coefficient of lateral pressure (expansion), which determines the ratio of transverse compressive stresses to longitudinal ones, the consolidation equation has the form

$$\frac{\partial \theta}{\partial t} = -\frac{(1+e)(1+\xi)}{a}\left(\frac{\partial^2}{\partial x^2}\left(D^{\alpha}_{a_+}H\right)(x) + \frac{\partial^2}{\partial z^2}\left(D^{\beta}_{0_+}H\right)(z)\right) \tag{2.9}$$

To link the sum of the principal stresses sigma with the head function H, we assume the assumption of V.A. Florin [6], according to which the stress state of the ground environment as a whole at any time coincides with the stress state of the ground environment, assuming its instantaneous consolidation, i.e.

$$\sigma + 2p = \sigma^* + 2p^*, \tag{2.10}$$

where σ^* and p^*- is the sum of the additional stresses in the ground skeleton and the pressure in the water that would occur if the water filling the pores did not prevent the change in the pore volume. With the instantaneous application of subsequently unchanged loads or boundary pressures, the values σ^* and p^* do not change in time and, then from (2.10) and (2.3) it follows

$$\frac{\partial \sigma}{\partial t} = -2\frac{\partial p}{\partial t} = -2\gamma\frac{\partial H}{\partial t}$$

Taking this into account in (2.9), we finally obtain the following equation for the two-dimensional consolidation of a fully water-saturated ground mass under the fractal filtration law

$$\frac{\partial H}{\partial t} = C_v\left(\frac{\partial^2}{\partial x^2}\left(D^{\alpha}_{a_+}H\right)(x) + \frac{\partial^2}{\partial z^2}\left(D^{\beta}_{0_+}H\right)(z)\right), \tag{2.11}$$

where,

$$C_v = \frac{(1+e)(1+\xi)k}{2\gamma a} \tag{2.12}$$

it is called the consolidation coefficient.

3 Numerical results

We will evaluate the influence of the fractality of the filtration law on the consolidation process by using the example of solving the one-dimensional problem of consolidating the ground layer of fully water-saturated ground.

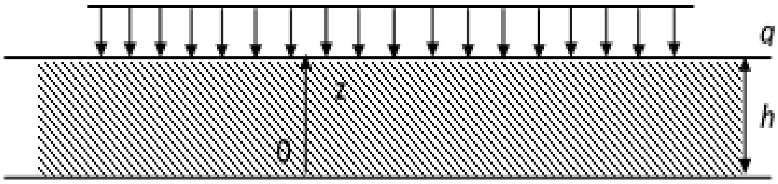

Fig. 1 Scheme of ground layer loading

Let's point the z-axis vertically upwards, placing the origin on the sole of the layer with thickness h.

Let the ground layer be compressed by a uniformly distributed load q, instantly applied at time $t = 0$. The boundary surfaces of the $z = 0$; h - layer will be considered completely permeable.

The consolidation equation (2.11) for the one-dimensional case will have the form:

$$\frac{\partial H}{\partial t} = C_v \frac{\partial^2}{\partial z^2}\left(D^{\beta}_{0_+} H\right)(z) \tag{3.1}$$

or

$$\frac{\partial H}{\partial t} = \frac{C_v}{\Gamma(1-\beta)} \frac{\partial^3}{\partial z^3} \int_0^z \frac{H(\xi,t)}{(z-\xi)^{\beta}} d\xi \tag{3.2}$$

Under the term "instantaneously applied load" we will understand such a rate of static application of the load, at which it can be assumed that there is no time to occur any outflow of water in the ground, and even more so, mineral particles of the ground, its porosity or porosity coefficient at the initial moment of application of the load $t = 0$ does not change.

Taking the dependence of the porosity coefficient on the stresses in the ground skeleton as a compression dependence $e = e_0 - a\sigma_z$ and, taking into account that at $t = 0$ $e - e_0 = 0$ we get at the initial moment $\sigma_z = \sigma_{z0} = 0$. Since in the case under consideration, the equilibrium equation for any moment of time has the form $q = \sigma_{z+p}$ we get $p_0 = q$. Where the additional excess pressure in the water at the initial time will be $H_0 = \frac{p_0}{\gamma} = \frac{q}{\gamma}$.

Thus, the solution of the problem under consideration is reduced to finding a solution to equation (3.2) under the following initial and boundary conditions:

$$t = 0 \text{ and } 0 \le z \le hH_0 = p_0/\gamma = q/\gamma \tag{3.3}$$

$$t \ne 0 \text{ and } z = 0 \text{ or } z = hH = 0 \tag{3.4}$$

We introduce the following dimensionless quantities

$$\tilde{H} = \frac{H}{h}; \tilde{z} = \frac{z}{h}; \overset{\sim}{\xi} = \frac{\xi}{h}; \tilde{t} = \frac{C_v}{h^{2+\beta}} t \tag{3.5}$$

In dimensionless quantities, the consolidation equation (2.11) and the initial and boundary conditions (3.3) and (3.4) are written in the form (where the $\sim$ sign above the letters is omitted to simplify the writing):

$$\frac{\partial H}{\partial t} = \frac{1}{\Gamma(1-\beta)} \frac{\partial^3}{\partial z^3} \int_0^z \frac{H(\xi, t)}{(z-\xi)^\beta} d\xi \tag{3.6}$$

$$t = 0 \text{ and } 0 \le z \le h H = H_0 \tag{3.7}$$

$$t \ne 0; \text{and } z = 0; \text{or } z = 1 H = 0 \tag{3.8}$$

To solve the problem, we use the numerical method of finite differences. For the convenience of further calculations, we denote:

$$f(z, t) = \int_0^z \frac{H(\xi, t)}{(z-\ \xi)^\beta} d\xi \tag{3.9}$$

Then, instead of (3.6), we get:

$$\frac{\partial H(z, t)}{\partial t} = \frac{1}{\Gamma(1-\beta)} \frac{\partial^3}{\partial z^3} f(z, t) \tag{3.10}$$

We apply the finite difference method. To do this, we cover the entire consolidation area with a coordinate grid in steps Δz and Δt, which has node points with coordinates $t_k = k\Delta t, z_i = i\Delta z.$

The value of the pressure function H at the nodal point with the number (i, k) is denoted by $H_{i,k}$. Replacing the partial derivatives included in equation (3.10) with the corresponding differences:

$$\frac{\partial H}{\partial t} \approx \frac{H_{i,k+1} - H_{i,k}}{\Delta t}$$

$$\frac{\partial^3 f}{\partial z^3} \approx \left(\frac{1}{\Delta z}\right)^3 (f_{i+1,k} - 3f_{i,k} + 3f_{i-1,k} - f_{i-2,k}),$$

where,

$$f(z, t) \approx f_{i,k} = \sum_{j=0}^{i-1} \frac{H_{j,k}}{(i-j)^\beta} \Delta z$$

and, substituting the obtained expressions in equation (3.10), we get the following numerical analogue of it:

$$H_{i,k+1} = H_{i,k} + \frac{\Delta t}{\Delta z^3} \frac{1}{\Gamma(1-\beta)} (f_{i+1,k} - 3f_{i,k} + 3f_{i-1,k} - f_{i-2,k}) \tag{3.11}$$

The numerical analog of the initial and boundary conditions will be:

$$k = 0 \text{ and } k = 0 \text{ and } 0 \le i \le n H_{i,k} = H_0 \tag{3.12}$$

$$k \ne 0 \text{ and } i = 0 \text{ or } i = n H_{i,k} = 0 \tag{3.13}$$

Based on the available formula (3.11), the initial level (3.12), and the boundary condition (3.13), a calculation program for the numerical values of the pressures $H_{i,k}$ at the nodal points of the numerical grid is compiled and implemented.

The calculation was performed for the following values of the initial parameters: $H_0 = 1$, $\beta = 0.5$ (fractality of the filtration law), $\beta = 0$ (classical filtration law), $\Delta \mathrm{t} = 0.01$, $\Delta \mathrm{z} = 0.1$, $k_* = 20$, $n = 10$. The calculation results are presented in Table 1. The same values are used to plot the distribution of the pressure function over the thickness of the ground layer for several time values.

Table 1.

		i=1	i=2	i=3	i=4	i=5	i=6	i=7	i=8	i=9
k=1	β=0,5	0,9271	1,0092	1,0030	1,0014	1,0008	1,0005	1,0003	1,0002	0,9438
	β=0	0,9000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,9000
k=2	β=0,5	0,8641	1,0114	1,0068	1,0029	1,0016	1,0010	1,0007	0,9973	0,8948
	β=0	0,8200	0,9900	1,0000	1,0000	1,0000	1,0000	1,0000	0,9900	0,8200
k=3	β=0,5	0,8093	1,0083	1,0108	1,0047	1,0025	1,0015	1,0008	0,9920	0,8519
	β=0	0,7550	0,9740	0,9990	1,0000	1,0000	1,0000	0,9990	0,9740	0,7550
k=4	β=0,5	0,7615	1,0014	1,0145	1,0068	1,0035	1,0021	1,0007	1,9850	0,8142
	β=0	0,7014	0,9546	0,9966	0,9999	1,0000	0,9999	0,9966	0,9546	0,7014
k=5	β=0,5	0,7195	0,9917	1,0176	1,0091	1,0045	1,0026	1,0003	0,9767	0,7808
	β=0	0,6566	0,9335	0,9927	0,9996	1,0000	0,9996	0,9927	0,9335	0,6566
k=6	β=0,5	0,6824	0,9799	1,0199	1,0115	1,0057	1,0032	0,9995	0,9676	0,7510
	β=0	0,6186	0,9117	0,9875	0,9989	0,9999	0,9989	0,9875	0,9117	0,6186
k=7	β=0,5	0,6494	0,9667	1.0212	1.0139	1.0070	1,0037	0,9983	0,9580	0,7243
	β=0	0,5861	0,8900	0,9811	0,9979	0,9997	0,9979	0,9811	0,8900	0,5861
k=8	β=0,5	0,6199	0,9526	1,0217	1,0164	1,0083	1,0042	0,9968	0,9479	0,7003
	β=0	0,5578	0,8687	0,9736	0,9964	0,9993	0,9964	0,9736	0,8687	0,5578
k=9	β=0,5	0,5935	0,9380	1,0212	1,0188	1,0098	1,0046	0,9949	0,9377	0,6786
	β=0	0,5331	0,8481	0,9564	0,9944	0,9988	0,9944	0,9564	0,8481	0,5331
k=10	β=0,5	0,5696	0,9230	1,0200	1,0211	1,0113	1,0050	0,9927	0,9274	0,6588
	β=0	0,5113	0,8283	0,9566	0,9919	0,9979	0,9919	0,9566	0,8283	0,5113
k=11	β=0,5	0,5480	0,9079	1,0179	1,0232	1,0129	1,0054	0,9903	0,9172	0,6408
	β=0	0,4919	0,8095	0,9473	0,9890	0,9967	0,9890	0,9473	0,8095	0,4919
k=12	β=0,5	0,5283	0,8929	1,0151	1,0251	1,0145	1,0057	0,9876	0,9071	0,6243
	β=0	0,4745	0,7915	0,9377	0,9856	0,9952	0,9356	0,9377	0,7915	0,4745
k=13	β=0,5	0,5104	0,8781	1,0117	1,0267	1,0162	1,0060	0,9848	0,8971	0,6091
	β=0	0,4587	0,7744	0,9279	0,9818	0,9932	0,9818	0,9279	0,7744	0,4587
k=14	β=0,5	0,4939	0,8635	1,0077	1,0280	1,0178	1,0062	0,9817	0,8873	0,5951
	β=0	0,4444	0,7582	0,9179	0,9775	0,9909	0,9775	0,9179	0,7582	0,4444
k=15	β=0,5	0,4787	0,8492	1,0032	1,0291	1,0194	1,0064	0,9785	0,8778	0,5821
	β=0	0,4314	0,7428	0,9079	0,9729	0,9883	0,9729	0,9079	0,7428	0,4314
k=16	β=0,5	0,4647	0,8353	0,9982	1,0298	1,0210	1,0065	0,9752	0,8685	0,5700
	β=0	0,4194	0,7281	0,8979	0,9679	0,9852	0,9679	0,8979	0,7281	0,4194
k=17	β=0,5	0,4517	0,8217	0,9929	1,0302	1,0224	1,0065	0,9718	0,8594	0,5588
	β=0	0,4083	0,7142	0,8879	0,9627	0.9817	0,9627	0,8879	0,7142	0,4083
k=18	β=0,5	0,4396	0,8085	0,9873	1,0304	1,0238	1,0065	0,9684	0,8506	0,5482
	β=0	0,3981	0,7010	0,8780	0,9571	0,9779	0,9571	0,8780	0,7010	0,3981
k=19	β=0,5	0,4284	0,7958	0,9814	1,0302	1,0251	1,0064	0,9648	0,8421	0,5384
	β=0	0,3886	0,6884	0,8682	0,9513	0,9738	0,9513	0,8682	0,6884	0,3886
k=20	β=0,5	0,4179	0,7834	0,9752	1,0297	1,0263	1,0063	0,9613	0,8338	0,5291
	β=0	0,3797	0,6764	0,8585	0,9452	0,9693	0,9452	0,8585	0,6764	0,3797

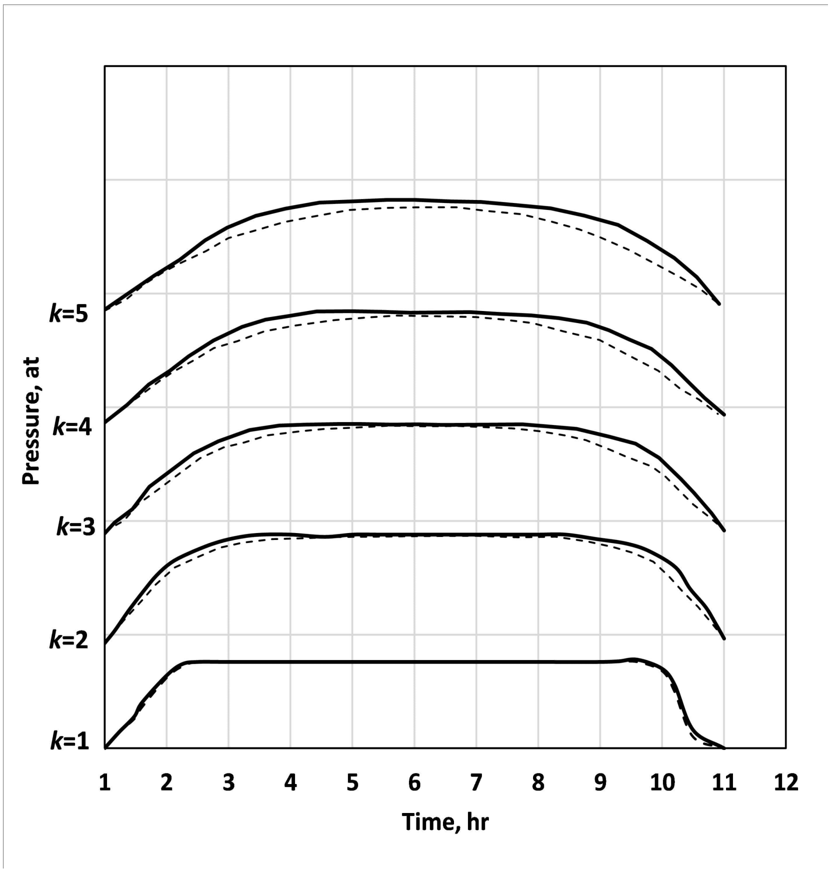

Fig. 2. Plots of the depth distribution of the pressure H for different time values $t_k = k\Delta t = 0.01t,\ -\beta = 0.5; -\beta = 0$

From the table and diagrams follows that taking into account the fractality of the filtration law gives an overestimation of the calculated values of the head function, and this overestimation becomes more and more noticeable over time.

Of particular interest is the effect of the fractality of the filtration law on the total layer sediment. For this precipitation, $S(t)$, according to the well-known formula [6], we have:

$$S(t) = \frac{a}{1+e}\int_0^h (q - \gamma H)\, dz$$

Or by means of $H_{i,k}$ - values of the head function at the nodal points of the numerical grid:

$$S_k = \frac{a}{1+e}\left(qh - \gamma\sum_{j=1}^{9} H_{i,k}\Delta z\right)$$

Since $H_0 = \frac{q}{h}\gamma = 1$, for a dimensionless precipitation $\tilde{S}\sim_k = S_k/h$, taking into account the dimensionless z coordinate and the dimensionless value $a^{\sim} = aq/(1+e)$, omitting, as before, the sign $\sim$ over the dimensionless values, we get:

$$S_k = a\left(1 - \sum_{j=1}^{9} H_{i,k}\Delta z\right)$$

4 Conclusions

The precipitation calculated by this formula for the time values $\Delta t, 5\Delta t, 10\Delta t, 15\Delta t, 20\Delta t$ or the fractal parameter $\beta = 0.5$ is summarized in Table 2. The values of precipitation obtained under the classical filtration law are also given there for comparison:

Table 2.

	S_k		$\frac{S_k^{\beta=0} - S_k^{\beta=0.5}}{S_k^{\alpha=0}} \cdot 100\%$
t_k	$\beta = 0$	$\beta = 0.5$	
0.001	0.0222a	0.0126a	43,24%
0.005	0.0970a	0.0552a	43,09%
0.01	0.1584a	0.0967a	38,95%
0.015	0.2113a	0.2113a	38,19%
0.02	0.2567a	0.1597a	37,78%

The last column of the table shows the differences between these results, expressed as a percentage. They indicate the significant influence of the fractality of the filtration law on the calculated values of the sediment of the ground layer.

References

1. Adler, P. M., Thovert, J.-F., Bekri, S., and Yousefian, F.: Real Porous Media: Local Geometry and Transports, *J. Eng. Mech.* 0733-9399, 128(8),2002 829–839.
2. Alexander S. Balankin, Benjamin Espinoza Elizarraraz Map of fluid flow in fractal porous medium into fractal continuum flow. May 2012, Physical Review DOI: 10.1103/PhysRevE.85.056314
3. Dinariev O.YU. Dvizhenie zhidkostej i gazov v poristyh sredah s fraktal'noj geometriej / Mekhanika zhidkosti i gaza, ^{1}5, 1992. – S. 101 – 109.
4. Feder E. *Fraktaly.* Izd-vo Mir, M. - 1991.
5. Huang, B. et al. "Experimental Study on the Permeability Characteristic of Fused Quartz Sand and Mixed Oil as a Transparent Soil." Water 11 (2019): 2514.
6. Ivanov L.P. Grunty i osnovaniya gidrotekhnicheskih sooruzhenij. Mekhanika gruntov, Vysshaya shkola, Moskva, 1991. – 447 s.
7. Jinsui Wu Boming, Yu Boming Yu A fractal resistance model for flow through porous media // September, 2007. International Journal of Heat and Mass Transfer 50(19):3925-3932 DOI: 10.1016/j.ijheatmasstransfer.2007.02.009
8. Nigmatulin R.I. Drobnyj integral i ego fizicheskaya interpretaciya // Teoreticheskaya i matematicheskaya fizika, T. 90, ^{1}3, 1992. – S. 354 – 367.
9. Sahimi, M., 1993, "Flow Phenomena in Rocks: From Continuum Models to Fractals, Percolation, Cellular Automata, and Simulated Annealing," Rev. Mod. Phys. 0034-6861, 65, pp. 1393–1534.
10. Yu, B. (July 30, 2008). "Analysis of Flow in Fractal Porous Media." ASME. Appl. Mech. Rev. September 2008; 61(5): 050801. https://doi.org/10.1115/1.2955849