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The compare of elastic parameters of materials and their wave propagation velocities at extremely high deformations in the case of harmonic potential in initially strained media

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Abstract. *Numerical results of pressure parameter and velocity parameters for longitudinal and share waves were obtained using the ratios of the harmonic type of elastic potential in the framework of nonclassical-linearized approach (NLA) for the materials: plexiglas, steel, plagiogranite and sandstone. These results were compared and their extremal values were defined.*

Keywords. high pressure · wave propagation velocities · elastic parameters.

Mathematics Subject Classification (2010): 74J30

1 Introduction

The determination of mechanical properties of materials is very important in many fields of science, including geophysics. It is known that the main geospheres of the Earth - the Crust, Mantle and the Earth's Core are considered deformable solids [2]. This implies the presence of different mechanical properties of geomaterials - density, velocity of propagation of elastic waves, deformability, heat capacity, viscosity, etc. Despite the diversity of modern science and technology, there are not many areas where the phenomena of ultrahigh deformations are closely considered. The main obstacle to this is that most materials under the action of high pressures are destroyed or change their structure, for example, allotropic modification (graphite-diamond). However, in the depths of most astronomical objects, and not least of all the Earth, the existence of ultrahigh deformations caused by colossal values of deforming forces in a confined space is very likely. To try to determine these phenomena, it is necessary, first of all, to study the properties of materials as best as possible. In order to determine the mechanical properties of the medium, their chemical composition and physical processes occurring at different depths of the Earth, seismological data are processed. The seismic velocities are compared and matched with the laboratory experimental data of geological materials (of relevant rocks and minerals), taking into account the Birch law. [1] The empirical relation that indicates a linear ratio between the wave velocity vp and the density ρ of rocks and minerals is known as Birch's law, and was established by Francis Birch in early fifties of twenty's century:

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$$v_p = a\,(M) + b\rho$$

Here, v_p - is the velocity of compression waves, M - average atomic mass, $a(x)$ and b are experimentally determined function and number respectively.
However these ratios cease to be fair with increasing pressures. The materials begin to exhibit inelastic properties at high pressures and temperatures [13].
To obtain analytical formulae that would express the change of the ratio between the parameter of propagation of elastic waves and the density or deformation parameter, we should use elastic potentials. We emphasize that under the term "parameter" of velocity is meant the velocity of propagation of elastic waves in a medium that is under deformation. The elastic potential can be defined as a quantity identical to the free energy of the body. One of the simple and relatively convenient elastic potentials is the harmonic type of potential. This potential is shown as follows:

$$\Phi^S = \frac{1}{2}\lambda S_1^2 + \mu S_2$$

Expressions for other type of potential in homogeneous deformable isotropic media within nonclassically linearized approach (NLA) were obtained in the work [13].
If we introduce the system of invariants $s_1; s_2; s_3$ as follows [3]:

$$s_1^0 = \left(\sqrt{1+2\varepsilon_1^0}-1\right) + \left(\sqrt{1+2\varepsilon_2^0}-1\right) + \left(\sqrt{1+2\varepsilon_3^0}-1\right)$$

$$s_2^0 = \left(\sqrt{1+2\varepsilon_1^0}-1\right)^2 + \left(\sqrt{1+2\varepsilon_2^0}-1\right)^2 + \left(\sqrt{1+2\varepsilon_3^0}-1\right)^2$$

$$s_3^0 = \left(\sqrt{1+2\varepsilon_1^0}-1\right)^3 + \left(\sqrt{1+2\varepsilon_2^0}-1\right)^3 + \left(\sqrt{1+2\varepsilon_3^0}-1\right)^3$$

Then the expression for the harmonic potential can be written in the following form:

$$\Phi^{S\varepsilon}{}_0 = \left(\frac{1}{2}\lambda + \mu\right)\left[\left(\sqrt{1+2\varepsilon_1^0}-1\right)^3 + \left(\sqrt{1+2\varepsilon_2^0}-1\right)^2 + \left(\sqrt{1+2\varepsilon_3^0}-1\right)^2\right] +$$
$$+\lambda\left[\left(\sqrt{1+2\varepsilon_1^0}-1\right)\left(\sqrt{1+2\varepsilon_2^0}-1\right) + \left(\sqrt{1+2\varepsilon_2^0}-1\right)\left(\sqrt{1+2\varepsilon_3^0}-1\right) + \right.$$
$$\left. + \left(\sqrt{1+2\varepsilon_3^0}-1\right)\left(\sqrt{1+2\varepsilon_1^0}-1\right)\right]$$

In turn, given this expression in:

$$A_{i\beta} = \frac{\partial^2}{\partial\varepsilon_i^0 \partial\varepsilon_\beta^0}\Phi^{S\varepsilon}{}_0$$

$$\mu_{ij} = \frac{1}{2\left(\varepsilon_i^0 - \varepsilon_j^0\right)}\left(\frac{\partial}{\partial\varepsilon_i^0} - \frac{\partial}{\partial\varepsilon_j^0}\right)\Phi^{S\varepsilon}{}_0$$

$$S_0^{\beta\beta} = \frac{\partial}{\partial\varepsilon_\beta^0}\Phi^{S\varepsilon}{}_0$$

An expression can be defined for determining the components of the tensor ω in the cases of the theory of large initial deformations in the following form:

$$\omega_{ij\alpha\beta}\frac{\partial^2 u_\alpha}{\partial x_i \partial x_\beta}-\rho\frac{\partial^2 u_j}{\partial t^2}=0$$

$$\omega_{ij\alpha\beta}=\lambda_j\lambda_a\left[\delta_{ij}\delta_{\alpha\beta}A_{i\beta}+(1-\delta_{ij})\left(\delta_{i\alpha}\delta_{j\beta}+\delta_{i\beta}\delta_{j\alpha}\right)\mu_{ij}\right]+\delta_{i\beta}\delta_{ja}\sigma^0_{\beta\beta}$$

Assuming that a harmonic wave propagates in an elastic isotropic compressible body with initial uniform stresses, u_j is presented as:

$$u_j=u_j^\wedge e^{i(k*r-\widehat{\omega}t)},\ \ i,\ j,m=1,2,3,\ u_j^\wedge=const$$

Where k –wave vector, $r=x_n\lambda_n i_n$ – radius vector of a point in a deformed body with respect to a Cartesian coordinate system, i_n-unit vectors of the Cartesian coordinate system, $\widehat{\omega}$- is the frequency.

Substituting this expression instead of u_α and u_j respectively, it was obtained [3] a system of homogeneous algebraic equations with respect to $u_j^\wedge$ the determinant of which has the form:

$$\det\left|\left|b_{mj}-\rho\widehat{\omega}^2\delta_{mj}\right|\right|=0,$$

where

$$b_{mj}=\delta_{im}a_{ij}\lambda_j^2\lambda_m\lambda_i k_i k_j+(1-\delta_{jm})\mu_{jm}\lambda_j^2\lambda_m^2 k_i k_m+(1-\delta_{im})\mu_{im}\delta_{jm}\lambda_j^2\lambda_i^2 k_i^2+\sigma_{ij}^{*0}\delta_{jm}\lambda_i^2 k_i^2$$

Considering elastic waves propagating along the ox_1 axis and $C=\omega k_1^{-1}$, according to these equations in the case of the theory of large initial deformations, it follows [3]:

$$\begin{aligned}\rho {C_{l_{x_1}}}^2&=\lambda_1^4 A_{11}+\lambda_1^2 S_{11}^0\\ \rho {C_{S_{x_2}}}^2&=\lambda_1^2\lambda_2^2\mu_{12}+\lambda_1^2 S_{11}^0\\ \rho {C_{S_{x_3}}}^2&=\lambda_1^2\lambda_3^2\mu_{13}+\lambda_1^2 S_{11}^0\end{aligned}\tag{1.1}$$

Here ${C_{l_{x_1}}}^2$, ${C_{S_{x_2}}}^2$ and ${C_{S_{x_3}}}^2$ are understood as the parameters of velocities of the propagation of longitudinal and share waves with mutually perpendicular polarization, respectively. λ_1,λ_2 - are elongation coefficients. S_{11}^0 - components of the stress tensor referred to the unit area in the natural state, ρ- is the density of the medium in the natural state, ρ_0- is the initial density of the medium . Values $A_{11},\ \mu_{12},\mu_{13},\ S_{11}^0$ – algebraic expressions, the form of which is specified depending on the version of the problem statement. It should be noted that the considered elastic potentials have the property of reversibility of the ongoing processes [3].

As we consider the case of the theory of large initial strains and all-round compression, the following relation is true:

$$\lambda_1=\lambda_2=\lambda_3=\lambda_0=\sqrt{1+2\varepsilon_0}\tag{1.2}$$

Here ε_0– is a parameter of comprehensive deformation.

For harmonic potential the values of $A_{11},\mu_{12},\mu_{13},\ S_{11}^0$ are determined as follows [3]:

$$\begin{aligned}A_{i\beta}&=\lambda(\lambda_i\lambda_\beta)^{-1}+\delta_{i\beta}[2\mu-\lambda\left(\lambda_1+\lambda_2+\lambda_3-3\right)]\lambda_i^{-3}\\ \mu_{ij}&=[2\mu-\lambda(\lambda_1+\lambda_2+\lambda_3-3](\lambda_i\lambda_j)^{-1}(\lambda_i+\lambda_j)^{-1}\\ S_\beta^0\beta&=[\lambda\left(\lambda_1+\lambda_2+\lambda_3-3\right)+2\mu(\lambda_\beta-1)]\lambda_\beta^{-1}\end{aligned}\tag{1.3}$$

Here λ, μ - are Lame's parameters.

Substituting the expression (1.2) and (1.3) in (1.1), and taking into account that:

$$C_{l_0} = \sqrt{\frac{\lambda + 2\mu}{\rho_0}}$$

$$C_{s_0} = \sqrt{\frac{\mu}{\rho_0}}$$

We finally obtain the following expressions:

$$C_{l_{x_1}} = C_{l_0}\sqrt{\frac{\rho_0}{\rho}(1+2\varepsilon_0)^{\frac{5}{2}}}$$

$$C_{S_{x_2}} = C_{S_{x_3}} = C_{S_{x0}}\sqrt{\frac{\rho_0}{\rho}(1+2\varepsilon_0)^{\frac{3}{2}}\left[(1+2\varepsilon_0)\left(\frac{2-\nu}{1-2\nu}\right) - \sqrt{1+2\varepsilon_0}\left(\frac{1+\nu}{1-2\nu}\right)\right]}$$

$$\frac{\mathrm{P}_0}{\mu} = \frac{2+2\nu}{1-2\nu}\left(\sqrt{1+2\varepsilon_0} - 1\right)$$

Parameter $\frac{P_0}{\mu}$– characterizes the pressure.

2 Numerical results

The calculations used data from Table 1. These data are taken from [5, 6, 8, 10, 11, 13, 14].

Table 1.

Parameters of the media	$10^{-3}\lambda, MPa$	$10^{-3}\mu, MPa$	ν	$\rho_0, g/sm^3$
Plexiglas	4,04	1,9	0,3401	1,1
Steel	94,4	79,0	0,2722	7,79
Plagiogranite	35,67	39,95	0,235	2,723
Sandstone	6,4	4,22	0,3013	1,5

Fig. 1-6 show the results of calculations for parameters: $C_{lx1}; C_{Sx2}; C_{Sx3}$ and $\frac{P_0}{\mu}$ of the materials according to the values of the deformation and the density respectively.

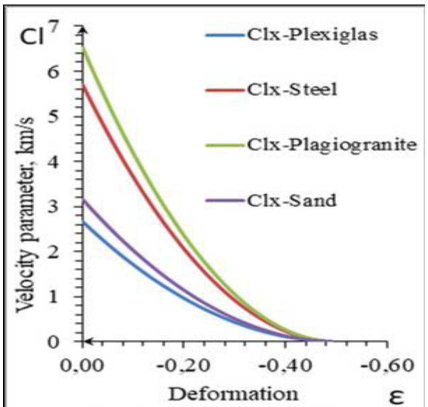

Fig. 1. The change of the longitudinal wave propagation parameter Cl_x of different materials according to the deformation values ε

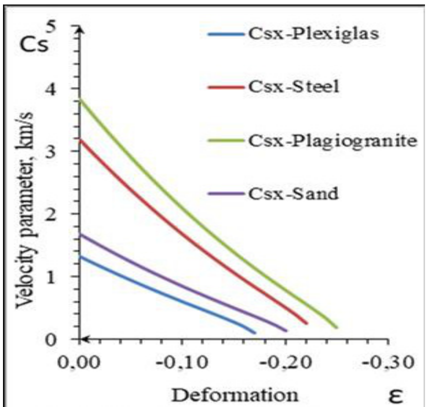

Fig. 2. The change of the share wave propagation parameter Cs_x of different materials according to the deformation values ε.

Considering the first graph (Fig. 1), it could be seen a smooth decrease in the parameter of the propagation velocity of longitudinal waves Cl_x with an increase in the values of deformations ε. A decrease in these parameters starts from their initial values of the Cl_0 parameter. These values are unique to the respective material. So for plexiglas it is equal to 2.6696 km/s , for steel 5.6921 km/s, for plagiogranite 6.5147 km/s and for sandstone 3.1477 km/s. The same values are extreme values of the corresponding graphs. It is interesting to note that these values smoothly approach zero at ε approximately equal to 0.5. Passing to the second graph (Fig. 2), there is also a smooth decrease in the values of the parameters of the shear wave propagation velocities Cs_x. In this case, the initial values of the parameter Cs_x from which this smooth decrease occurs, namely, Cs_0 for plexiglas is 1.31425 km/s, for steel 3.18452 km/s, for plagiogranite 3.83031 km/s and for sandstone 1.67859 km/s . And in this case, these values are the largest extreme values. The smallest values are achieved for plexiglas Cs_x = 0.10299 km/s at ε = -0.17, for steel Cs_x = 0.24647 km/s at ε = -0.22, for plagiogranite Cs_x = 0.17864 km/s at ε = -0.25 and for sandstone Cs_x = 0.13118 km/s at $\varepsilon = -0.2$. Thus, it can be assumed that the propagation of elastic waves in plagiogranite is subject to the smallest attenuation with an increase in extremely high

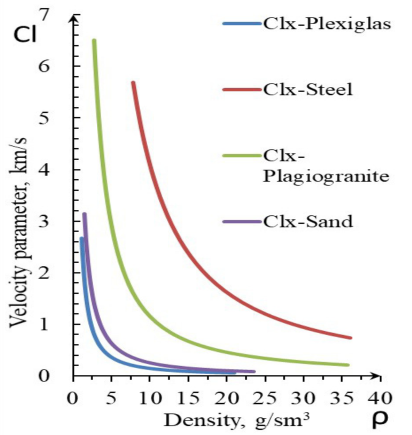

Fig. 3. The change of the longitudinal wave propagation parameter Cl_x of different materials according to the density values ρ

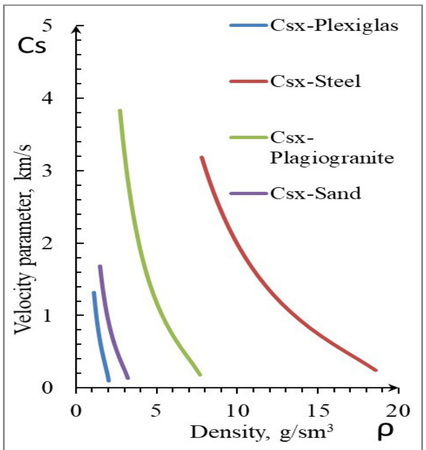

Fig. 4. The change of the share wave propagation parameter Cs_x of different materials according to the density values ρ

deformation values in the case of using the harmonic type of potential.

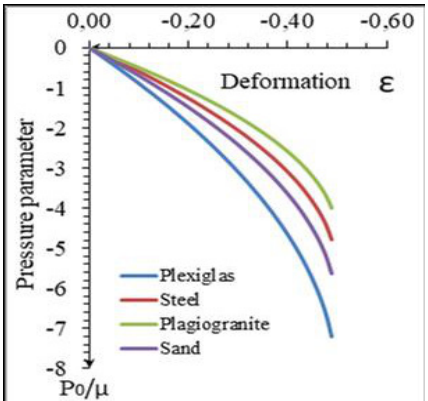

Fig. 5. The change of the pressure parameter Po/μ of different materials according to the deformation values ε

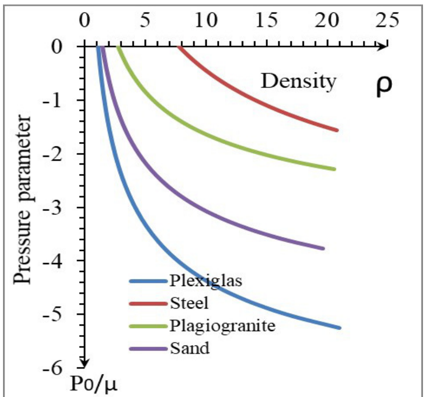

Fig. 6. The change of the pressure parameter Po/μ of different materials according to the density values ρ

Now let us consider the ratios of the parameters of propagation of velocities of materials and the values of density ρ. Considering the following two graphs (Fig. 3 and 4), one can notice a more significant hyperbolic decrease in the values of the propagation parameters of the longitudinal velocities Cl_x with an increase in the value of the density ρ and less noticeable decrease in the case of the propagation parameters of the share velocities Cs_x. The highest values of the Cl_x parameter in the first graph (Fig. 3) are also equal to the initial value of the Cl_0 parameter (for plexiglass 2.6696 km/s, for steel 5.6921 km/s, for plagiogranite 6.5147 km/s and for sandstone 3.1477 km/s). Accordingly, along the density axis, each graph begins with the corresponding initial density value for a given material ρ_0 that is given in Table 1. The same is true for the second graph (Fig. 4), where let's remind Cs_0 for plexiglass is 1.31425 km/s, for steel 3.18452km/s, for plagiogranite 3.83031 km/s and for sandstone 1.67859 km/s. As for the smallest values, the Cl_x values reach them at extremely high densities. So for plexiglass this value will be 0.001067 km/s at $\rho = 388$ g/sm^3, for steel 0.0022 km/s at $\rho = 2754 g/sm^3$, for plagiogranite 0.0026 km/s at ρ = 962 g/sm^3 and for sandstone 0.00125 km/s at $\rho = 530 g/sm^3$. In turn, the minimum values of Cs_x are reached at more modest values of density, namely, for plexiglass Cs_x will be 0.1029 km/s at $\rho = 2.0515 g/sm^3$, for steel 0.2464 km/s at $\rho = 18.588 g/sm^3$, for plagiogranite 0.1786 km/s at $\rho = 7.701 g/sm^3$ and for sandstone 0.131 km/s at $\rho = 3.227 g/sm^3$. In this case, it is obvious that the increase in the density value least of all affects the parameter of propagation of velocities Cl_x and Cs_x for the steel material.

The next two graphs (Fig. 5 and 6) show the change in the parameter P_0/ μ, which represents the internal pressure of the material as a tangential stress on the external action by changing the values of deformations and densities. So in the first graph (Fig. 5) it is seen an increase of this parameter (in absolute value) with an increase of the deformation values. The initial and lowest value of this parameter is obviously zero for all materials. This parameter reaches its maximum value at $\varepsilon = 0.49$ and for plexiglas it is -7.1956, for steel -4.7949, for plagiogranite -4.0013 and for sandstone -5.6228. In the second graph (Fig. 6), the parameter value also increases starting from the corresponding values of the initial density ρ_0 indicated in Table 1 for each material. This parameter reaches its largest values at $\rho = 388 g/sm^3$ for plexiglas, $\rho = 2754 g/sm^3$ for steel, $\rho = 962 g/sm^3$ for plagiogranite and $\rho = 530 g/sm^3$ for sandstone, just as in the case of parameter Cl_x. Obviously, plexiglas is subjected to the greatest increase in the parameter of P_0/μ with a change in values of deformation and density.

Conclusion

Under conditions of extremely high values of deformations and, at the same time, densities, the parameters Cl_x and Cs_x showed a decrease, and the parameter P_0 / μ increased their values, in absolute value, for all types of materials considered. The difference in reaching their extreme values can be associated with the structure and natural characteristics of the material. Also, the absence of actual values of the velocity parameters Cl_x and Cs_x may indicate complete attenuation and the impossibility of waves passing through the material under these conditions and or destruction of the material in essence, in other words, the material completely loses its stability.

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