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On the dispersion of torsional waves in a "hollow cylinder + surrounding medium" system with inhomogeneous initial stresses

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Abstract. *The present paper within the framework of the piecewise homogeneous body model by utilizing the three-dimensional linearized theory of elastic waves in initially stressed bodies studies the influence of the inhomogeneous initial stresses in the system consisting of the hollow cylinder and surrounding elastic medium on the torsional wave velocities propagated in this system. It is assumed that the initial inhomogeneous initial stresses are caused by uniformly distributed compressional radial normal forces acting at infinity. The corresponding eigenvalue problem is solved by employing the discrete-analytical solution method and it is obtained corresponding dispersion equation which is solved numerically. As a result of this solution, it is constructed dispersion curves, according to which, the corresponding conclusions on the influence of the inhomogeneous initial stresses on the torsional wave propagation velocity are made. In particular it is established that the considered type inhomogeneous initial stresses cause to decrease the torsional wave propagation velocity in the system under consideration.*

Keywords. stress · cylinder · torsional waves · inhomogeneous · velocity

Mathematics Subject Classification (2010): 74H55

1 Introduction

The present paper deals with the study of the influence of inhomogeneous initial stresses in the system "hollow cylinder + surrounding elastic medium" caused by the uniformly distributed radial compressional forces acting at infinity on the dispersion of the torsional waves propagated in the mentioned system. Note that the corresponding problem for the case where the initial stresses in the system under consideration are homogeneous one was studied in the paper [4] and discussed in the monograph [1]. Moreover, note that the corresponding problems for the longitudinal axisymmetric wave dispersions in the bi-layered hollow cylinder and in the system under consideration was investigated in the papers [2] and [3], respectively. However, up to now, there are not any investigations related to the influence of the aforementioned inhomogeneous initial stresses on the torsional wave dispersion propagated in the system "hollow cylinder + surrounding elastic medium". The present investigations concern, namely, on this topic.

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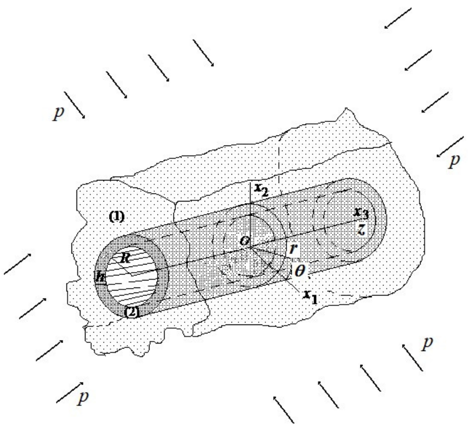

Fig. 1 The sketch of the "hollow cylinder + surrounding medium" system and the forces causing the initial stresses

2 Formulation of the problem

The system consisting of the hollow cylinder with h thickness and R radius of the internal circle of the cross-section and surrounding elastic medium, a sketch of which is shown in Fig. 1, is considered. It is assumed that in the initial state at infinity, the static radial compressional axisymmetric forces with intensity p act on the surrounding medium and as a result of this action the initial stress state appears in the constituents of the system and these stresses are determined through the following expressions.

$$\sigma_{rr}^{(1)0} = -p - 2\mu^{(1)}B^{(1)}\frac{1}{r^2}, \sigma_{\theta\theta}^{(1)0} = -p + 2\mu^{(1)}B^{(1)}\frac{1}{r^2},$$

$$\sigma_{zz}^{(1)0} = \nu^{(1)}(\sigma_{rr}^{(1)0} + \sigma_{\theta\theta}^{(1)0}),$$

$$\sigma_{rr}^{(2)0} = 2(\lambda^{(2)} + \mu^{(2)})A^{(2)} - 2\mu^{(2)}B^{(2)}\frac{1}{r^2},$$

$$\sigma_{\theta\theta}^{(2)0} = 2(\lambda^{(2)} + \mu^{(2)})A^{(2)} + 2\mu^{(2)}B^{(2)}\frac{1}{r^2},$$

$$\sigma_{zz}^{(2)0} = \nu^{(2)}(\sigma_{rr}^{(2)0} + \sigma_{\theta\theta}^{(2)0}), \tag{2.1}$$

where $\lambda^{(n)}$ and $\mu^{(n)}$ are the Lame constants and $\nu^{(n)}$ is Poisson's ratio of the $n-th$ material. The unknown constants $B^{(2.1)}$**,** $A^{(2.2)}$ and $B^{(2.2)}$ in (2.1) are determined from the system of algebraic equations

$$2\mu^{(1)}B^{(1)}\frac{1}{(R+h)^2} + 2(\lambda^{(2)} + \mu^{(2)})A^{(2)} - 2\mu^{(2)}B^{(2)}\frac{1}{(R+h)^2} = p,$$

$$\frac{p}{2(\lambda^{(1)} + \mu^{(1)})}(R + h^{(1)}) + B^{(1)}\frac{1}{(R+h)} - A^{(2)}(R + h^{(1)}) - B^{(2)}\frac{1}{(R+h)} = 0,$$

$$2(\lambda^{(2)} + \mu^{(2)})A^{(2)} - 2\mu^{(2)}B^{(2)}\frac{1}{R^2} = 0. \tag{2.2}$$

which are obtained from the boundary $\sigma_{rr}^{(2.1)0}\big|_{r\to\infty} \to -p$, $\sigma_{rr}^{(2.2)0}\big|_{r=R} = 0$ and contact $\sigma_{rr}^{(2.1)0}\big|_{r=R+h} = \sigma_{rr}^{(2.2)0}\big|_{r=R+h}$, $u_r^{(2.1)0}\big|_{r=R+h} = u_{rr}^{(2.2)0}\big|_{r=R+h}$ conditions between the cylinder and surrounding elastic medium.

This completes the determination of the initial stresses and the notation used in (2.1) and (2.2) is conventional.

Attempt to formulate the problem related to the torsional wave propagation in the foregoing system consisting of the hollow cylinder and surrounding elastic medium which have the inhomogeneous initial stresses determined through the expressions given in (2.1) and (2.2). As we consider the torsional waves propagated along the cylinder's axis, therefore it can be written the following expressions for the perturbations of the displacements

$$u_\theta^{(n)} = u_\theta^{(n)}(r,z,t), u_r^{(n)} = 0, u_z^{(n)} = 0, \tag{2.3}$$

where $u_r^{(n)}$, $u_\theta^{(n)}$ and $u_z^{(n)}$ are the components of the displacement vector in the hollow cylinder ($n = 2$) and surrounding medium ($n = 1$).

According to the expressions in (2.3) and the monographs [1, 5], it is obtained the following linearized field equations for the torsional wave propagation in the foregoing system.

$$\frac{\partial t_{r\theta}^{(n)}}{\partial r} + \frac{1}{r}\left(t_{r\theta}^{(n)} + t_{\theta r}^{(n)}\right) + \frac{\partial t_{z\theta}^{(n)}}{\partial z} = \rho^{(n)}\frac{\partial^2 u_\theta^{(n)}}{\partial t^2}, \tag{2.4}$$

where

$$t_{r\theta}^{(n)} = \sigma_{r\theta}^{(n)} + \sigma_{rr}^{(n)0}\frac{\partial u_\theta^{(n)}}{\partial r}, t_{\theta r}^{(n)} = \sigma_{r\theta}^{(n)} - \sigma_{\theta\theta}^{(n)0}\frac{u_\theta^{(n)}}{r},$$

$$t_{z\theta}^{(n)} = \sigma_{z\theta}^{(n)} + \sigma_{zz}^{(n)0}\frac{\partial u_\theta^{(n)}}{\partial z}, t_{\theta z}^{(n)} = \sigma_{\theta z}^{(n)}. \tag{2.5}$$

The elasticity relations:

$$\sigma_{r\theta}^{(n)} = \mu^{(n)}\left(\frac{\partial u_\theta^{(n)}}{\partial r} - \frac{u_\theta^{(n)}}{r}\right), \sigma_{\theta z}^{(n)} = \mu^{(n)}\frac{\partial u_\theta^{(n)}}{\partial z}. \tag{2.6}$$

Also, it is assumed the satisfaction of the following boundary and contact conditions.

$$t_{r\theta}^{(2)}\Big|_{r=R} = 0, t_{r\theta}^{(2)}\Big|_{r=R+h^{(1)}} = t_{r\theta}^{(1)}\Big|_{r=R+h^{(1)}}, u_\theta^{(2)}\Big|_{r=R+h^{(1)}} = u_\theta^{(1)}\Big|_{r=R+h^{(1)}}, \tag{2.7}$$

and the conditions

$$\left|t_{r\theta}^1\right|; \left|u_\theta^1\right| < const \,\text{as}\, r \to \infty. \tag{2.8}$$

Note that in (2.4), (2.5) and (2.7) the notation $t_{r\theta}^{(n)}$, $t_{\theta r}^{(n)}$ ($t_{\theta r}^{(n)} \neq t_{r\theta}^{(n)}$) and $t_{z\theta}^{(n)}$ show the linearized expressions of the components of the non-symmetric Kirchhoff stress tensor which are expressed through the corresponding components of the ordinary symmetric stress tensor $\sigma_{r\theta}^{(n)}$, $\sigma_{\theta r}^{(n)}$ and $\sigma_{z\theta}^{(n)}$, and through the corresponding components of the ordinary symmetric stress tensor of the classical linear theory of elasticity related to the initial state and through the components of the displacement vector. It follows from the expressions in (2.5) that in the case under consideration only the components$t_{r\theta}^{(n)}$, $t_{\theta r}^{(n)}$, $t_{z\theta}^{(n)}$ and $t_{\theta z}^{(n)}$ of the non-symmetric Kirchhoff stress tensor are differ from zero, however, the remain components of this tensor, i.e. the components $t_{rr}^{(n)}$, $t_{\theta\theta}^{(n)}$, $t_{zz}^{(n)}$, $t_{rz}^{(n)}$ and $t_{zr}^{(n)}$are equal to zero. According to the references [1, 5], within the non-linear and linearized theory of elasticity the components P_r, P_θ and P_z of the force vector P= $P_r e_r$+$P_\theta e_\theta$+$P_z e_z$, where e_r, e_θ and $e_z$are the unit ort vectors in the selected cylindrical system of coordinates on the surfaces $r = const$ are expressed through the components of the non-symmetric Kirchhoff stress tensor by the expressions $P_r^{(n)} = t_{rr}^{(n)}n_r + t_{\theta r}^{(n)}n_\theta + t_{zr}^{(n)}n_z$, $P_\theta^{(n)} = t_{r\theta}^{(n)}n_r +$

$t_{\theta\theta}^{(n)}n_\theta + t_{z\theta}^{(n)}n_z$and $P_z^{(n)} = t_{rz}^{(n)}n_r + t_{\theta z}^{(n)}n_\theta + t_{zz}^{(n)}n_z$, where n_r, n_θ and n_z are the components of the unite external normal vector to the surface $r = const$. Consequently, for the surfaces $r = const$it is obtained that $n_r = 1$, $n_\theta = 0$ and $n_z = 0$, and, according to which, $P_r^{(n)} = t_{rr}^{(n)}$, $P_\theta^{(n)} = t_{r\theta}^{(n)}$ and $P_z^{(n)} = t_{rz}^{(n)}$. As, in the case under consideration, according to the foregoing discussions on the non-zero components of the non-symmetric Kirchhoff stress tensor, the conditions $P_r^{(n)} = 0$and $P_z^{(n)} = 0$satisfy automatically and therefore the boundary and contact conditions with respect to the forces are satisfied only through the $P_\theta^{(n)} = t_{r\theta}^{(n)} = 0$ in (2.7). Note that the first lower index in the components of the non-symmetric Kirchhoff stress tensor shows the direction of the normal vector to the area on which acts this component, however, the second lower index the action direction of this component. Consequently, the conditions in (2.7) are based on the foregoing discussions.

This completes formulation of the problem.

3 Method of solution

Using the expressions in (2.5) and in (2.6) we can rewrite the equation of motion (2.4) in the $u_\theta^{(n)}(r,z,t)$displacement term as follows:

$$\mu^{(n)}\left(\frac{\partial^2 u_\theta^{(n)}}{\partial r^2} - \frac{u_\theta^{(n)}}{r^2} + \frac{1}{r}\frac{\partial u_\theta^{(n)}}{\partial r} + \frac{\partial^2 u_\theta^{(n)}}{\partial z^2}\right) + \frac{\partial}{\partial r}\left(\sigma_{rr}^{(n)0}\frac{\partial u_\theta^{(n)}}{\partial r}\right) + \frac{1}{r}\sigma_{rr}^{(n)0}\frac{\partial u_\theta^{(n)}}{\partial r} -$$

$$-\frac{1}{r}\sigma_{\theta\theta}^{(n)0}\frac{u_\theta^{(n)}}{r} + \sigma_{zz}^{(n)0}\frac{\partial^2 u_\theta^{(n)}}{\partial z^2} = \rho^{(n)}\frac{\partial^2 u_\theta^{(n)}}{\partial t^2}. \quad (3.1)$$

Moreover, using the presentation

$$u_\theta^{(n)} = U^{(n)}(r)\cos(kz - \omega t) \quad (3.2)$$

we obtain the following equation for the unknown function $U^{(n)}(r)$ from equation (3.1) and (3.2).

$$\mu^{(n)}\left(\frac{d^2U^{(n)}}{dr^2} - \frac{U^{(n)}}{r^2} + \frac{1}{r}\frac{dU^{(n)}}{dr} - k^2U^{(n)}\right) + \frac{d}{dr}\left(\sigma_{rr}^{(n)0}(r)\frac{dU^{(n)}}{dr}\right) +$$

$$+\frac{1}{r}\sigma_{rr}^{(n)0}(r)\frac{dU^{(n)}}{dr} - \frac{1}{r}\sigma_{\theta\theta}^{(n)0}(r)\frac{U^{(n)}}{r} - k^2\sigma_{zz}^{(n)0}(r)U^{(n)} = -\omega^2\rho^{(n)}U^{(n)}. \quad (3.3)$$

It is evident that to find the analytical solution to the equation (3.3) it is very difficult, therefore, according to [1 – 3], for solution to this equation we employ the discrete-analytical solution method. The employing this method is based on reducing the differential equation (3.3) with variable coefficients to the series corresponding equations with constant coefficients. As the describing of the mentioned reducing is detailed in the works [1 – 3], therefore, here we note briefly some principal moments of this method.

First of all, the regions $[R, R+h]$ occupied by the hollow cylinder and $[R+h, \infty]$ occupied by the surrounding medium are divided into a certain number the corresponding sub-regions. These sub-regions for the region $[R, R+h]$ are $[R + (n_2 - 1)h/N_2)$,$(R + n_2h/N_2)]$, where $n_2 = 1, 2, ..., N_2$; and for the region $[R+h, \infty]$ are $[R + h + (R_M - R - h)(n_1 - 1)/N_1$,$R + h + (R_M - R - h)n_1/N_1]$and infinite sub-regions $[R_M, \infty]$, where $n_1 = 1, 2, ..., N_1$ and the numbers N_2 and N_1, and the value for R_M are determined in the solution procedure from the convergence requirement of the numerical results. In each sub-regions the inhomogeneous initial stresses determined through the expressions (2.1) and (2.2) are taken as constants, the values of which are defined by the following relations:

In the $n_2 - th$ sub-region occupied by the hollow cylinder

$$\sigma_{rr}^{(2)0}(r) \approx \sigma_{rr}^{(2)0}(r_{n\,2}), \sigma_{\theta\theta}^{(2)0}(r) \approx \sigma_{\theta\theta}^{(2)0}(r_{n\,2}),$$

$$\sigma_{zz}^{(2)0}(r) \approx \sigma_{zz}^{(2)0}(r_{n\,2}), r_{n_2} = R + (n_2 - 1)h/N_2 + h/(2N_2). \tag{3.4}$$

In the $n_1 - th$ finite sub-region occupied by the surrounding medium

$$\sigma_{rr}^{(1)0}(r) \approx \sigma_{rr}^{(1)0}(r_{n\,1}), \sigma_{\theta\theta}^{(1)0}(r) \approx \sigma_{\theta\theta}^{(1)0}(r_{n\,1}), \sigma_{zz}^{(1)0}(r) \approx \sigma_{zz}^{(1)0}(r_{n\,1}),$$

$$r_{n\,1} = R + h + (R_M - R - h)(n_1 - 1)/N_1 + (R_M - R - h)/(2N_1), \tag{3.5}$$

and in the infinite sub-region $[R_M, \infty]$ also occupied by the surrounding medium

$$\sigma_{rr}^{(1)0}(r) \approx \sigma_{rr}^{(1)0}(R_M), \sigma_{\theta\theta}^{(1)0}(r) \approx \sigma_{\theta\theta}^{(1)0}(R_M), \sigma_{zz}^{(1)0}(r) \approx \sigma_{zz}^{(1)0}(R_M). \tag{3.6}$$

After foregoing procedures, within each sub-region it is obtained corresponding equation of motion and other related relations. In other words, the equation (3.3) is replaced with the following one.

$$\frac{d^2 U_\theta^{(i)n_i}}{d(kr)^2} + \frac{1}{kr}\frac{dU_\theta^{(i)n_i}}{d(kr)} - \frac{\alpha^{(i)n_i}}{(kr)^2} U_\theta^{(i)n_i} +$$

$$+ \left(\frac{c^2}{(c_2^{(i)})^2} \left(1 + \frac{\sigma_{rr}^{(i)0}(r_{n_i})}{\mu^{(i)}} \right)^{-1} - \beta^{(i)n_i} \right) U_\theta^{(i)n_i} = 0, \tag{3.7}$$

where

$$\alpha^{(i)n_i} = \left(1 + \frac{\sigma_{\theta\theta}^{(i)0}(r_{n_i})}{\mu^{(i)}} \right) \left(1 + \frac{\sigma_{rr}^{(i)0}(r_{n_i})}{\mu^{(i)}} \right)^{-1},$$

$$\beta^{(i)n_i} = \left(1 + \frac{\sigma_{zz}^{(i)0}(r_{n_i})}{\mu^{(i)}} \right) \left(1 + \frac{\sigma_{rr}^{(i)0}(r_{n_i})}{\mu^{(i)}} \right)^{-1}. \tag{3.8}$$

Using the notation

$$r_{1n_i} = kr \sqrt{ \frac{c^2}{(c_2^{(i)})^2} \left(1 + \frac{\sigma_{rr}^{(i)0}(r_{n_i})}{\mu^{(i)}} \right)^{-1} - \beta^{(i)n_i} } \tag{3.9}$$

we obtain

$$\frac{d^2 U_\theta^{(i)n_i}}{dr_{1n_i}^2} + \frac{1}{r_{1n_i}} \frac{dU_\theta^{(i)n_i}}{dr_{1n_i}} + \left(1 - \frac{\alpha^{(i)n_i}}{r_{1n_i}^2} \right) U_\theta^{(i)n_i} = 0. \tag{3.10}$$

from equation (3.7), the solution to which for the sub-regions $[R + (n_2 - 1)h/N_2), (R + n_2 h/N_2)]$ and $[R + h + (R_M - R - h)(n_1 - 1)/N_1, R + h + (R_M - R - h)n_1/N_1]$ are found as follows

$$U_\theta^{(i)n_i} = \begin{cases} A_1^{(i)n_i} J_{\gamma^{(i)n_i}}(r_{1n_i}) + A_2^{(i)n_i} Y_{\gamma^{(i)n_i}}(r_{1n_i}), \text{ if } r_{1n_i}^2 > 0, \\ A_1^{(i)n_i} I_{\gamma^{(i)n_i}}(r_{1n_i}) + A_2^{(i)n_i} K_{\gamma^{(i)n_i}}(r_{1n_i}), \text{ if } r_{1n_i}^2 < 0. \end{cases} \tag{3.11}$$

where $J_\delta(x)$ and $I_\delta(x)$ ($Y_\delta(x)$ and $K_\delta(x)$) are the Bessel and modified Bessel function of the first (second) order and

$$\gamma^{(i)n_i} = \sqrt{\alpha^{(i)n_i}} \tag{3.12}$$

However, the solution to the equation (3.10) for the region $[R_M, \infty]$, according to the conditions in (2.8), is

$$U_\theta^{(1)\infty} = \begin{cases} A_2^{(1)\infty} Y_{\gamma^{(1)\infty}}(r_{1\infty}), \ \text{if } r_{1\infty}^2 > 0, \\ A_2^{(1)\infty} K_{\gamma^{(1)\infty}}(r_{1\infty}), \ \text{if } r_{1\infty}^2 < 0. \end{cases} \tag{3.13}$$

where

$$r_1^{(1)\infty} = kr \times \sqrt{\frac{c^2}{(c_2^{(1)})^2(1+\sigma_{rr}^{(1)0}(R_M)/\mu^{(1)})} - (\beta^{(1)\infty}(R_M))^2}, \gamma^{(1)\infty}(R_M) =$$

$$= \frac{1-\alpha^{(1)\infty}(R_M)}{2},$$

$$\alpha^{(1)\infty}(R_M) = \frac{1+\sigma_{\theta\theta}^{(1)0}(R_M)/\mu^{(1)}}{1+\sigma_{rr}^{(1)0}(R_M)/\mu^{(1)}}, \beta^{(1)\infty}(R_M) = \frac{1+\sigma_{zz}^{(1)0}(R_M)/\mu^{(1)}}{1+\sigma_{rr}^{(1)0}(R_M)/\mu^{(1)}}. \tag{3.14}$$

The solutions in (3.11) and in (3.14) contain the unknown constants $A_1^{(i)n_i}$, $A_2^{(i)n_i}$ ($i = 1, 2$, $n_1 = 1, 2, ..., N_1$, $n_2 = 1, 2, ..., N_2$) and $A_2^{(2.1)\infty}$. For determination these constants it is used the conditions in (2.7) and the contact conditions written between each pare of the neighboring sub-regions. We do not write here these conditions which can be easily formulated from the contact conditions in (2.7) and it is also can be used the conditions formulated in [3]. Thus, using the mentioned conditions we obtain the system of homogeneous linear equations with respect to the aforementioned unknown constants. According to the well-known procedure, equating to zero determinant of the coefficient matrix of this system we obtain the dispersion equation, which can be presented as

$$\det(a_{nm}(c, kR, p/\mu^{(1)}, \mu^{(1)}/\mu^{(2)}, h^{(1)}/R, h^{(2)}/R)) = 0,$$

$$n; m = 1, 2, ..., 2(N_1 + N_2) + 1. \tag{3.15}$$

We do not give here the explicit expressions of the components of the matrix $(a_{nm})$and these expressions can be easily obtained from the foregoing related expressions.

Thus, solving the equation (3.15) numerically we obtain the dispersion curves, i.e. the graphs of the function $c = c(kR)$ and according to these graphs, determine the influence of the problem parameters, especially, the influence of the initial inhomogeneous initial stresses, the magnitude of which will be characterized through the ratio$p/\mu^{(2.1)}$, on these curves.

This completes the consideration of the solution procedure of the problem under consideration.

4 Numerical results and discussions

Consider numerical results which are obtained from the numerical solution of the dispersion equation (3.15) for various values of the ratios $E^{(2.1)}/E^{(2.2)}$ and h/R under $\nu^{(2.1)} = \nu^{(2.2)} = 0.3$ and $\rho^{(2.1)}/\rho^{(2.2)} = 1$ where $E^{(2.1)}(E^{(2.2)})$,$\nu^{(2.1)}(\nu^{(2)})$ and $\rho^{(2.1)}(\rho^{(2)})$are modulus of elasticity, Poisson's ratio and density, respectively, of the surrounding medium (cylinder material), Under obtaining the numerical results it is assumed that $N_1 = N_2 = 15$ and $R_M = 3(R + h)$which are selected from the convergence requirement of the numerical results and this convergence is satisfied with the accuracy 10^{-7} between the results obtained for $N_1 = N_2 = 10$ and for $N_1 = N_2 = 15$. Moreover, under obtaining numerical results the influence of the initial inhomogeneous initial stresses on the dispersion curves is estimated through the parameter $p/\mu^{(2.1)}$ where $\mu^{(2.1)}$ is the shear modulus of the surrounding medium and p is the intensity of the external forces acting at infinity and caused the initial inhomogeneous initial stresses in the system under consideration.

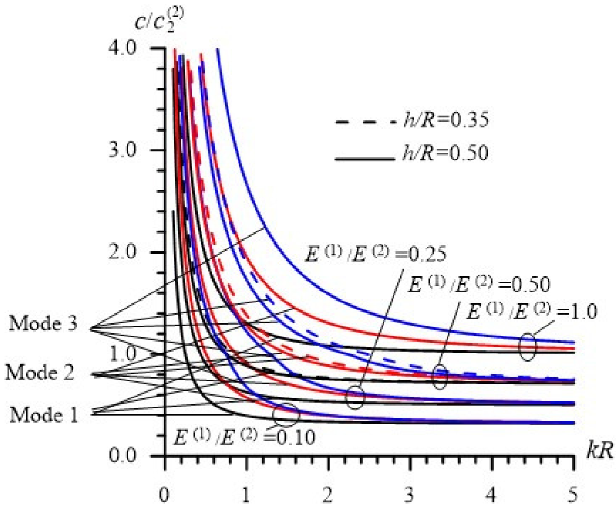

Fig. 2. Dispersion curves obtained in the case where the initial stresses in the system under consideration are absent.

First, we consider the dispersion curves which are obtained from the solution of the dispersion equation (3.15) in the case where the initial inhomogeneous initial stresses are absent in the system under consideration, i.e. in the case where $p/\mu^{(2.1)} = 0$. These curves are given in Fig. 2 for the first three modes and are obtained for various values of the ratios $E^{(2.1)}/E^{(2.2)}$ and h/R. It is also considered the case where the materials of the cylinder and surrounding medium are the same, i.e. the case where $E^{(2.1)}/E^{(2.2)} = 1.0$. Note that these graphs show the dependence between the ratio $c/c_2^{(2.2)}$ and kR where c is the torsional wave propagation velocity, $c_2^{(2.2)}$ is the shear wave propagation velocity in the cylinder material and kR is the dimensionless wave number. It follows from the graphs that a decrease in the values of the ratio $E^{(2.1)}/E^{(2.2)}$ causes to decrease in the values of the cut off frequency. Moreover, these graphs show that a decrease in the values of the ratio causes to decrease in the values of the ratio $c/c_2^{(2.2)}$. We recall that in Fig. 2 it is considered the cases $h/R = 0.35$ and $h/R = 0.50$only for the ratio $E^{(2.1)}/E^{(2.2)} = 0.5$ and from which follows that an increase in the h/R causes to decrease in the values of the ratio $c/c_2^{(2.2)}$.

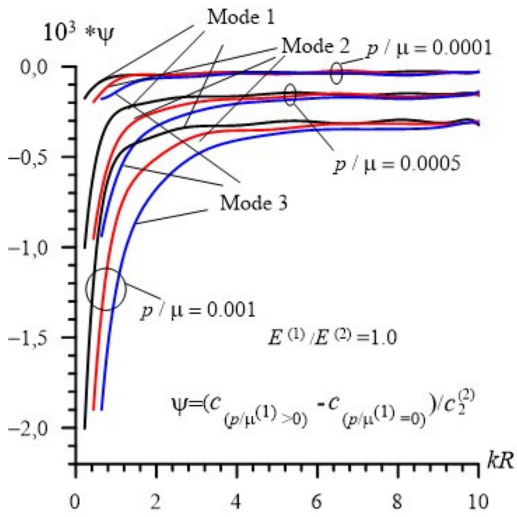

Fig. 3. The influence of the initial inhomogeneous initial stresses on the torsional wave propagation velocity in the case where $E^{(2.1)}/E^{(2.2)} = 1.0$

The results illustrated in Fig. 2 in the principal sense do not new ones and the corresponding analyses about these results are described in the monograph [1]. Moreover, in the paper [4] it was considered the case where the hollow cylinder and surrounding medium has homogeneous initial stresses $\sigma_{zz}^{(2.2)0} = Const_2$ and $\sigma_{zz}^{(2.1)0} = Const_1$. Consequently, the novelty of the results obtained in the present paper consists on the study the influence of the inhomogeneous initial stresses determined through the expressions (2.1) and (2.2) on the torsional wave propagation velocity on the dispersion curves considered in Fig. 2. For estimation the mentioned influence we introduce the parameter

$$\Psi= \left(c_{p/\mu^{(1)}>0} - \ c_{p/\mu^{(1)}=0}\right) \Big/ c_2^{(2)} \tag{4.1}$$

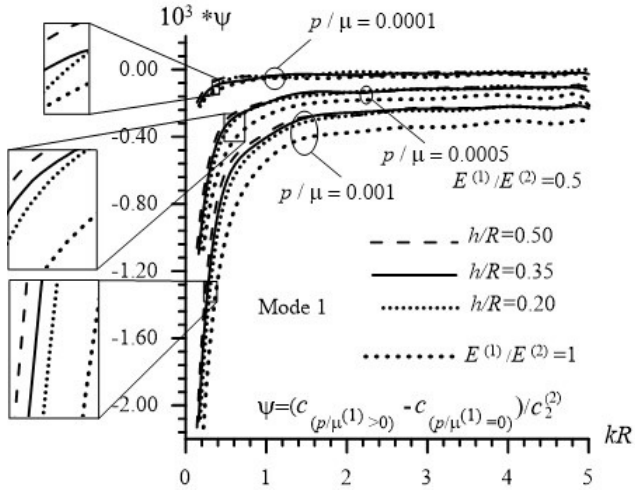

Fig.4. The influence of the ratio h/R on the dependence between Ψ(4.1) and kR in the first mode

and consider the influence of the problem parameters, especially, of the parameter $p/\mu^{(2.1)}$ on the values of the Ψcalculated for the various values of the dimensionless wavenumber kR.

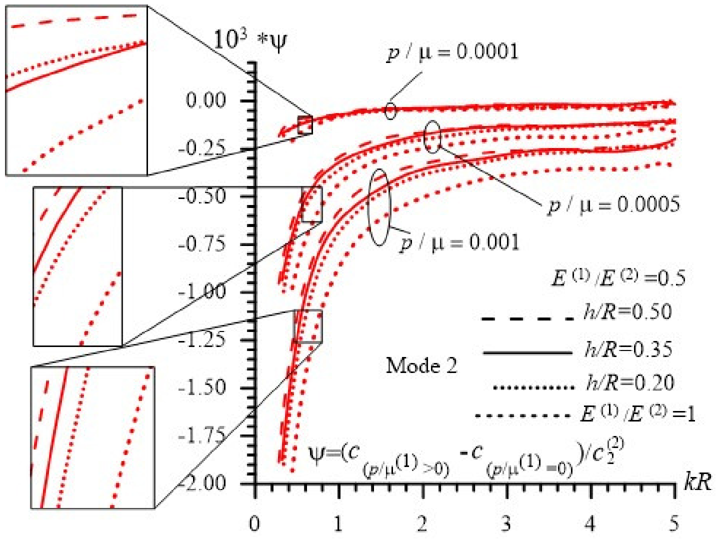

Fig. 5. The influence of the ratio h/R on the dependence between $\Psi(4.1)$ and kR in the second mode

Thus, we begin the analyses of the numerical results with the case where the materials the hollow cylinder and surrounding medium are the same, i.e. with the case where $E^{(2.1)}/E^{(2.2)} = 1.0$. These results which are obtained for the first three modes under various values of $p/\mu^{(2.1)}$ are given in Fig. 3. It follows from Fig. 3 that the considered type inhomogeneous initial stresses cause to decrease of the torsional wave propagation velocity and the main quantity of this influence appears under relatively low-wave number values, i.e. under $kR \leq (kR)*$ and the values of the $(kR)*$ increase with $p/\mu^{(2.1)}$.

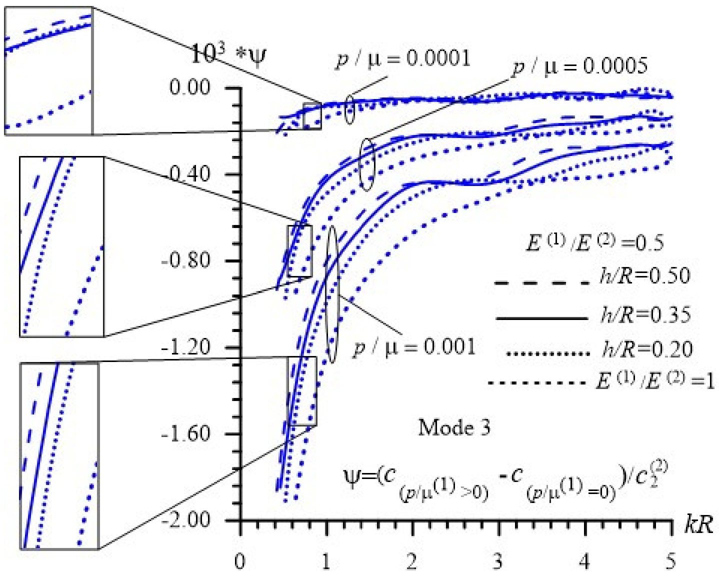

Fig. 6. The influence of the ratio h/R on the dependence between $\Psi(4.1)$ and kR in the third mode

Moreover, an increase in the values of the ratio $p/\mu^{(2.1)}$causes to decrease the torsional wave propagation velocity in the system under consideration. Besides all of these, the results given in Fig. 3, show that the magnitude of the influence of the initial stresses on the torsional wave propagation velocity increase with the number of the mode.

Now we consider numerical results related to the cases where $E^{(2.1)}/E^{(2.2)} < 1.0$ and investigate the influence of the ratio h/R on the graphs of the dependence between Ψ and kR constructed for various values of this ratio h/R under $E^{(2.1)}/E^{(2.2)} = 0.5$. The mentioned graphs are illustrated in Figs. 4, 5 and 6 for the first, second and third modes respectively. In these figures it is also given the corresponding graphs related to the case where $E^{(2.1)}/E^{(2.2)} = 1.0$. First of all, the analyses of the result given in these figures allow us to conclude that in the principal sense all the conclusions made above for the results obtained in the case where $E^{(2.1)}/E^{(2.2)} = 1.0$ hold also for the case where $E^{(2.1)}/E^{(2.2)} = 0.5$. Moreover, the comparison of the results given in these figures and obtained for various values of the h/R shows that an increase in the values of the h/R causes to decrease the influence of the initial stresses on the torsional wave propagation velocity. This fact can be explained with decreasing of the stress concentration around the hollow cylinder with the h/R, i.e. with decreasing of the initial stresses around the hollow cylinder with the h/R. According to this view of point, it must be observed the same effect with decreasing in the values of the ration $E^{(2.1)}/E^{(2.2)}$, a decrease in the values of the ration $E^{(2.1)}/E^{(2.2)}$ must cause to decrease the influence of the initial stresses on the wave propagation velocity. This prediction is proven with the numerical results illustrated in Fig. 7 which are obtained for various values of $E^{(2.1)}/E^{(2.2)}$under $p/\mu^{(2.1)} = 0.001$ and $h/R = 0.5$.

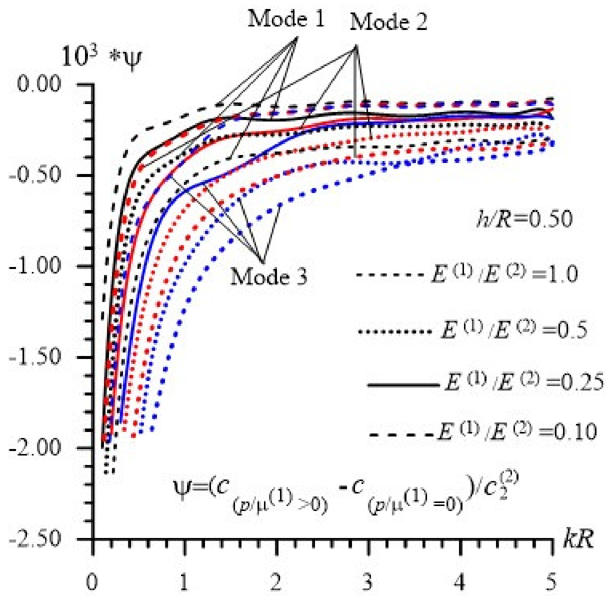

Fig. 7. The influence of the ratio $E^{(2.1)}/E^{(2.2)}$ on the dependence between $\Psi(4.1)$ and kR under $p/\mu^{(2.1)} = 0.001$ and $h/R = 0.5$

5 Conclusions

Thus, in the present paper within the framework of the piecewise homogeneous body model by utilizing the three-dimensional linearized theory of elastic waves in initially stressed bodies it has been studied the influence of the inhomogeneous initial stresses in the system consisting of the hollow cylinder and surrounding elastic medium on the torsional wave velocities propagated in this system. It is assumed that the initial inhomogeneous initial stresses are caused by uniformly distributed compressional radial normal forces acting at infinity.

The corresponding eigenvalue problem is solved by employing the discrete-analytical solution method and it is obtained corresponding dispersion equation which is solved numerically. As a result of this solution, it is constructed dispersion curves, according to which, the corresponding conclusions on the influence of the inhomogeneous initial stresses on the torsional wave propagation velocity are made. Some of these conclusions can be formulated as follows:

- the existence of the considered type inhomogeneous initial stresses causes to decrease of the torsional wave propagation velocity;

-an increase in the thickness of the hollow cylinder in the cases where the modulus of elasticity of the cylinder material is greater than that of the surrounding medium causes to decrease the influence of the initial stresses on the wave propagation velocity;

- the growing of the modulus of elasticity of the hollow cylinder also causes to decrease the magnitude of the influence of the initial stresses on the torsional wave velocity propagated in the system under consideration.

References

1. Akbarov, S.D.: *Dynamics of pre-strained bi-material elastic systems: linearized three-dimensional approach.* Springer, New York, USA (2015).
https://doi. 10.1007/978–3–319– 4460–3.
2. Akbarov, S.D. and Bagirov, E.T.: Axisymmetric longitudinal wave dispersion in a bi-layered circular cylinder with inhomogeneous initial stresses, *J. Sound Vib.* **450**, 1 - 27,(2019).
https://doi.org/10.1016/j.jsv.2019.03.003
3. Akbarov, S. D., Bagirov E. T.: Dispersion of axisymmetric longitudinal waves in a “hollow cylinder +surrounding medium” system with inhomogeneous initial stresses. *Structural Engineering and Mechanics*, 72(5),(2019), 597-615.
https://doi.org/10.12989/sem.2019.72.5.597.
4. Ozturk A. and Akbarov S.D.: Torsional wave propagation in a pre-stressed circular cylinder embedded in a pre-stressed elastic medium. Appl Math Model. 33: 3636 – 3649,(2009).
5. Guz, A.N. Elastic waves in bodies with initial (residual) stresses, A.C.K. Kiev (2004) (in Russian).