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STRENGTHENED OPTIMALITY CONDITION OF THE FIRST TYPE IN DISCRETE SYSTEMS OF CONTROL

Abstract

Using specific of a discrete system and new property of optimal control we obtain strengthened optimality condition of linearized type. Also the results illustrating the rich content of the obtained results are cited.

Introduction. Beginning with the paper [1], for discrete systems the necessary optimality conditions are being searched in the form similar to the maximum principle for continuous systems [2]. In the paper [3] the analogue of the maximum principle is proved for a linear (with respect to space variable) discrete control problem with linear quality criterion. Unifortunately, direct extension of L.S. Pontryagin's maximum principle to discrete systems in the general case is impossible [4]. After the paper [4] appeared, the researchers tried to prove the maximum principle in the weakened form (local maximum, stationary state) and to get higher order optimality conditions for various discrete optimization problems. Taking into account the abovementioned and analyzing the results of the papers [4-29 and others], we can say that theory of necessary optimality conditions in discrete systems remains an actual problem even today.

As it is noted in [5,6,22], unlike the continuous one, in the discrete case the linearized maximum principle in the general case is not the corollary of the discrete maximum principle. In this connection we can say that the linearized maximum principle has an independent value. Therefore, there arises theoretical and practical interest for obtaining a new optimality condition of linearized type with the large scope properties being constructively used and more strengthened. The present paper is devoted to the investigation of optimality of controls in such a statement. Here, using the method from [24] taking into account the specific character of the discrete system and the detected new properties of optimal controls (see lemma) we get linearized type optimality conditions satisfying the above properties. In the conclusion, the results illustrating the rich content of the obtained results are cited.

1. Problem statement. It is required to minimize the functional

$$\mathcal{S}(u) = \Phi(x(t_1)) \tag{1}$$

on the trajectories of the discrete system

$$x(t+1) = f(x(t), u(t), t), \quad t \in T, \quad x(t_0) = x^*, \tag{2}$$

under the constraint

$$u(t) \in U(t) \subset E^r, \quad t \in T, \tag{3}$$

where $x = (x_1, \ldots, x_n)'$ is a state vector ($'$ (the prime) is the transposition operation), $u = (u_1, \ldots, u_r)'$ is a control vector, t is time (discrete), x^* is the given vector, $T = \{t_0, t_0 + 1, \ldots, t_1 - 1\}$; E^r is the r dimensional Eulidean space; $f(x, u, t)$, $(x, u, t) \in E^n \times \times E^r \times [t_0, t_1]$ is a vector-function continuous in the union of variables together with partial derivatives with respect to x, u, and $\Phi(x)$, $x \in E^n$ is a continuously-differentiable function; $U(t)$, $t \in T \backslash \{t_1 - 1\}$ are the given convex sets, and $U(t_1 - 1)$ is the arbitrary set (not necessarily convex).

We call the controls satisfying condition (3) admissible. An admissible control $u(t)$, $t \in T$ minimizing the functional (1) under the constraint (2) is said to be an optimal control, and the corresponding trajectory $x(t)$, $t \in T \cup \{t_1\}$ of system (2) an optimal trajectory. Therewith the pair $(u(t), x(t))$ is called an optimal process.

2. Formula of increment in quality functional. Let $(u^0(t), x^0(t))$ be some process in problem (1)-(3). Introduce the sets [30]:

$$U[x^0(\cdot)](t) = \{u \in U(t) : \triangle_u f\left(x^0(t), u^0(t), t\right) =$$

$$= f\left(x^0(t), u, t\right) - f\left(x^0(t), u^0(t), t\right) = 0\}, t \in T. \tag{4}$$

Note that the sets $U[x^0(\cdot)](t)$, $t \in T$ are not empty and even if one set $U[x^0(\cdot)](\theta)$ consists of at least of two elements, then it permits to get extra information on the optimality of the control $u^0(t)$, $t \in T$ [23]. We also underline that the finding of the elements of the set $U[x^0(\cdot)](\theta)$, $\theta \in T$ in most cases is simple. For example, in problem (1)-(3) if $f(x(t), u(t), t) = g(x(t), t) + A(x(t), t)u(t)$, $t \in T$, then the finding of the elements of the set $U[x^0(\cdot)](\theta)$ is reduced to the solution of a linear algebraic system of equations.

Lemma. *If the control $u^0(t)$, $t \in T$ is optimal, then any control $\hat{u}(t) \in U[x^0(\cdot)](t)$, $t \in T$ is optimal, and the pair $(\hat{u}(t), x^0(t))$ is an optimal process.*

The proof of the lemma easily follows from the definition of the set (4).

Along with $u^0(t)$, $t \in T$ we consider another admissible control $u^*(t)$, $t \in T$ of the form

$$u^*(t) = \begin{cases} \tilde{u} + \varepsilon\alpha(\tilde{v} - \tilde{u}), t = \theta \in T \backslash \{t_1 - 1, t_1 - 2\}, \\ \hat{u} + \varepsilon\beta(\hat{v} - \hat{u}),\ t = \hat{\theta} \in \{\theta_1, \theta_2, \ldots\} \cap T \backslash \{t_1 - 1\}, \\ v, \qquad t = t_1 - 1, \\ u^0(t), \quad t \in T \backslash \left\{\theta, \widehat{\theta}, t_1 - 1\right\}, \end{cases} \tag{5}$$

where $\theta_i = \theta + i$, $i = 1, 2, \ldots$, $\tilde{u} \in U\left[x^0(\cdot)\right](\theta)$, $\hat{u} \in U\left[x^0(\cdot)\right]\left(\hat{\theta}\right)$, $\tilde{v} \in U(\theta)$, $\hat{v} \in U\left(\hat{\theta}\right)$, $v \in U(t_1 - 1)$, $\alpha \geq 0$, $\beta \geq 0$, $\max\{\alpha, \beta\} = \varepsilon_0 > 0$, $\varepsilon \in \left(0, \dfrac{1}{\varepsilon_0}\right]$.

Note that variation of the control $u^0(t)$, $t \in T$ in the form (5) is new and this is the basis of the scheme for investigation of the problem under consideration (1)-(3).

Denote by $\triangle^* x(t)$, $t \in T \cup \{t_1\}$ the increment of the trajectory $x^0(t)$, $t \in T \cup \{t_1\}$, responding to the special increment $\triangle^* u(t) = u^*(t) - u^0(t)$, $t \in T$ of the control

$u^0(t), t\in T$. It is clear that the increment $\triangle^* x(t)$, $t\in T\cup\{t_1\}$ is the solution of the system

$$\begin{cases} \triangle^* x(t+1) = f\left(x^0(t)+\triangle^* x(t), u^*(t), t\right) - f\left(x^0(t), u^0(t), t\right), \\ \triangle^* x(t) = 0, \quad t\in\{t_0,\dots,\theta\}. \end{cases} \tag{6}$$

Taking into account (5) and definition of the set $U\left[x^0(\cdot)\right](t), t\in T$, and using the Taylor formula from (6) by means of the steps method it is easy to show that

$$\|\triangle^* x(t)\| \le K_1\varepsilon, \quad t\in T, \tag{7}$$

where $K_1 = const > 0$, $\|\triangle^* x(t)\|$ is an Euclidean norm of the vector $\triangle^* x(t)$.

But the solution of the system (6) at the point $t=t_1$ is final with respect to $\varepsilon : \|\triangle^* x(t_1)\| \sim \varepsilon^0$. Distinguish the principal part of the increment $\triangle^* x(t_1)$. From (6) we have

$$\triangle^* x(t_1) = \triangle_v f\left(x^0(t_1-1), u^0(t_1-1), t_1-1\right) +$$
$$+ \triangle_{\tilde{x}} f\left(x^0(t_1-1), v, t_1-1\right), v\in U(t_1-1), \tag{8}$$

where

$$\triangle_{\tilde{x}} f\left(x^0(t_1-1), v, t_1-1\right) =$$
$$= f\left(x^0(t_1-1)+\triangle^* x(t_1-1), v, t_1-1\right) - f\left(x^0(t_1-1), v, t_1-1\right). \tag{9}$$

Taking into attention (7), (9), we get that the second term in (8) has order ε, i.e.

$$\left\|\triangle_{\tilde{x}} f\left(x^0(t_1-1), v, t_1-1\right)\right\| \le K_2\varepsilon, K_2 = const > 0. \tag{10}$$

Now derive the second order increment of the quality functional. The increment $\triangle^* S\left(u^0\right) = S(u^*(\cdot)) - S(u^0(\cdot))$ of the functional (1), caused by (5), allowing for (7)-(10) may be written using the Taylor formula as follows:

$$\triangle^* S\left(u^0\right) = \triangle_v \Phi(f(a^0\left(u^0, t_1-1\right))) + \triangle_1^* S\left(u^0\right) + o(\varepsilon), \tag{11}$$

where $\varepsilon^{-1} o(\varepsilon) \to 0$, for $\varepsilon\to 0$,

$$\triangle_v \Phi(f(a^0\left(u^0, t_1-1\right))) = \Phi(f(a^0(v, t_1-1))) - \Phi(f(a^0(u^0, t_1-1))), \tag{12}$$

$$a^0\left(u^0, t\right) = \left(x^0(t), u^0(t), t\right), a^0(v,t) = \left(x^0(t), v, t\right), t\in T, \tag{13}$$

$$\triangle_1^* S\left(u^0\right) = \Phi_x'(f(a^0(v, t_1-1)))\triangle_{\tilde{x}} f\left(x^0(t_1-1), v, t_1-1\right). \tag{14}$$

Consider the auxiliary-vector-function $\psi^0(t; u^0(t+1); v)$, $t\in T\backslash\{t_1-1\}$, as the solution of the:

$$\begin{cases} \psi^0\left(t-1; u^0(t); v\right) = f_x'\left(a^0\left(u^0, t\right)\right)\psi^0\left(t; u^0(t+1); v\right), \\ \psi^0\left(t_1-2; u^0(t_1-1); v\right) = f_x'\left(a^0(v, t_1-1)\right)\psi^0(t_1-1; v), \\ \psi^0(t_1-1; v) = -\Phi_x\left(f(a^0(v, t_1-1))\right); \end{cases} \tag{15}$$

where $a^0(\cdot)$ is determined from (13). According to (7), (9), (14), (15) for $\triangle_1^* S(u^0)$ by the Taylor formula it holds the representation

$$\triangle_1^* S\left(u^0\right) = -H_x'(\psi^0\left(t_1-1;v\right), a^0\left(v,t_1-1\right))\triangle^* x\left(t_1-1\right) + o(\varepsilon). \tag{16}$$

where $H(\psi,x,u,t)=\psi' f(x,u,t)$.

We substitute (16) in (11) and taking into account (15), have

$$\triangle^* S\left(u^0\right) = \triangle_v \Phi\left(f(a^0(u^0,t_1-1))\right) -$$

$$-\psi^{0\prime}\left(t_1-2;u^0(t_1-1);v\right)\triangle^* x\left(t_1-1\right) + o\left(\varepsilon\right). \tag{17}$$

Here and in sequel, $o(\varepsilon)$ means the total residual term.

At first calculate the second term in (17) for admissible control (5).

Using the identity

$$\psi^{0\prime}(t_1-2;u^0\left(t_1-1\right);v)\triangle^* x\left(t_1-1\right) =$$

$$= \sum_{t=\theta}^{t_1-2}\psi^{0\prime}\left(t;u^0\left(t+1\right);v\right)\triangle^* x\left(t+1\right) - \sum_{t=\theta_1}^{t_1-2}\psi^{0\prime}\left(t-1;u^0\left(t\right);v\right)\triangle^* x\left(t\right)$$

and taking into account (5)-(7), (15), by the Taylor formula we get

$$\psi^{0\prime}(t_1-2;u^0\left(t_1-1\right);v)\triangle^* x\left(t_1-1\right) =$$

$$= \varepsilon\left[\alpha H_u'\left(b^0\left(\tilde{u},\theta;v\right)\right)\left(\tilde{v}-\tilde{u}\right) + \beta H_u'(b^0(\hat{u},\hat{\theta};v))\left(\hat{v}-\hat{u}\right)\right] +$$

$$+\left[H_x'(b^0(\hat{u},\hat{\theta};v)) - H_x'(b^0(u^0,\hat{\theta};v))\right]\triangle^* x\left(\hat{\theta}\right) + o(\varepsilon) \tag{18}$$

where

$$b^0\left(u^0,t;v\right) = (\psi^0\left(t;u^0\left(t+1\right);v\right), x^0\left(t\right), u^0\left(t\right), t), \quad t\in T\backslash\{t_1-1\},$$

$$b^0\left(u,t;v\right) = (\psi^0(t;u^0\left(t+1\right);v), x^0\left(t\right), u, t), \quad t\in\left\{\theta,\hat{\theta}\right\}, \quad u\in\{\tilde{u},\hat{u}\}. \tag{19}$$

In what follows, solving system (6) by means of the steps method (sequentially with respect to $t: t=\theta_1,\theta_2,\ldots,\hat{\theta})$ and by the Taylor formula for $\triangle^* x\left(\hat{\theta}\right)$ it is not difficult to get the expansion of the form

$$\triangle^* x\left(\hat{\theta}\right) = \varepsilon\alpha Z(\hat{\theta};\theta,u^0) f_u\left(x^0\left(\theta\right),\tilde{u},\theta\right)\left(\tilde{v}-\tilde{u}\right) + o(\varepsilon), \tag{20}$$

where the matrix $Z(\hat{\theta};\theta,u^0)$ is the value of the solution $Z(t;\theta,u^0)$ of the following system at the point $\hat{\theta}$:

$$\left\{\begin{array}{l} Z\left(t+1;\theta,u^0\right) = f_x\left(x^0\left(t\right),u^0\left(t\right),t\right) Z\left(t;\theta,u^0\right), \quad t\in\{\theta_1,\theta_2,\ldots,t_1-3\}, \\ Z\left(\theta_1;\theta,u^0\right) = E, E - \text{is a unique } \ n\times n \ \text{ matrix.} \end{array}\right. \tag{21}$$

Thus, according to (12), (18), (20) from (17) for $\triangle^* S\left(u^0\right)$ we get a formula of the form:

$$\triangle^* S\left(u^0\right)=\triangle_v \Phi\left(f\left(x^0\left(t_1-1\right), u^0\left(t_1-1\right), t_1-1\right)\right)-$$

$$-\varepsilon[\alpha \psi^{0\prime}(\theta ; u^0\left(\theta_1\right) ; v) f_u\left(x^0(\theta), \tilde{u}, \theta\right)(\tilde{v}-\tilde{u})+\beta \psi^{0\prime}(\hat{\theta} ; u^0\left(\hat{\theta}+1\right) ; v) \times$$

$$\times f_u\left(x^0\left(\hat{\theta}\right), \hat{u}, \hat{\theta}\right)(\hat{v}-\hat{u})\Big]+\varepsilon \alpha\left[\psi^{0\prime}(\hat{\theta}-1 ; \hat{u} ; v)-\psi^{0\prime}(\hat{\theta}-1 ; u^0\left(\hat{\theta}\right) ; v)\right] \times$$

$$\times Z(\hat{\theta} ; \theta, u^0) f_u\left(x^0(\theta), \tilde{u}, \theta\right)(\tilde{v}-\tilde{u})+o(\varepsilon), \qquad (22)$$

where $\alpha \geq 0$, $\beta \geq 0$, $v \in U(t_1-1)$, $\hat{v} \in U(\hat{\theta})$, $\tilde{v} \in U(\theta)$,

$$\tilde{u} \in U[x^0(\cdot)](\theta), \quad \hat{u} \in U[x^0(\cdot)](\hat{\theta}), \quad \theta \in\{t_0, \ldots, t_1-3\};$$

$Z\left(\hat{\theta} ; \theta ; u^0\right)$ is determined from the system (21), $\psi^0\left(t ; u^0(t+1) ; v\right)$ is the solution of the system (15), $\psi^0\left(\hat{\theta}-1 ; \hat{u} ; v\right)$ are determined according to (15) as follows:

$$\psi^0\left(\hat{\theta}-1 ; \hat{u} ; v\right)=f_x^{\prime}\left(x^0\left(\hat{\theta}\right), \hat{u} ; \hat{\theta}\right) \psi^0\left(\hat{\theta} ; u^0\left(\hat{\theta}+1\right) ; v\right), \qquad (23)$$

3. Optimality conditions. Consider the sets:

$$U_0\left(t_1-1\right)=\left\{v \in U(t_1-1): \triangle_v \Phi\left(f\left(x_0\left(t_1-1\right), u^0\left(t_1-1\right), t_1-1\right)\right)=0\right\}. \qquad (24)$$

Obviously $U[x^0(\cdot)]\left(t_1-1\right) \subset U_0\left(t_1-1\right)$ and $u^0(t_1-1) \in U_0(t_1-1)$.

Theorem 1. *Let $\left(u^0(t), x^0(t)\right)$ be an optimal process in problem (1)-(3), $\psi^0\left(t ; u^0(t+1) ; v\right)$ and $Z\left(t ; \theta, u^0\right)$ be the solutions of the systems (15) and (21), respectively. Then the following inequalities are fulfilled*

$$\triangle_v \Phi\left(f\left(x_0\left(t_1-1\right), u^0\left(t_1-1\right), t_1-1\right)\right) \geq 0, \forall v \in U\left(t_1-1\right); \qquad (25)$$

$$\psi^{0\prime}\left(t_1-2 ; u^0\left(t_1-1\right) ; v\right) f_u\left(x^0\left(t_1-2\right), \hat{u}, t_1-2\right)(\hat{v}-\hat{u}) \leq 0, \qquad (26)$$

$$\forall v \in U_0\left(t_1-1\right), \forall \hat{v} \in U\left(t_1-2\right), \forall \hat{u} \in U\left[x^0(\cdot)\right]\left(t_1-2\right);$$

$$\{\psi^{0\prime}\left(\theta ; u^0\left(\theta_1\right) ; v\right)+[\psi^{0\prime}\left(\hat{\theta}-1 ; \hat{u} ; v\right)-$$

$$-\psi^{0\prime}\left(\hat{\theta}-1 ;\ u^0\left(\hat{\theta}\right) ; v\right)] Z(\hat{\theta} ; \theta ; u^0)\} f_u\left(x^0(\theta),\ \tilde{u}, \theta\right)(\tilde{v}-\tilde{u}) \leq 0, \qquad (27)$$

$$\forall \theta \in\left\{t_0, \ldots, t_1-3\right\}, \forall \hat{\theta} \in\left\{\theta_1, \ldots, t_1-2\right\}, \forall v \in U_0\left(t_1-1\right),$$

$$\forall \tilde{v} \in U(\theta), \forall \tilde{u} \in U\left[x^0(\cdot)\right](\theta), \forall \hat{u} \in U\left[x^0(\cdot)\right]\left(\hat{\theta}\right),$$

where $\psi^0(\hat{\theta}-1 ; \hat{u} ; v)$ are determined by (23), and $\triangle_v \Phi(f(\cdot))$ by (12), (13).

Proof. As along the optimal process $\left(u^0(t), x^0(t)\right)$ in the representation (22) the left-hand side is non-negative, then we directly get the proof of optimality condition (25). In what follows, let in (22) $\alpha=0$ and $\hat{\theta}=t_1-2$ then according to (20), (24) from (22) the validity of condition (26) follows. If in (22) $\beta=0$, then with regard to (24), from (22) we get the proof of optimality condition (27). The theorem is proved.

Cite a more effective (both in verification and calculational aspects) corollary of the theorem.

Corollary 1. *For optimality of the admissible control* $u^0(t), t \in T$ *it is necessary that inequalities* (25), (26) *and the following inequality be fulfilled*

$$\psi^{0\prime}\left(\theta_1; u^0(\theta_2); v\right) f_x(x^0(\theta_1), \hat{u}, \theta_1) f_u(x^0(\theta), \tilde{u}, \theta)(\tilde{v} - \tilde{u}) \le 0, \tag{28}$$

$$\forall \theta \in \{t_0, \ldots, t_1 - 3\}, \forall v \in U_0(t_1 - 1), \forall \tilde{v} \in U(\theta), \forall \tilde{u} \in U\left[x^0(\cdot)\right](\theta),$$
$$\forall \hat{u} \in U\left[x^0(\cdot)\right](\theta_1).$$

For proving Corollary 1, in (27) it suffices to take into account (15), (21), (23) and $\hat{\theta} = \theta_1$.

It should be noted 1) that if in addition to conjectures of point 1 the set $U(t_1-1)$ in convex, then necessary optimality conditions established in [5,23,29] follow from the Theorem. The theorem differs from the earlier known ones [5,29] with more complete account of information on specific character of system (2) and properties of the optimal control (see the lemma), and in this sense is the strengthening of the last ones (see [23] and also the example); 2) If in problem (1)-(3) the sets $U(t)$, $t \in T \backslash \{t_1 - 1\}$ are open, the assertion of the theorem in another form was established in [24].

Consider the illustrative examples indicating the rich content of the obtained results.

4. Examples. Consider an example indicating the rich content of the obtained results.

Example 1. $x_1(t+1) = u_1(t)$, $x_2(t) = x_1^{\frac{3}{2}}(t) - u_2(t)$, $x_3(t+1) = x_2(t) u_2(t)$, $x_i(0) = 0$, $i = 1,2,3$; $T = \{0,1,2\}$, $t_1 = 3$; $U(t) = U_1(t) \times U_2(t)$, $t \in T$, where $U_1(t) = [0,1]$, $t \in T$, $U_2(t) = [-1,1]$, $t \in T$, $\Phi(x_1, x_2, x_3) = x_2(x_3 + x_1)$, $S(u) = \Phi(x(3))$.

Study the control $u^0(t) = (0,0)'$, $t \in T$ for optimality. It is easy to calculate $x^0(t) \equiv (0,0,0)'$, $\Phi_x\left(x^0(3)\right) = (0,0,0)'$ (degenerated case [5,29, p.89]), $f\left(x^0(2), u^0(2), 2\right) = (0,0,0)'$, $f\left(x^0(2), v, 2\right) = (v_1, -v_2, 0)'$, where

$$(v_1, v_2) \in U(2) = [0,1] \times [-1,1]; \Phi\left(f\left(x^0(2), v, 2\right)\right) = -v_1 v_2,$$

$$\Phi\left(f\left(x^0(2), u^0(2), 2\right)\right) = 0, \Phi_x\left(f\left(x^0(2), v, 2\right)\right) = (-v_2, v_1, -v_2)',$$

$f_x\left(x^0(2), v, 2\right) = (q_{ij})$, where $q_{32} = v_2$, $q_{ij} = 0, (i,j) \neq (3,2)$.

Further, $U\left[x^0(\cdot)\right](t) = \left\{(0,0)'\right\} = u^0(t)$, $t \in T$; $f_u\left(x^0(t), u^0(t), t\right) = (p_{ij})$, $t \in \{0,1,2\}$, where $p_{11} = 1$, $p_{22} = -1$, $p_{ij} = 0$, $i \in \{1,2,3\}$, $j \in \{1,2\}$.

Note that the previously known optimality conditions, for instance, from [4, 5, 9, 22, 29] are ineffective for the given example. Use the optimality condition (25). It takes the form $-v_1 v_2 \ge 0$, for all $v_1 \in [0,1]$ and $v_2 \in [-1,1]$, that is impossible. Consequently, the control $u^0(t) = (0,0)', t \in T$ is not optimal by condition (25).

Now let's consider a new problem in which in Example 1 we accept the set $U(2)$ in the form: $U(2) = [0,1] \times [-1,0]$. Then condition (25) is fulfilled, and leaves the control $u^0(t) \equiv (0,0)'$ among the pretendes for an optimal one. Continuing investigations for optimality of the control $u^0(t) \equiv (0,0)'$, we use condition (26). Using the above calculations and taking into account (15), (24), we have:

$$\psi^{0'}(2;v) f_x\left(x^0(2), v, 2\right) f_u\left(x^0(1), u^0(1), 1\right)\left(\hat{v} - u^0(1)\right) = -v_2^2 \hat{v}_2 \leq 0,$$

for all $(0, v_2)' \in U_0(2) = \{(v_1, v_2)' : v_1 v_2 = 0, v_1 \in [0,1], v_2 \in [-1,0]\}$ and $\hat{v}_2 \in U_2(1) = [-1,1]$. This is impossible. Thus, the control $u^0(t) \equiv (0,0)'$ is not optimal now by optimality condition (26).

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