Besarion DOCHVIRI, Omar GLONTI,
Omar PURTUKHIA, Grigol SOKHADZE

RADON-NIKODYM DERIVATIVE OF SOLUTION OF NONLINEAR EQUATIONS WITH RANDOM RIGHT SIDE AND APPLICATIONS

Abstract

In Hilbert space H consider the equation

$$Ay + B(y) = \xi,$$

where A is an unbounded linear operator, B is a bounded smooth operator and ξ is a random element in H with smooth distribution measure μ_ξ Specifically we suppose that μ_ξ possesses a logarithmical derivative along the directions of vectors from the dense subspace $H_+ \subset H$.

We study the problem: when the distribution μ_y of the solution of the given equation y possesses a logarithmical derivative , and under what conditions this measure is equivalent with respect to a simpler measure. In the case of equivalence we calculate the Radon- Nikodym density. We cite examples when A is a differential operator.

Before going on to the main problem we cite a theorem on nonlinear transformation of smooth measure in Banach space from the paper [1], that we'll need in future.

Let B be a separable real Banach space; H be Hilbert space compactly imbedded on B and the imbedding $i : H \to B$ be the Hilbert-Schmidt operator. Therewith $i^* : B^* \to H^* \sim H$ and therefore we'll assume that $B^* \subset H \subset B$. Denote by $\langle \cdot, \cdot \rangle$ a coupling of elements from B and B^*. Let μ be a measure (a real- valued finite function of the sets) on a Borel σ -algebra $\mathfrak{B}$, and $z(x) : B \to B^*$ be a vector field in B.

It is said that (see [2]) μ possesses a logarith mical derivative along the vector field z of the form $\rho_\mu(z, x)$ if for any function $\varphi \in C_b^1(B)$ it is valid the equality (the integration by parts formula)

$$\int_B \left\langle \varphi'(x), z(x) \right\rangle \mu(dx) = \int_B \varphi(x) \rho_\mu(z, x) \mu(dx).$$

We'll denote by $\mathfrak{M}$ a set of measures possessing a logarithmic derivative along any constant directions $z(x) = h \in B^*$ of the form $\rho_\mu(z, x) = \langle \lambda(x), h \rangle$, where $\lambda(x) : B \to B$ is a continuous function. In particular, Gauss measures and also their smooth images belong to $\mathfrak{M}$.

Theorem 1. **[1]** *Let a nonlinear transformation $f : B \to B$ having the inverse of the form $f^{-1} : x \to y = x + F(x)$ where $F(x) : B \to B^*$ is differentiable, act on B. Then if the operator $I + tF'(x)$ is inversible for each $t \in [0, 1]$: then the* *measures* $\mu \in \mathfrak{M}$ and $\mu^f = \mu(f^{-1})$ *are equivalent and we can represent the Radon -Nikodym derivative in the form:*

$$\frac{d\mu^f}{d\mu}(x) = \left|\det\left(I + F'(x)\right)\right| \exp\left\langle \int_0^1 \lambda(x + tF(x)dt, F(x) \right\rangle \tag{1}$$

Remark. Consider the expression

$$\beta(t, F, x) = \langle \lambda(x + tF(x), F(x)\rangle + trF'(x).$$

As it is shown in [2] it has sense also when: $F : B \to H$, and therefore we can strengthen the theorem, and require this condition on the function F instead of $F(x) : B \to B^*$. Therewith (1) takes the form

$$\frac{d\mu^f}{d\mu}(x) = \left|\det\left(I + F'(x)\right)\right| \exp \int_0^1 \beta(t, F, x)dt$$

In a separable real Hilbert space H consider the equation

$$A\eta + g(\eta) = \xi, \tag{2}$$

for which the following conditions are fulfilled:

a) A is a linear unbounded operator with domain of definition $\mathbb{D}(A)$ densely imbedded in H Suppose that there exists a bounded inverse A^{-1} being the Hilbert Schimdt operator. In the domain $\mathbb{D}(A)$ introduce a scalar derivative by the formula $(x, y)_{\mathbb{D}} = (Ax, Ay)_H$. We get an equipped Hilbert space $X_+ \subset X \subset X_-$,where $X_+ = \mathbb{D}(A), X = H$;

b) g is a differentiable nonlinear mapping, and the operator $I + tA^{-1}g'(x)$ is inversible for all $t \in [0, 1]$;

c) the random element ξ in X_- has the distribution $\mu_\xi \in \mathfrak{M}$, i.e..

$$E(\varphi'(\xi), h)_H = E\varphi(\xi)(\lambda(\xi), h)_H, \varphi \in C_b^1(X_-).$$

In addition to (2) consider the linear equation

$$A\varsigma = \xi. \tag{3}$$

Let μ_η and μ_ς be measures corresponding to random elements η and ς .

Theorem 2. *Let conditions a) b) c) be fulfilled for equations (2) and (3) .Then* $\mu_\eta \sim \mu_\varsigma$ *and*

$$\frac{d\mu_\eta}{d\mu_\varsigma}(v) = \left|\det(I + A^{-1}g'(v))\right| \exp \int_0^1 \beta(t, A^{-1}, g, v)dt, \tag{4}$$

if $g'(v)$ *is a Hilbert-Schmidt operator, then (4) takes the form*

$$\frac{d\mu_\eta}{d\mu_\varsigma}(v) = \left|\det(I + A^{-1}g'(v))\right| \exp\left(\int_0^1 \lambda(Av + tg(v))dt, g(v)\right)_H.$$

Let Δ be an open, bounded domain in finite -dimensional Euclidean space R^n. Denote the boundary Δ by $\partial\Delta$ Everywhere we suppose that $\partial\Delta$ is a smooth surface. Under this we understand the following one: to each point $x \in \partial\Delta$ we can assign n-dimensional ball $\Gamma(x)$ centered at the point x such that the part $\partial\Delta$ contained in $\Gamma(x)$ admits the representation with respect to some system of coordinates $(t_1, .., t_n)$ with origin at the point x, of the form

$$t_n = \psi\left(t_1, .., t_n\right), \tag{5}$$

where the function ψ is determined in some domain, where it belongs to the class $C^1$and $\psi(x) = \frac{\partial\psi}{\partial x_i} = 0$, $i = 1, ...n$. Therewith at each point $x \in \partial\Delta$ there exists a definite tangential hyper plane T_x, given by the equation t_n. Say that the domain $\Delta \cup \partial\Delta$ belongs to the class $A^{(k)}$ if the function ψ contained in (5) belongs to the class C^k.

For the derivatives (ordinary or generalized) we apply the following denotation:

$$D_j = \frac{\partial}{\partial x_j},\ j = 1, ..., n,\ D^\alpha = D_1^{\alpha_1} ... D_n^{\alpha_n},\quad \alpha = (\alpha_1, ..., \alpha_n),\ |\alpha| = \alpha_1 + ... + \alpha_n.$$

The linear differential expression of order r is written as follows:

$$Lu = \sum_{|\alpha| \le r} a_\alpha(x) D^\alpha u,$$

where $a_\alpha(x)$ are real coefficients that are smooth. Under this we mean

$$a_\alpha(x) \in C^{|\alpha|}(\Delta \cup \partial\Delta).$$

Denote the conjugation to L by L^*.Thus,

$$L^* u = \sum_{|\alpha| \le r} (-1)^{|\alpha|} D^\alpha (a_\alpha(x) u) = \sum_{|\alpha| \le r} b_\alpha(x) D^\alpha u.$$

Denote by $\mathcal{L}_2(G)$ a space of real valued functions that are integrable together with own sguare with respect to Lebesgue measure and with a scalar product

$$(u, v)_{\mathcal{L}_2(G)} = \int_G u(x) v(x) dx, \qquad u, v \in \mathcal{L}$$

As usually $W_2^l(G)$ denotes a Sobolev space with a scalar product

$$(u, v)_{W_2^l(G)} = (u, v)_{\mathcal{L}_2(G)} + \sum_{|\alpha| = l} (D^\alpha u, D^\alpha v)_{\mathcal{L}_2(G)}.$$

In order to cover a more general situation we follow [3] and introduce the notion of boundary conditions. Denote the set of functions finite with respect to G and ∞ from $C^l (l = 0, 1, ..., \infty)$ by $C_0^l(G)$, the space $W_2^{0l}(G)$, $l = 0, 1, ...$is determined as a subspace of $W_2^l(G)$ obtained by the closure in $W_2^l(G)$ of the linear set $C_0^\infty \subset W_2^l(G)$.

It is known that $\overset{0}{W}{}_2^{l}(G)$ for $l \geq 1$ coincides with the totality of all functions $u(x) \in W_2^l(G)$ for which $(D^\alpha u)(x) = 0, (x \in \Gamma)$ for $|\alpha| \leq l-1$.

Any subspace from $W_2^l(G)$ containing $\overset{0}{W}{}_2^{l}(G)$ is called a subspace of functions satisfying definite boundary conditions and is denoted as $\overline{W}{}_2^l(\partial G)$.

Let's consider a triple of equipped Hilbert spaces

$$W_2^{2p} \subset W_2^p \subset \mathcal{L}_2(G), \tag{6}$$

where G is an open bounded domain of the class $A^{(1)}$. Let $\xi = \xi(x)$, $x \in G$ be a random field with probability 1 belonging to $\mathcal{L}_2(G)$ and let the distribution μ_ξ in $\mathcal{L}_2(G)$ possess a logarithmic derivative along W_2^{2p} of the form $\lambda(x) : \mathcal{L}_2(G) \to \mathcal{L}_2(G)$. Take a general differential expression

$$Lu = \sum_{|\alpha| \leq r} a_\alpha(x) D^\alpha u, \tag{7}$$

and suppose that for differential operators L and L^* the following energetic inequalities are fulfilled:

$$\|Lu\|_{\mathcal{L}_2(G)} \geq c\,\|u\|_{\mathcal{L}_2(G)}\,, \quad \|L^* v\|_{\mathcal{L}_2(G)} \geq c\,\|v\|_{\mathcal{L}_2(G)}\,, \tag{8}$$

where $c > 0,\ \ u, v \in C_0^\infty(G)$.

We understand the solvability of the boundary value problem

$$L\varsigma(x) = \xi(x), \quad \varsigma \in \overline{W}{}_2^\alpha(\partial G) \tag{9}$$

in the following sense: as is known (details in [3] ,subject to energetic inequalities (8) there exists a resolvable extension of L having a continuous inverse determined on all $\mathcal{L}_2(G)$. We'll again write the resolvable extension of L by L . Under the solution of the stated problem it is natural to understand $\varsigma = L^{-1}\xi$. In the similar sense we should also understand the solvability of the nonlinear boundary value problem

$$(L\eta)(x) + g(x, \eta(x)) = \xi(x), \quad \eta \in \overline{W}{}_2^\alpha(\partial G)$$

as the solvability of the equation

$$\eta(x) + L^{-1} g(x, \eta(x)) = L^{-1}\xi(x),$$

where $g(x, y)$ is a smooth function in $\mathcal{L}_2(G)$.

Consider the nonlinear boundary value problem

$$(L\eta)(x) + g(x, \eta(x)) = \xi(x), \quad \eta \in \overline{W}{}_2^\alpha(\partial G) \tag{10}$$

In (9) and (10) $\xi(x)$ is a random field satisfying the conditions:

$$\int_G E\xi^2(x)dx < \infty \tag{11}$$

and its distribution μ_ξ possesses a logarithmic derivative along the constant directions W_2^{2p}. This means that for amy smooth functional $\varphi \in C^{-1}(\mathcal{L}_2(G))$ we have

$$E(\varphi'(\xi), h)_{W_2^p(G)} = E\varphi(\xi)(\lambda(\xi), h)_{W_2^p(G)}, \qquad (12)$$

where $\lambda : \mathcal{L}_2(G) \to \mathcal{L}_2(G)$, $h \in W_2^{2p}$.

Let μ_ξ be the distribution in $\mathcal{L}_2(G)$ of problem (10) and μ_ς be the distribution in $\mathcal{L}_2(G)$ of problem (9). From theorem 2 it followers.

Theorem 3. *Let Δ be an open bounded domain of the class $A^{(1)}$ with the boundary $\partial\Delta$, L be a differential operator determined by equality (7) with smooth coefficients $a_\alpha(x) \in C^{|\alpha|}(\Delta \cup \partial\Delta)$, $\xi(x)$ be a random field satisfying conditions (11) and (12), $g(x,u)$ determined on $G \times \mathcal{L}_2(G)$ for each u possess derivatives generalized in Sobolev's sense and of order $2p$ and there exist an operator $F = \dfrac{\partial g}{\partial u}$ satisfying the relation $\|F\| < \gamma$ where $\gamma = \left\|L^{-1}\right\|^{-1}$.*

Then, if for any $u, v \in C_0^\infty(G)$. and for some $C > 0$ the energetic inequalities (8) are fulfilled, then $\mu_\eta \sim \mu_\varsigma$ and

$$\frac{d\mu_\eta}{d\mu_\varsigma}(u) = \left|\det(I + L^{-1}(u))\right| \exp\left\{ \int_0^1 \int_G \lambda\left(\sum_{|\alpha|\le p} a_\alpha(x) D^\alpha u + tg(x,u)\right) g(x,u) ds dt + \right.$$

$$\left. +(-1)^p \int_0^1 \int_G \lambda\left(\sum_{|\alpha|\le p} a_\alpha(x) D^\alpha u + tg(x,u)\right) \sum_{|\alpha|=p} D^{2\alpha} u g(x,u) dx dt \right\}, \qquad (13)$$

for $u \in W_2^p(G)$.

In the special case when $\xi(x)$ is a Gaussion random field, whose correlation operator in the scalar product of the space $W_2^p(G)$ is $\theta > 0$ we have

$$\frac{d\mu_\eta}{d\mu_\varsigma}(u) = \left|\det(I + L^{-1}F(u))\right| \exp\Big\{ -\frac{1}{\theta} \int_G \sum_{|\alpha|\le p} a_\alpha(x) D^\alpha u \cdot g(x,u) dx +$$

$$+(-1)^{p+1} \int_G \sum_{|\beta|\le p} a_\alpha(x) D^\alpha u \cdot D^{2\beta} g(x,u) dx(x,u) dx - \frac{1}{2\alpha} \int_G \sum_{|\alpha|=p} (D^\alpha u)^2 dx \Big\}.$$

Cite application of theorem 3 to theory of prediction and filtration of random fields. Let X be a Hilbert space, ξ be a random variable with values in X, $\Phi : X \to R$ be a measurable functional. Let E be some linear space with $\sigma-$ algebra of its subsets $\mathfrak{E}$, and $\mathbb{Q} = X \to E$ be some linear operator. The problem is in calculation of optimal meansquare estimation $\Phi^*(\xi)$ of the function Φ from the random variable ξ by observations of $\mathbb{Q}\xi$. It is known well that such an estimation is given by the equality

$$\Phi^*(\xi) = E\left\{ \frac{\Phi(\xi)}{\mathfrak{E}_{\mathbb{Q}}^{\xi}} \right\},$$

where $\mathfrak{E}_{\mathbb{Q}}^{\xi}$ is $\sigma-$ algebra generated by the random element $\mathbb{Q}\xi$.

Suppose that on the Borel σ -algebra of $\mathfrak{B}$ space X another random variable η is given such that the distributions μ_ξ and μ_η are equivalent $\mu_\xi \sim \mu_\eta$ and $\rho(x) = \dfrac{d\mu_\xi}{d\mu_\eta}(x)$.

Lemma. *It $\Phi(x)$ is a bounded μ_η -measurable function, then the following formula is valid*

$$\Phi^*(\xi) = \left.\frac{E\left\{\Phi(\eta)\rho(\eta)/\mathfrak{E}^\xi_{\mathbb{Q}}\right\}}{E\left\{\rho(\eta)/\mathfrak{E}^\xi_{\mathbb{Q}}\right\}}\right|_{\eta=\xi}. \qquad (14)$$

Proof. By definition of conditional mean, for any measurable bounded function h on E we have

$$E\Phi(\xi)h(\mathbb{Q}\xi) = E\left\{E\left[\Phi(\xi)/\mathfrak{E}^\xi_{\mathbb{Q}}\right]\right\}h(\mathbb{Q}\xi),$$

hence

$$E\Phi(\eta)\rho(\eta)h(\mathbb{Q}\eta) = E\left\{E\left[\Phi(\xi)/\mathfrak{E}^\xi_{\mathbb{Q}}\right]_{\xi=\eta}\right\}E\{\rho(\eta)/\mathfrak{E}^\eta_{\mathbb{Q}}h(\mathbb{Q}\eta),$$

but as

$$E\Phi(\eta)\rho(\eta)h(\mathbb{Q}\xi) = E\left\{E\left[\Phi(\eta)\rho(\eta)/\mathfrak{E}^\xi_{\mathbb{Q}}\right]\right\}h(\mathbb{Q}\eta),$$

then because of arbitrariness of $h(x)$ we get

$$E\left[\Phi(\xi)/\mathfrak{E}^\xi_{\mathbb{Q}}\right]_{\xi=\eta}E\left\{\rho(\eta)/\mathfrak{E}^\xi_{\mathbb{Q}}\right\} = E\left[\Phi(\eta)\rho(\eta)/\mathfrak{E}^\xi_{\mathbb{Q}}\right].$$

hence we get (14).

We can simplify formula (14) if η is a Gaussian variable in X and $\mathbb{Q}$ is a continuous linear mapping of the space X in X. For that we represent η in the form $\eta = \eta^* + \overline{\eta}$,where $\eta^* = E\{\eta/\mathfrak{E}^\eta_{\mathbb{Q}}\}$ is an optimal in the meansquare sense linear prediction of Gaussian random variable η by observations $\mathbb{Q}\eta$, while $\overline{\eta}$ is a Gaussian variable independent of $\mathfrak{E}^\eta_{\mathbb{Q}}$. Then from (14) we can write

$$\Phi^*(\xi) = \left.\frac{E\left\{\Phi(\eta)\rho(\eta)/\mathfrak{E}^\eta_{\mathbb{Q}}\right\}}{E\left\{\rho(\eta)/\mathfrak{E}^\eta_{\mathbb{Q}}\right\}}\right|_{\eta=\xi} = \left.\frac{E\left\{\Phi(\eta^*+\overline{\eta})\rho(\eta^*+\overline{\eta})/\mathfrak{E}^\xi_{\mathbb{Q}}\right\}}{E\left\{\rho(\eta^*+\overline{\eta})/\mathfrak{E}^\eta_{\mathbb{Q}}\right\}}\right|_{\eta=\xi} =$$

$$= \left.\frac{E\{\Phi(x+\overline{\eta})\rho(x+\overline{\eta})\}}{E\{\rho(x+\overline{\eta})\}}\right|_{x=\eta^*=E\{\eta/\mathfrak{E}^\eta_{\mathbb{Q}}\},\eta=\xi}, \qquad (15)$$

where (unconditional) mean value is taken with respect to $\overline{\eta}$ and is substitutied by turns $x = \eta^* = E\{\eta/\mathfrak{E}^\eta_{\mathbb{Q}}\}$ and $\eta = \xi$ (this last substitution is assumed to be a substitution of observation ξ).

Let the solution of problem (10)- $\eta(x)$ be observed in some subdomain $G_1 \subset G$. It is required to find an optimal in the meanquadratic sense estimation of the functional Φ from the solution of $\eta(x)$ at the point $x = x_0 \in G_2 = G - G_1$.

To this end , in addition to problem (10) we consider the linear problem (9)

$$L\varsigma(x) = \xi(x), \quad \varsigma \in \overline{W}^\alpha_2(\partial G).$$

By combining theorem 3 with the lemma we get

Theorem 4. *Let in open, bounded domain G of the class $A^{(1)}$ with the boundary ∂G we consider a partial equation with boundary conditions (10) in which the coefficients of the operator L are sufficiently smooth, $a_\alpha(x) \in C^{|\alpha|}(\Delta \cup \partial\Delta)$, $\xi(x)$ is a Gaussian random field whose correlation operator in the scalar product of the space $W_2^p(G)$ is $\theta I, \theta > 0; g(x,u)$ is a function determined on $G \times \mathcal{L}_2(G)$ and possessing for each x generalized in the Sobolev sense derivatives of order $2p$, the operators $F = \frac{\partial g}{\partial u}$ satisfy the relation $\|F\| < \gamma$, where $\gamma = \left\|L^{-1}\right\|^{-1}$. Then if for any $u, v \in C_0^\infty(G)$ and some $C > 0$ the energetic inequalities (8) are fulfilled, then optimal prediction $\Phi^*(\eta)(x_0)$ is given by the formula:*

$$\Phi^*(\eta)(x_0) = \left\{E\Phi(z(x_0) + \overline{v}(x_0)) \left|\det(I + L^{-1}F(z(x) + \overline{v}(x)))\right| \times\right.$$

$$\times \exp\left\{-\frac{1}{\theta}\int_G \sum_{|\alpha|\le p} a_\alpha(x) D^\alpha (z + \overline{v}(x) g(x, z(x) + \overline{v}(x))dx + \right.$$

$$+(-1)^{p+1}\int_G \sum_{|\beta|\le p} a_\alpha(x) D^\alpha (z(x) + \overline{v}(x)) \cdot D^{2\beta} g(x, z(x) + \overline{v}(x))dx -$$

$$\left. -\frac{1}{2\alpha}\int_G \sum_{|\alpha|=p} (D^\alpha(z(x) + \overline{v}(x)))^2 dx \right\} \cdot \left\{E \left|\det(I + L^{-1}F(z(x) + \overline{v}(x)))\right| \times\right.$$

$$\times \exp\left\{-\frac{1}{\theta}\int_G \sum_{|\alpha|\le p} a_\alpha(x) D^\alpha (z + \overline{v}(x) g(x, z(x) + \overline{v}(x))dx + \right.$$

$$+(-1)^{p+1}\int_G \sum_{|\beta|\le p} a_\alpha(x) D^\alpha (z(x) + \overline{v}(x)) \cdot D^{2\beta} g(x, z(x) + \overline{v}(x))dx -$$

$$\left.\left. -\frac{1}{2\alpha}\int_G \sum_{|\alpha|=p} (D^\alpha(z(x) + \overline{v}(x)))^2 dx \right\}^{-1} \right|_{\eta=\xi}.$$

References

[1]. Daletskii Yu. L., Sokhadze G. *Absolute Continuity of Smooth Measures.* Functional Analysis and Its Applications. 1988, vol. 22, No. 2/April, pp. 149-150.

[2]. Belopolskaya Ia. I., Daletskii Yu. L. *Stochastic Equations and Differential Geometry.* Springer. 1990. 280 p.

[3].Berezanskii Ju. M. *Expansions in Eigenfunctions of Selfadjoint Operators.* Translations of Mathematical Monographs, 1968, vol 17, 809 p.

[4].Daletskii Yu. L., Shatashvili A. D. *On Optimal Prediction of Random Variables Nonlinearly related to Gaussian.* Theory of Random Processes, 1975, Issue 3, pp. 30-33.

Besarion Dochviri, Omar Glonti, Omar Purtukhia,
I.Javakhishvili Tbilisi State University, Tbilisi, Georgia

Grigol Sokhadze
Applied Mathematical Institute of TSU, Tbilisi, Georgia
Tel.: (99412) 539-47-20 (off.).

Received February 26, 2014; Revised April 24, 2014.