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MATHEMATICS

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ON THE EXISTENCE AND UNIQUENESS OF THE GENERALIZED SOLUTION OF A BOUNDARY VALUE PROBLEM FOR SECOND ORDER OPERATOR- DIFFERENTIAL EQUATIONS

Abstract

In the paper, the sufficient conditions providing the existence and uniqueness of a boundary value problem on a finite interval for elliptic type, second order operator- differential equations with variable coefficients are obtained. These conditions were expressed by the properties of operator coefficients of the equations.

In a separable Hilbert space H consider the boundary value problem

$$P\left(d/dt\right)u\left(t\right)=-u''\left(t\right)+\rho(t)A^{2}u\left(t\right)+$$

$$+A_{1}\left(t\right)u'\left(t\right)+A_{2}\left(t\right)u'\left(t\right)=f\left(t\right),t\in\left(0,T\right) \tag{1}$$

$$u\left(0\right)=0,\quad u\left(T\right)=0, \tag{2}$$

where $f\left(t\right),u\left(t\right)$ are the functions determined almost everywhere in $\left(0,T\right)$ and the operator coefficients satisfy the conditions:

1) A is a positive-definite self –adjoint operator in H with domain of definition $D\left(A\right)$;

2) $\rho\left(t\right)$ is a measurable, bounded function in $\left(0,T\right)$ and $0<\alpha\leq\rho\left(t\right)\leq\beta<\infty$;

3) $A_{1}\left(t\right),A_{2}\left(t\right)$are linear operators for $t\in\left(0,T\right)$, and $A_{1}\left(t\right)$ has a strongly continuous derivative, the operators $B_{1}\left(t\right)=A_{1}\left(t\right)A^{-1}$, $C_{1}\left(t\right)=A^{-1}A_{1}'\left(t\right)A^{-1}$ and $B_{2}\left(t\right)=A^{-1}A_{2}\left(t\right)A^{-1}$ are bounded in H, moreover

$$\sup_{t\in(0,T)}\left\|B_{1}\left(t\right)\right\|+\sup_{t\in(0,T)}\left\|C_{1}\left(t\right)\right\|+\sup_{t\in(0,T)}\left\|B_{2}\left(t\right)\right\|\leq const.$$

Denote by $L_{2}\left(\left(0,T\right):H\right)$ a Hilbert space of functions $f\left(t\right)$ determined almost everywhere in $\left(0,T\right)$, with the values in H, quadratically integrable by Bochner, for which

$$\left\|f\right\|_{L_{2}((0,T):H)}=\left(\int_{0}^{T}\left\|f\left(t\right)\right\|^{2}dt\right)^{1/2}.$$

For $m=1,m=2$ determine the Hilbert spaces [1]

$$W_{2}^{m}\left(\left(0,T\right):H\right)=\left\{u\left(t\right):u^{(m)}\in L_{2}\left(\left(0,T\right):H\right),A^{m}u\in L_{2}\left(\left(0,T\right):H\right)\right\}$$

with the norm

$$\|u\|_{W_2^m((0,T):H)} = \left(\left\| u^{(m)} \right\|^2_{L_2((0,T):H)} + \|A^m u\|^2_{L_2((0,T):H)} \right)^{1/2}$$

In what follows denote by $D([0,T]:H_1)$ a linear set of infinitely differentiable in $[0,T]$ vector- functions with the values in H, where

$$H_1 = D(A), (x,y)_{H_1} = (Ax, Ay)$$

and

$$\overset{\circ}{D}([0,T]:H_1) = \{u : u \in D([0,T]:H_1), u(0) = u(T) = 0\}.$$

Note that $D([0,T]:H_1)$ and $\overset{\circ}{D}([0,T]:H_1)$ are everywhere dense in the spaces $W_2^1((0,T):H)$ and $\overset{\circ}{W}{}_2^1((0,T):H)$, respectively [1], where

$$\overset{\circ}{W}{}_2^1((0,T):H) = \left\{ u \in W_2^1((0,T):H), u(0) = u(T) = 0 \right\}.$$

Denote by

$$P_0(d/dt)u = -u''(t) + \rho(t) A^2 u(t), \; P_1(d/dt)u(t) =$$
$$= A_1(t) u'(t) + A^2(t) u(t).$$

At first prove the following

Lemma. *Let conditions 1) -3) be fulfilled. Then the bilinear function*

$$P(u,\psi) = (P_0(d/dt)u, \psi)_{L_2((0,T):H)} + (P_1(d/dt)u, \psi)_{L_2((0,T):H)}$$

determined for all functions $u, \psi \in \overset{\circ}{D}([0,T]:H_1)$ *continues by continuity on the space* $\overset{\circ}{W}{}_2^1((0,T):H) \oplus W_2^1((0,T):H)$ *to the bilinear functional*

$$\widetilde{P}(u,\psi) = \widetilde{P}_0(u,\psi) + \widetilde{P_1}(u,\psi),$$

where

$$\widetilde{P}_0(u,\psi) = (u', \psi')_{L_2((0,T):H)} + \left(\rho^{1/2} A u, \rho^{1/2} A \psi \right)_{L_2((0,T):H)},$$

$$\widetilde{P}_1(u,\psi) = -\left(A_1(t) u', \psi' \right)_{L_2((0,T):H)} - \left(A_1'(t) u, \psi \right)_{L_2((0,T):H)} +$$
$$+ (A_2(t) u, \psi)_{L_2((0,T):H)}$$

Proof. Let $u, \psi \in \overset{\circ}{D}([0,T]:H_1)$.Then integrating by parts, we get

$$(P_0(d/dt)u, \psi)_{L_2((0,T):H)} = -(u'', \psi)_{L_2((0,T):H)} +$$
$$+ \left(\rho(t) A^2 u, \psi \right)_{L_2((0,T):H)} =$$
$$= (u, \psi')_{L_2((0,T):H)} + \left(\rho^{1/2} A u, \rho^{1/2} A \psi \right)_{L_2((0,T):H)},$$
$$(P_1(d/dt)u, \psi)_{L_2((0,T):H)} = -\left(A_1'(t) u, \psi' \right)_{L_2((0,T):H)} -$$

$$-\left(A_1'(t)\,u,\psi\right)_{L_2((0,T):H)}+\left(A_2(t)\,u,\psi\right)_{L_2((0,T):H)}.$$

Then

$$\left(P_0(d/dt)\,u,\psi\right)_{L_2((0,T):H)}\le\left\|u'\right\|_{L_2((0,T):H)}\cdot\left\|\psi'\right\|_{L_2((0,T):H)}+$$
$$+\beta\left\|Au\right\|_{L_2((0,T):H)}\cdot\left\|A\psi\right\|_{L_2((0,T):H)}\le$$
$$\le const\cdot\left\|u\right\|_{W_2^1((0,T):H)}\cdot\left\|\psi\right\|_{W_2^1((0,T):H)}, \tag{3}$$

$$\left(P_1(d/dt)\,u,\psi\right)_{L_2((0,T):H)}\le\left|\left(B_1(t)\,Au,\psi\right)_{L_2((0,T):H)}\right|+$$
$$+\left|\left(C_1(t)\,Au,A\psi\right)_{L_2((0,T):H)}\right|+\left|\left(B_2(t)\,Au,A\psi\right)_{L_2((0,T):H)}\right|\le$$
$$\le\sup_{t\in(0,T)}\left\|B_1(t)\right\|\cdot\left\|Au\right\|_{L_2((0,T):H)}\cdot\left\|\psi'\right\|_{W_2^1((0,T):H)}+$$
$$+\sup_{t\in(0,T)}\left\|C_1(t)\right\|\cdot\left\|Au\right\|_{L_2((0,T):H)}\cdot\left\|A\psi\right\|_{L_2((0,T):H)}+$$
$$+\sup_{t\in(0,T)}\left\|B_2(t)\right\|\cdot\left\|Au\right\|_{L_2((0,T):H)}\cdot\left\|A\psi\right\|_{L_2((0,T):H)}\le$$
$$\le const\cdot\left\|u\right\|_{W_2^1((0,T):H)}\cdot\left\|\psi\right\|_{W_2^1((0,T):H)}, \tag{4}$$

Then statement of the lemma follows from the density of $\overset{\circ}{D}([0,T]:H_1)$ in $\overset{\circ}{W}{}_2^1((0,T):H)$ and from inequalities 3), 4).

Definition. *If the function* $u\in\overset{\circ}{W}{}_2^1((0,T):H)$ *satisfies the identity* $\left(\widetilde{P}u,\psi\right)=(f,\psi)_{L_2((0,T):H)}$ *for any* $\psi\in\overset{\circ}{W}{}_2^1((0,T):H)$, *it is called a generalized solution of problem (1) (2).*

In this paper we'll indicate sufficient conditions on the coefficients of equation (1) that ensure the existence and uniqueness of the generalized solution of problem (1) (2). Note that similar problems in a half-space were studied for instance in the papers [2- 4], in the finite interval in [5,6]

It holds the following

Theorem 1. *Let conditions 1) and 2) be fulfilled. Then for any* $f\in L_2((0,T):H)$ *there exists a unique function* $u(t)\in W_2^2((0,T):H)$ *that satisfies the equation* $P_0(d/dt)\,u=-u''(t)+\rho(t)\,A^2u(t)=f(t)$, $t\in(0,T)$ *almost everywhere in* $(0,T)$ *and the boundary condition* $u(0)=u(T)=0$.

Proof. Determine the operator P_0 in $L_2((0,T):H)$ generated by the differential expression $P_0(d/dt)$ with domain of definition

$$D(P_0)=\left\{u:u\in W_2^2((0,T):H)\,,u(0)=u'(0)\right\}.$$

Obviously, P_0 is a self- adjoint operator in $L_2((0,T):H)$. On the hand, for any $u\in D(P_0)$ it holds the relation

$$(P_0u,u)_{L_2((0,T):H)}=-\left(u'',u\right)_{L_2((0,T):H)}+\left(\rho(t)\,A^2u,u\right)_{L_2((0,T):H)}=$$

$$= \|u'\|^2_{L_2((0,T):H)} + \left(\rho^{1/2}Au, \rho^{1/2}A\psi\right)_{L_2((0,T):H)} \geq \alpha\mu_0^2 \|u\|^2_{L_2((0,T):H)},$$

where μ_0 is the lower bound of the spectrum of the operator A. Thus the operator P_0 is a positive-definite self-adjoint operator in $L_2((0,T):H)$. Then the equation $P_0u = f$ has a unique solution $u \in D(P_0)$ for any $f \in L_2((0,T):H)$. Hence it follows the statement of the theorem.

Corollary. *The solution of the equation* $P_0u = f$ *satisfied the relation*

$$\left(\widehat{P}_0 u, \psi\right) = (f, \psi)_{L_2((0,T):H)}.$$

Indeed, $u \in W_2^2((0,T):H)$ *and*

$$\widetilde{P}_0(u,\psi) = (u', \psi')_{L_2((0,T):H)} + \left(\rho^{1/2}Au, \rho^{1/2}Au\right)_{L_2((0,T):H)} =$$

$$= \left(-u'' + \rho A^2 u, \psi\right)_{L_2((0,T):H)} = (f,\psi)_{L_2((0,T):H)}.$$

now prove the main theorem.

Theorem 2. *Let conditions 1)-3) be fulfilled and it hold the inequality*

$$q = \frac{1}{2}\sup_{t\in(0,T)} \|B_1(t)\| + \left(\sup_{t\in(0,T)} \|C_1(t)\| + \sup_{t\in(0,T)} \|B_2(t)\|\right) < \min(1,\alpha).$$

Then problem(1),(2) has a unique regular solution for any $f \in L_2((0,T):H)$

Proof. Let $\psi \in \overset{\circ}{D}([0,T]:H_1)$. Then

$$\left(\widetilde{P}\psi, \psi\right) = \left(\widetilde{P}_0\psi, \psi\right) + \left(\widetilde{P}_1\psi, \psi\right) > \left(\widetilde{P}_0\psi, \psi\right) - \left|\left(\widetilde{P}_1\psi, \psi\right)\right|.$$

Since

$$\left(\widetilde{P}_0\psi, \psi\right) = \|\psi'\|^2_{L_2((0,T):H)} + \left\|\rho^{1/2}A\psi\right\|^2_{L_2((0,T):H)} \geq$$

$$\geq \|\psi'\|^2_{L_2((0,T):H)} + \alpha \|A\psi\|^2_{L_2((0,T):H)}$$

and (see the proof of the lemma)

$$\left|\left(\widetilde{P}_1\psi, \psi\right)\right| \leq \left|\left(A_1(t)\psi, \psi'\right)_{L_2((0,T):H)}\right| + \left|\left(A_1'(t)\psi, \psi\right)_{L_2((0,T):H)}\right| +$$

$$+\left|\left(A_2(t)\psi, \psi'\right)_{L_2((0,T):H)}\right| \leq \sup_t \|B_1(t)\| \cdot \|A\psi\|_{L_2((0,T):H)} \cdot \|\psi'\|_{L_2((0,T):H)} +$$

$$+\sup_t \|C_1(t)\| \cdot \|A\psi\|^2_{L_2((0,T):H)} + \sup_t \|B_2(t)\| \cdot \|A\psi\|^2_{L_2((0,T):H)} \leq$$

$$\leq \frac{1}{2}\sup_t \|B_1(t)\| \cdot \left(\|A\psi\|^2_{L_2((0,T):H)} + \|\psi\|^2_{L_2((0,T):H)}\right) +$$

$$+\left(\sup_t \|C_1(t)\| + \sup_t \|B_2(t)\|\right) \left\|A^2\psi\right\| \leq$$

$$\leq \left(\frac{1}{2}\sup_t \|B_1(t)\| + \sup_t \|C_1(t)\| + \|B_2(t)\|\right) \times$$

$$\times \|\psi\|^2_{W_2^1((0,T):H)} \leq q \|\psi\|^2_{W_2^1((0,T):H)} ,$$

then

$$\left|\left(\widetilde{P}_1\psi, \psi\right)\right| \geq \|\psi'\|^2_{L_2((0,T):H)} + \alpha \|A\psi\|^2_{L_2((0,T):H)} -$$
$$-q \|\psi\|^2_{W_2^1((0,T):H)} \geq (\min(1,\alpha) - q) \|\psi\|^2_{W_2^1((0,T):H)} .$$

Thus, for $\psi \in \overset{\circ}{W}{}_2^1((0,T):H)$ $\left(\widetilde{P}\psi, \psi\right) \geq C \|\psi\|^2_{W_2^1((0,T):H)}$ i.e. the bilinear form is positive-definite . Now look for the generalized solution $u(t)$ in the form of $u(t) = u_0(t) + u_1(t)$, where $u_1(t) \in \overset{\circ}{W}{}_2^1((0,T):H)$ is a still unknown function, and $u_0(t)$ is the solution of the equation $P_0 u = f$ from theorem 1.Then using corollary (1) with respect to u_1, we get the relation

$$\left(\widetilde{P}_0(u_0 + u_1), \psi\right) + \left(\widetilde{P}_1(u_0 + u_1), \psi\right) = (f, \psi)$$

or

$$\left(\widetilde{P}_0 u_1, \psi\right) + \widetilde{P}_1(u_1, \psi) = -\left(\widetilde{P}_1 u_0, \psi\right) \tag{5}$$

As the right side is a linear functional with respect to ψ, and taking into account that $\left(\widetilde{P}\widetilde{\psi}, \widetilde{\psi}\right) \geq C \|\psi\|^2_{W_2^1((0,T):H)}$, by applying the Lax-Milgram theorem we get that there exists a unique function $u_1(t) \in \overset{\circ}{W}{}_2^1((0,T):H)$ that satisfies relation (5). Then the function $u_0(t) + u_1(t)$ will be a unique generalized solution of problem (1), (2).

The theorem is proved.

References

[1]. Lions J.L. , Majenes E. *Inhomogeneous boundary value problems and their applications.* Moscow, Mir, 1971 371 p. (Russian)

[2].Mirzoev S.S. *On the generalized solution of boundary value, problems for operator-differential equations* //Prikladniye voprosi funksionalnogo analiza.Baku, ASU, 1987, pp.71-80 (Russian)

[3]. Aliev A.R. *On the generalized solution of the boundary-volue problem for the operator-differntial equations of the second order with variable coefficients* // Journal of Mathematical Physics,Analysis,Geometry,2006,v.2,N1,pp.87-93

[4]. Gambataliev R.Z. *On completeness of elementary generalized solution of a class of operator-differential equations of a higher order* // Turkish Journal of Mathematics,2009,v.33,N4,pp.383-396

[5]. Mirzoev S.S., Gambataliev R.Z. *On completeness of elementary solutions of a clean of operator- differential equations on a finite interval.* "Doklady RAN 2010, vol. 431, No 4,pp. 454-456 (Russian)

[6]. Arazov B.O. *On completeness of elementary solutions for some operator-differential equations on a send-axis and interval.* // DAN of USSR, 1979, vol.245, No pp.788-792.(Russian)

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