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ON REPRESENTABILITY OF FUNCTIONS ANALYTIC ON A HALF PLANE WITH RESPECT TO OWN BOUNDARY CONDITIONS

Abstract

In the paper, using the notion of $A-$ integration we prove that the Cauchy-type integrals of Lebesgue integrable functions on a real axis R on upper and lower half-plane are the Cauchy $A-$ integrals and the moments of nontangential limit values of Cauchy type integrals in the sense of $A-$ integration equal zero.

Introduction. Let Γ be a simple closed rectifiable contour, G^+ be the bounded and G^- the unbounded domains with boundary Γ and $f \in L(\Gamma)$. The functions

$$F^+(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\tau)d\tau}{\tau - z},\ z \in G^+,\quad F^-(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\tau)d\tau}{\tau - z},\ z \in G^-,$$

are called Cauchy- type integrals of the function f over Γ.

V.I. Smirnoff [1] proved that the analytic functions $F^+(z)$ and $F^-(z)$ have finite nontangential boundary values $F^+(\tau)$ and $F^-(\tau)$ for almost all points $\tau \in \Gamma$ (see V.P.Khavin's paper [2]).

It follows from A.Zigmund's theorem (see, for example, [3]) that, if $f \in L\log L(\Gamma)$, then the Cauchy type integral is the Cauchy integral, that is it holds the equality

$$F^+(z) = \frac{1}{2\pi i}\int_\Gamma \frac{F^+(\tau)}{\tau - z}d\tau,\ z \in G^+. \tag{1}$$

If $f \in L(\Gamma)$ and $f \notin L\log L(\Gamma)$, then even in the cricle case $T = \{z \in C : |z| = 1\}$ it can happen that the boundary values $F^+(\tau)$ and $F^-(\tau)$ are not Lebesgue integradle on T, and therefore the equality (1) in this case is not satisfied.

Using the notion of A-integration P.L.Ul'yanov ([4] in the case of a unit circle; and [5] in the case of prime closed Lyapunov contour) established that, if $f \in L(\Gamma)$, then it holds the equality

$$F^+(z) = \frac{1}{2\pi i}(A)\int_\Gamma \frac{F^+(\tau)}{\tau - z}d\tau,\ z \in G^+, \tag{2}$$

that is, the Cauchy- type integral of the Lebesgue integrable function is a Cauchy A-integral. A.B.Alexandrov [6] established the validity of equality (2) for analytic functions in the unit circle, which belonging to V.I.Smirnoff's class and satisfying the condition $\lambda m\{\tau \in T : |F^+(\tau)| > \lambda\} = o(1)$ as $\lambda \to +\infty$.

For analytic functions in bounded simply connected domains G with simple rectifiable boundary ∂G and satisfying the condition

$$\lambda m\{\tau \in \partial G : |F^*_\alpha(\tau)| > \lambda\} = o(1),\ \lambda \to +\infty$$

for some $\alpha > 1$, equality (2) was proved by T.S.Salimov [7] for $\alpha > 2$ and by the author [8] for $\alpha > 1$,where

$F^*_\alpha(\tau) = \sup\{|F(z)| : z \in G, |z-\tau| < \alpha\rho(z,\partial G)\}$ is an analogue of a nontangential maximum function, and $\rho(z,\partial G)$ is the distance from the point $z \in G$ to the boundary ∂G .

Absence of formula of change of variables for A- integral with respect to unbounded sets requires new methods for obtaining similar results on domain with unbounded boundaries.

In the paper, using the notion of A-integration, it is proved that if the function f is Lebesgue integrable on a real axis R, then Cauchy - type integral of the function f on the upper and lower half -plane are the Cauchy $A-$ integrals, and the moments of nontangential boundary values of Cauchy- type integrals in the sense of $A-$ integration equal zero.

Representability of analytic on a half plane functions with respect to own boundary values. Let $f \in L_1(R)$, that is the function $f(t)$ be Lebesgue integrable on a real axis R. The analytic functions

$$F^+(z) = \frac{1}{2\pi i}\int\limits_R \frac{f(t)}{t-z}dt,\ z \in G^+,\quad F^-(z) = \frac{1}{2\pi i}\int\limits_R \frac{f(t)}{t-z}dt,\ z \in G^-,$$

where $G^+ = \{z \in C : \operatorname{Im} z > 0\}$, $G^- = \{z \in C : \operatorname{Im} z < 0\}$ are called Cauchy - type integrals of the function $f \in L_1(R)$.

The analytic functions $F^\pm(z)$ have finite nontangential boundary values $F^\pm(t)$ almost for all $t \in R$ (see, for example, [3],[9]). But the boundary values $F^\pm(t)$, $t \in R$ generally speaking, may not even belong to the class of functions $L_1^{(loc)}(R)$.

Definition 1. T*he complex-valued function $g(t)$ measurable on a real axis R is said to be A- integrable on R if the following condition is fulfilled*

$$\lambda m\{t \in R : |g(t)| > \lambda\} = o(1), \quad \lambda \to +\infty,$$

and there exists a finite limit

$$\lim_{\lambda\to+\infty} \int\limits_{\{t\in R:|g(t)|\le\lambda\}} g(t)dt.$$

This limit is called the A-integral of the function $g(t)$ with respect, to R and is denoted by $(A)\int\limits_R g(t)dt$.

Note that A-integral possesses the additivity property with respect to functions, that is if the functions $g_1(t)$ and $g_2(t)$ are A -integrable on R, then their sum $g_1(t)+g_2(t)$ also is A-integrable on R, and the following equality is valid:

$$(A)\int\limits_R (g_1(t)+g_2(t))dt = (A)\int\limits_R g_1(t)dt + (A)\int\limits_R g_2(t)dt.$$

Definition 2. *Let $f \in L_1(R)$. The function*

$$(Hf)(t) = \frac{1}{\pi} v.p.\int\limits_R \frac{f(\tau)}{t-\tau}d\tau = -\frac{1}{\pi}\lim_{\varepsilon\to0}\int\limits_\varepsilon^{+\infty} \frac{f(t+\tau)-f(t-\tau)}{\tau}d\tau$$

is called Hilbert transformation of the function f on R.

Note that for $f \in L_1(R)$ the function $(Hf)(t)$ was determined almost every where on R, but , generally speaking, may not even belong to the class of functions $L_1^{(loc)}(R)$. Therewith, if $f \in L_p(R), p > 1$ then the function $(Hf)(t)$ also belongs to the class of functions $L_p(R)$ (see, for example, [3])

It is known that (see, for example, [3]) if $f \in L_p(R)$, $g \in L_q(R)$, $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p,\ q < \infty$, then the following equality is valid

$$\int_R g(t)(Hf)(t)dt = -\int_R f(t)(Hg)(t)dt.$$

Anter Ali Al Saiyad [10] proved the following theorem that we'll need in future.

Theorem A. *If $g(t)$ is a bounded function, $g \in L_p(R)$* for some $p \geq 1$, *and its Hilbert transformation $(Hg)(t)$ is also bounded, and $f \in L_1(R)$, then the function $(Hf)(t)g(t)$ is A-integrable on R and the following equality is valid :*

$$(A)\int_R g(t)(Hf)(t)dt = -\int_R f(t)(Hg)(t)dt. \tag{3}$$

Theorem 1. *Let $f \in L_1(R)$ and*

$$F^+(z) = \frac{1}{2\pi i}\int_R \frac{f(t)}{t-z}dt,\ z \in G^+$$

be the Cauchy- type integral of the function $f(t)$ on the upper half-plane. Then for any $z \in G^+$ the following equality is valid

$$F^+(z) = \frac{1}{2\pi i}(A)\int_R \frac{F^+(t)}{t-z}dt, \tag{4}$$

where $F^+(t)$ are nontangential boundary values of the function $F^+(z)$ as $z \to t \in R$.

Proof. Take any point $z = x + iy$ from the upper half-plane G^+ and represent $F^+(z)$ in the from:

$$F^+(z) = \frac{1}{2\pi i}\int_R \frac{f(t)}{t-z}dt = \frac{1}{2\pi i}\int_R \frac{f(t)}{t-x-iy}dt =$$

$$= \frac{1}{2\pi i}\int_R \frac{t-x+iy}{(t-x)^2+y^2}f(t)dt = \frac{1}{2\pi i}\int_R \frac{t-x}{(t-x)^2+y^2}f(t)dt+$$

$$+\frac{1}{2\pi}\int_R \frac{y}{(t-x)^2+y^2}f(t)dt. \tag{5}$$

Consider the function $g(t) = \frac{y}{(t-x)^2+y^2}$.Find the Hilbert transformation of this function:

$$(Hg)(t) = \frac{1}{\pi}v.p.\int_R \frac{g(\tau)}{t-\tau}d\tau = \frac{1}{\pi}v.p.\int_R \frac{y}{(\tau-x)^2+y^2}\cdot\frac{1}{t-\tau}d\tau =$$

$$=\frac{1}{\pi}v.p.\int\limits_R\left[\frac{\tau+t-2x}{(\tau-x)^2+y^2}+\frac{1}{t-\tau}\right]\frac{y}{(t-x)^2+y^2}d\tau=$$

$$=\frac{y}{(t-x)^2+y^2}\left[\frac{t-x}{y}\arctan\left.\frac{\tau-x}{y}\right|_{-\infty}^{+\infty}+\lim_{\varepsilon\to 0+}\left(\ln\left.\frac{\sqrt{(x-\tau)^2+y^2}}{|\tau-t|}\right|_{-\infty}^{t-\varepsilon}+\right.\right.$$

$$\left.\left.+\ \ln\left.\frac{\sqrt{(x-\tau)^2+y^2}}{|\tau-t|}\right|_{t+\varepsilon}^{+\infty}\right)\right]=\frac{t-x}{(t-x)^2+y^2}.$$

Taking into attention equality (3) (see Theorem A) and taking into account the additivity of A- integral with respect to functions, by equality (5) we have

$$F^+(z)=\frac{1}{2\pi i}\int\limits_R(Hg)(t)f(t)dt+\frac{1}{2\pi}\int\limits_R g(t)f(t)dt=$$

$$=-\frac{1}{2\pi i}(A)\int\limits_R g(t)(Hf)(t)dt+\frac{1}{2\pi}\int\limits_R g(t)f(t)dt=$$

$$=\frac{1}{2\pi}(A)\int\limits_R g(t)\left[f(t)+i(Hf)(t)\right]dt. \tag{6}$$

Since the nontangential boundary values of the function $F^+(z)$ as $z\to t\in R$ are calculated from formula (see [3])

$$F^+(t)=\frac{1}{2}\left[f(t)+i(Hf)(t)\right],$$

then from equality (6) it follows that

$$F^+(z)=\frac{1}{\pi}(A)\int\limits_R g(t)F^+(t)dt. \tag{7}$$

On the other hand, based on the same equality (3), from theorem A the following relations are valid

$$\int\limits_R f(t)(Hg)(t)dt=-(A)\int\limits_R g(t)(Hf)(t)dt,$$

$$\int\limits_R f(t)g(t)dt=(A)\int\limits_R(Hg)(t)(Hf)(t)dt.$$

From these equalities, by the additivity with respect to the functions of A-integral, it follows that

$$(A)\int\limits_R\left[(Hg)(t)-ig(t)\right]F^+(t)dt=(A)\int\limits_R\left[(Hg)(t)-ig(t)\right]\times$$

$$\times\left[f(t)+i(Hf)(t)\right]dt=\int\limits_R f(t)(Hg)(t)dt-i\int\limits_R f(t)g(t)dt+$$

$$+i(A)\int_R (Hf)(t)(Hg)(t)dt+(A)\int_R (Hf)(t)g(t)dt=0. \qquad (8)$$

Taking into account equality (8), from equality (7) we get

$$F^+(z)=\frac{1}{\pi}(A)\int_R g(t)F^+(t)dt=\frac{1}{\pi}(A)\int_R g(t)F^+(t)dt+$$

$$+\frac{1}{2\pi i}(A)\int_R \left[(Hg)(t)-ig(t)\right]F^+(t)dt=$$

$$=\frac{1}{2\pi i}(A)\int_R \left[(Hg)(t)+ig(t)\right]F^+(t)dt=$$

$$=\frac{1}{2\pi i}(A)\int_R \frac{t-x+iy}{(t-x)^2+y^2}F^+(t)dt=$$

$$=\frac{1}{2\pi i}(A)\int_R \frac{F^+(t)}{t-x-iy}dt=\frac{1}{2\pi i}(A)\int_R \frac{F^+(t)}{t-z}dt,$$

that is formula (4) is valid. Theorem 1 is proved.

Definition 3. *If for a function $\Phi(z)$ analytic in the half-plane G^+ there exists a constant $C>0$ such that for any $y>0$ the condition*

$$\int_R |\Phi(x+iy)|^p\,dx\le C,$$

is fulfilled, then the function $\Phi(z)$ belongs to the space $H_p(G^+)$, where $p\ge 1$.

Theorem 2. *Let $f\in L_1(R)$, and*

$$F^+(z)=\frac{1}{2\pi i}\int_R \frac{f(t)}{t-z}dt,\ z\in G^+$$

be a Cauchy - type integral of the function $f(t)$ on the upper half-plane G^+ . If the function $\Phi(z)$ analytic and bounded on the upper half-plane G^+ belongs to the space $H_p(G^+)$ for some $p\ge 1$ it is valid the equality

$$(A)\int_R F^+(t)\Phi(t)dt=0, \qquad (9)$$

where $F^+(t)$ and $\Phi(t)$ are nontangential boundary values of the functions $F^+(z)$ and $\Phi(z)$, respectively , as $z\to t\in R$.

Proof. Denote $g(t)=\operatorname{Re}\Phi(t), t\in R$. Then from the condition $\Phi\in H_p(G^+)$ if follows that $(Hg)(t)=\operatorname{Im}\Phi(t), t\in R$. Hence we get that the functions $g(t)$ and $(Hg)(t)$ are bounded, $g\in L_p(R)$, $Hg\in L_p(R)$. Applying theorem A to the functions $g(t)$ and $f(t)$ and also to the functions $(Hg)(t)$ and $f(t)$, we have equalities

$$\int_R f(t)(Hg)(t)dt=-(A)\int_R g(t)(Hf)(t)dt,$$

$$\int_R f(t)g(t)dt = (A)\int_R (Hg)(t)(Hf)(t)dt.$$

Taking into account the additivity with respect to the functions of A-integral, from these equalities it holds

$$2(A)\int_R F^+(t)\Phi(t)dt = (A)\int_R [f(t)+i(Hf)(t)]\,[g\,(t)+i(Hg)(t)]dt =$$

$$= \int_R f(t)g(t)dt + i\int_R f(t)(Hg)(t)dt + i(A)\int_R g(t)(Hf)(t)dt -$$

$$-(A)\int_R (Hf)(t)(Hg)(t)dt = 0,$$

i.e. formula (9) is valid. Theorem 2 is proved.

Corollary 1. *Let $f \in L_1(R)$, and*

$$F^+(z) = \frac{1}{2\pi i}\int_R \frac{f(t)}{t-z}dt,\ \ z \in G^+,$$

be a Cauchy type integral of the function $f(t)$ on the upper half-plane G^+. Then for any point $z \in G^+$ it is valid the equality

$$(A)\int_R \frac{F^+(t)}{t-\overline{z}}dt = 0,$$

where $F^+(t)$ are nontangential boundary values of the function $F^+(z)$ as $z \to t \in R, \overline{z} = x - iy$.

For the lower half-plane $G^-$the analogs of theorems 1,2 are formulated in the following way:

Theorem 3. *Let $f \in L_1(R)$, and*

$$F^-(z) = \frac{1}{2\pi i}\int_R \frac{f(t)}{t-z}dt,\ \ z \in G^-,$$

be a Cauchy- type integral of the function $f(t)$ on the lower half-plane G^-. Then for any $z \in G^-$ it is valid the equality

$$F^-(z) = \frac{1}{2\pi i}(A)\int_R \frac{F^-(t)}{t-z}dt,$$

where $F^-(t)$ are nontangential boundary values of the function $F^-(z)$ as $z \to t \in R$.

Theorem 4. *Let $f \in L_1(R)$, and*

$$F^-(z) = \frac{1}{2\pi i}\int_R \frac{f(t)}{t-z}dt,\ \ z \in G^-$$

be a Cauchy- type integral of the function $f(t)$ on the lower half-plane G^- . If the function $\Phi(z)$ analytic and bounded on the lower half-plane G^- belongs to the space $H_p(G^-)$ for some $p \geq 1$ it is valid the equality

$$(A)\int_R F^-(t)\Phi(t)dt = 0,$$

where $F^-(t)$ and $\Phi(t)$ are nontangential boundary values of the functions $F^-(z)$ and $\Phi(z)$, respectively, as $z \to t \in R$.

The proofs of theorems 3 and 4 are similar to the proofs of theorems 1 and 2.

Corollary 2. *Let $f \in L_1(R)$, and*

$$F^-(z) = \frac{1}{2\pi i}\int_R \frac{f(t)}{t-z}dt, \ z \in G^-,$$

be a Cauchy- type integral of the function $f(t)$ on the lower half-plane G^-. Then for any point $z \in G^-$ it is valid the equality

$$(A)\int_R \frac{F^-(t)}{t-\overline{z}}dt = 0,$$

where $F^-(t)$ are nontangential boundary values of the function $F^-(z)$ as $z \to t \in R$, $\overline{z} = x - iy$.

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