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ERROR ESTIMATION USING THE VARIATION STEP OF LATTICE

Abstract

Exact solutions of boundary value problems for a mixed type equation can be obtained only in special cases. Therefore these problems may be solved approximately. The quality of the differenence scheme is determined by its accuracy and efficiency. For the problems may be solved approximately. For the problems considered in practical calculation work, theoretical error estimations $u - u_h$ *don't exist or as a rule are very high. In practice for estimating the error estimation* $u - u_h$ *comparison of approximate solutions obtained at different steps of lattice spacings are usually used.*

Study of heat and mass exchange processes has always played an important role in development of engineering and natural science. At the end of the last and the beginning of this century the investigations in this field were stimulated mainly by the needs of heat power engineering. Recent years, development of aviation, nuclear power, rocket-space technology put forth new statemens of heat and mass exchange problems and at the same time more rigid requirements to completeness and reliability of the theory and experiment data. For the last ten years the scope of intensive study and application of heat and mass exchange phenomena has extended. It includes both key directions of engineering (chemical technology, oil development, machine-building and so on) and main natural sciences (biology, physics of atmosphere and ocean, etc) At preset theoretical investigation of heat and mass change processes in the large extent is based on their numerical simulation using a computer. This became possible owing to significant progress in development of computational methods of the solutions for partial equations problems and increase of power of up-to-date computers. The last years significant successes in development of numerical methods of theory of heat and mass exchange methods were attained and a great experience of partically important problems was stored numerical simulation heat mass exchange processes at present acquires a significant role in the connection that for modern science and engineering data on the processes whose experimental study in laboratory or life conditions is very complicated and in expensive, and in some cases just is impossible, are necessary. Numerical simulation of heat and mass exchange processes very successfully go into practice of the work of various scientific research, design and industrials institutions. At present numerical solution methods of heat conductivity and diffusion processes in fixed media have been developed and rather widely applied. The same numerical methods may be applied in such frequently encountered cases when heat and mass exchange doesn't influent on motion of liquid or gaseous media, and the motion of the medium itself is known.

Transition from the mathematical model of this other heat and mass exchange process to numerical algorithm realized using a computer, at present more often is performed by the lattice method. A lattice aggregate of nodal points is introduced into the range of independent variables.

Differential equations with corresponding boundary conditions are replaced by approximate defference equations connecting the values of the sought-for function at the nodal points of the lattice. A system of algebraic equations that may be solved by this or other method using a computer, is obtained in such a way. There are different methods for constructing grid equations approximating the original boundary value problem. On regular grids the derivatives may be replaced by finite- difference expressions approximating them (finite differences method). In the present paper only the netpoint method in its simplest form, i.e. the finite- differences method, and for increasing accuracy of approximation the Runge method is used.

According to the net-point method , a differential equation and boundary conditions are replaced by difference equations connecting the values of the sought -for function at the nodal points of the grid (grid boundary value problem or scheme). Proximity of the scheme of original boundary value problem is estimated by the quantity discrepancy obtained when substituting the exact solution to the equation and boundary conditions of the grid boundary value problem .

Denote by

$$R(u) = 0 \qquad (I)$$

the set of equations contained in the boundary value problem, i.e. the principal differential equations and boundary (initial and boundary) conditions.

In the similar way write the grid boundary value problem

$$R_h(u_h) = 0 \qquad (II)$$

For brevity of notation here and in the sequel we assume $\tau = \tau(h), \tau \to 0$ as $h \to 0$ so that the grid is determined by me parameter h.

Approximation error of the scheme (II) on the exact solution of problem (I) is said be a the grid function

$$R_h(u) = \alpha_h \qquad (III)$$

The scheme is said to be approximate on the exact solution if as $h \to 0$ the approximate error tends to zero: $\alpha_h \to 0$ as $h \to 0$

If $\alpha_h = O(h^p)$ it is said that approximation order equals p. Associating (II) and (III) we can consider $u - u_h$ as perturbation of the solution of the grid problem caused by the small perturbations α_h in the right hand side of (II). For the convergence to follow from the approximation property, i.e. from tendency of $u - u_h$ to zero, it suffices additionally to require the scheme to be stable with respect to small perturbations.

It is significant that the stability be uniform as $h \to 0$, i.e. not be deteriorated as $h \to 0$. Recall that notation (III) denotes the system of equations unboundedly increasing for $h \to 0$. Therefore, sensitivity of the system to small perturbations may increase unboundedly as $h \to 0$ that reduces to lack of convergence.

For the problems considered in the practical calculation work, theoretical error estimations $u - u_h$ either don't exist or as a rule, are very high.

In proactice, for error estimation $u - u_h$ usually comparisonof approximate solutions obtained for different steps of the grid (Runge method) are used.

Frequently we have every reason that the approximate solution error may be written in form:

$$u - u_h = h^p \omega + ...,$$

where ω is a function independent of h; the dots means the members of a higher order of smallness that in the sequel are omitted. Substituting ch for h, where c is a positive multiplier, we get

$$u - u_{ch} \approx c^p h^p \omega.$$

Having eliminated the unknown function ω we find

$$u - u_h \approx \frac{u_h - u_{ch}}{c^p - 1}.$$

In particular, for the schemes of second order accuracy ($p = 2$) assuming for example $c = 2$, we have $u - u_h \approx (u_h - u_{2h}) \backslash 3$.

For applying the Runge method we consider numerical solution of boundary value problems for degenerate hyperbolic equations of second order

$$Lu \equiv y\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \tag{1}$$

in the characteristical triangle by the finite differences method. In domain D bounded by the characteristics

$$OC : x - \frac{2}{3}(-y)^{\frac{3}{2}} = 0,$$

$$AC : x + \frac{2}{3}(-y)^{\frac{3}{2}} = 1$$

and the segments AC of the axis Ox consider the following problems for equation (1):

$$u\,|_{OA} = \varphi(x), \tag{2}$$

$$u\,|_{OC} = \psi(x), \tag{3}$$

where $\varphi(0) = \psi(0)$, $\varphi(x) \in C^{(2)}[OA]$, $\psi(x) \in C^{(2)}[OC]$, $f(x, y) \in C^{(2)}\left(\overline{D}\right)$.

Let the boundary of domain D intersect the abscissa axis at the points $O(0, 0)$ and $A(1, 0)$. Divide the segment OA into equal parts. Through the partition point we draw characteristics

$$x - \frac{2}{3}(-y)^{\frac{3}{2}} = 2nh, \quad x + \frac{2}{3}(-y)^{\frac{3}{2}} = 2nh, \ n = 0, 1, 2, ...$$

of equation (1) Having taken for $y < 0$ the intersection points of these lines, occurring interior or on the boundary of domain D, we get a characteristical non-uniform grid $\overline{\omega}_h \subset \overline{D}$, where

$$\overline{\omega}_h = \left\{ (x_k, y_m) / x_k = kh, \ y_m = -y_m, \ y_m = \left(\frac{3}{2}mh\right)^{2/3}, \ k, m = 0, 1, 2, , ... \right\}$$

All the points of the grid are located on the straight lines $y_m = -y_m, m = 0, 1, 2, ...$ At ony point $(x_k, -y_m)$ equation (1) is approximated by the following difference equation:

$$R_h u_h \equiv \frac{1}{l_m l_{m+1}} \left[\frac{2l_m}{l_m + l_{m+1}} u_h(x_k, -y_{m-1}) + \frac{2l_m}{l_m + l_{m+1}} u_h(x_{k,} - y_{m+1}) - \right.$$

$$-u_h\left(x_k+h,-y_m\right)-u_h\left(x_k-h,-y_m\right)]=f\left(x_k,-y_m\right), \tag{4}$$

where $l_m=y_m-y_{m-1}$. Boundary conditions (2), (3) are approximated with zero discrepancy i.e.

$$u_h\Big|_{(OA)_h}=\varphi_h, \tag{5}$$

$$u_h\Big|_{(OC)_h}=\psi_h, \tag{6}$$

Assuming $u\in C^{(4)}\left(\overline{D}\right)$, by the Taylor formula for discrepancy we have the estimation:

$$\psi=R_h u-Lu=O\left(h^{(2/3)}\right).$$

For increasing the accracy we use the Runge method. Let $\overline{\omega}_h$ be the main grid, $\overline{\omega}_{h_1},\overline{\omega}_{h_2}$ be the crushed grids, $h_2<h_1<h$, $\overline{\omega}_h\cap\overline{\omega}_{h_1}\cap\overline{\omega}_{h_2}\neq\varnothing$.

If the solution of the original problem $u\left(x,y\right)$ and $f\left(x,y\right)$ is a rather smooth function $\left(x,y\right)$, then increase of accuracy order of the grid solution may be achieved by calculations for one and the same problem(4)-(6) from the sequence of the girds $\overline{\omega}_h,\overline{\omega}_{h_1},\overline{\omega}_{h_2}$.

Assume that the following asymptotic expansion is true

$$u_h=u+\sum_{s=1}^{n-1}\alpha_s\left(t\right)h^s+\alpha_n\left(t,h\right)h^n, \tag{7}$$

where $\alpha_s\left(t\right)$ are independent of $h, \left|\alpha_n\left(t,h\right)\right|\leq M$ and we want to get the solution to within $O\left(h^2\right)$, then we should conduct calculations of the difference problem with the steps $h_1,h_2,...h_n$

By smoothens of $u\left(x,y\right)$, $f\left(x,y\right)$ for the error estimation we get the following expression

$$\psi=\sum_{s=1}^{n-1}\beta_s(t)h^s+\beta_n(t,h)h^n, \tag{8}$$

where $\beta_s\left(t\right)$ are independent of $h, \left|\beta_n\left(t,h\right)\right|\leq M$. Let $\alpha_s\left(t\right)$ be the solutions of the equations

$$L\alpha_s=\beta_s(t)h^s-\sum_{m=1}^{s-1}\gamma_m\left(t\right),\quad for\quad s=1,2,...,n-1.$$

Notice that coefficients (8) may contain not all powers of h^s,then the corresponding the coefficients $\beta_s(t)\equiv 0$.

Applying (7) to $\overline{\omega}_h,\overline{\omega}_{h_1},\overline{\omega}_{h_2}$, respectively, for the solution of difference scheme (4)-(6) we get

$$\begin{aligned} u_h&=u+\alpha_1(t)h^{2/3}+\alpha_2(t)h^{4/3}+O(h^2),\\ u_{h_1}&=u+\alpha_1(t)h_1^{2/3}+\alpha_2(t)h_1^{4/3}+O(h_1^2),\\ u_{h_2}&=u+\alpha_1(t)h_2^{2/3}+\alpha_2(t)h_2^{4/3}+O(h_2^2)\end{aligned} \tag{9}$$

By (9) in the grid domain $\overline{\omega}_{\widetilde{h}}=\overline{\omega}_h\cap\overline{\omega}_{h_1}\cap\overline{\omega}_{h_2}$ we can look for the solution of problem (4)-(6) in the form

$$u_{\widetilde{h}}=c_1u_h+c_2u_{h_1}+c_3u_{h_2}, \tag{10}$$

where c_1, c_2, c_3 are unknown coefficients that will be determined later. Taking into account (9),(10), we get:

$$u_h = c_1 \left[u + \alpha_1(t) h^{2/3} + \alpha_2(t) h^{4/3} + O(h^2) \right] +$$
$$+ c_2 \left[u + \alpha_1(t) h_1^{2/3} + \alpha_2(t) h_1^{4/3} + O(h_1^2) \right] +$$
$$+ c_3 \left[u + \alpha_1(t) h_2^{2/3} + \alpha_2(t) h_2^{4/3} + O(h_2^2) \right]$$

After corresponding grouping for determining c_1, c_2, c_3 we get the following equations

$$\begin{gathered} c_1 + c_2 + c_3 = 1 \\ c_1 h^{2/3} + c_2 h_1^{2/3} + c_3 h_2^{2/3} = 0, \\ c_1 h^{4/3} + c_2 h_1^{4/3} + c_3 h_2^{4/3} = 0. \end{gathered} \tag{11}$$

Obviously, the determinant of this system is not zero. Then

$$c_1 = \left[1 + \frac{h^{4/3} - h^{2/3} h_2^{2/3}}{h_1^{2/3} h_2^{2/3} - h_1^{4/3}} - \frac{h_1^{4/3} \left(h^{4/3} - h^{2/3} h_2^{2/3} \right)}{h^{4/3} \left(h_1^{2/3} h_2^{2/3} - h_1^{4/3} \right)} - \frac{h^{4/3}}{h_2^{4/3}} \right]^{-1},$$

$$\begin{gathered} c_2 = \left[1 + \frac{h^{4/3} - h^{2/3} h_2^{2/3}}{h_1^{2/3} h_2^{2/3} - h_1^{4/3}} - \frac{h_1^{4/3} \left(h^{4/3} - h^{2/3} h_2^{2/3} \right)}{h^{4/3} \left(h_1^{2/3} h_2^{2/3} - h_1^{4/3} \right)} - \frac{h^{4/3}}{h_2^{4/3}} \right]^{-1} \times \\ \times \frac{h^{4/3} - h^{2/3} h_2^{2/3}}{h_1^{2/3} h_2^{2/3} - h_1^{4/3}}, \end{gathered} \tag{12}$$

$$\begin{gathered} c_3 = \left[1 + \frac{h^{4/3} - h^{2/3} h_2^{2/3}}{h_1^{2/3} h_2^{2/3} - h_1^{4/3}} - \frac{h_1^{4/3} \left(h^{4/3} - h^{2/3} h_2^{2/3} \right)}{h^{4/3} \left(h_1^{2/3} h_2^{2/3} - h_1^{4/3} \right)} - \frac{h^{4/3}}{h_2^{4/3}} \right]^{-1} \times \\ \times \left[\frac{h^{4/3} - h^{2/3} h_2^{2/3}}{h_1^{2/3} h_2^{2/3} - h_1^{4/3}} \cdot \frac{h_1^{4/3}}{h_2^{4/3}} - \frac{h^{4/3}}{h_2^{4/3}} \right]. \end{gathered}$$

In particular we can take $h_1 = \frac{h}{2}$,$h_2 = \frac{h_1}{2}$ Then we have :

$$c_1 = 1,1077,$$

$$c_2 = -4,5381,$$

$$c_3 = 14,4304.$$

Taking into account (11), by (12) , from (10) we get:

$$u_{\tilde{h}} = u + O(h^2)$$

i.e.

$$u_{\tilde{h}} - u = O(h^2)$$

Hence it follows that the solution (10), of the difference problem (4)-(6) uniformly converges to the solution of the original problem (1)-(3) with velocity $O(h^2)$, i.e.

$$\left\| u_{\widetilde{h}} - u \right\| \leq O(h^2).$$

Now, study the stability of the problem. Assume that the difference problem (4) (6) is solvable. Then it is natural to require that under unrestricted grinding of the grid the solution of the difference problems to tend to the solution of the original problem for a differential equation. In these reasonings we assume that the difference problem is solved exactly and the solution may be found with any number of signs. All the calculations are conducted with the finite number of sings and at each step of calculations the rounding -off error are committed. Therefore it is necessary to prove that the scheme under consideration is stable.

As a rule it is impossible to justify strictly the stability schemes for the equations that are encountered in contemporary applied investigations. This is explained by the following reasons. In spite of log-term efforts of outstanding mathematicians for the most nonlinear equations of continuum mechanics today there is no rather complete mathematical theory, in particular, theorems on existence of non uniqueness of the solution and its continuous dependence on the problem data have not been proved yet. Usually grid approximations are no less difficult for studying, tham corresponding differential equations. Furthermore when passing from differential equations to grid approximations,the fundamential propertics lying on the basis of the corresponding mathematical theory, for example the property of maximum for parabolic and elliptic equations may be lost or put on a mask . The reason last in succession but not in importance is lack of time compelling to relinquish strong investigations even in the cases when it is impossible.

Consider problem (4)-(6) . From difference equations (4) we get:

$$u_{k,m} = (1 - a_m)\, u_{k,m-1} + (1 + a_m)\, u_{k-1,m+1} - u_{k-1,m} = f_h l_m l_{m+1} \tag{13}$$

where $u_{k,m}$ is the value of the function u_h at the point with the indices (k, m),

$$a_m = \frac{l_m - l_{m+1}}{l_m + l_{m+1}}$$

Let's consider any point with the indices $(k-1, m)$. For simplicity assume that this point is the knot of sequences $\overline{\omega_h}$ of the grids as $h \to 0$.

Equation (13) for the point $\overline{\omega_h}$ has the form $(k-1, m)$:

$$u_{k-1,m} = (1 - a_m)\, u_{k-1,m-1} + (1 + a_m)\, u_{k-2,m+1} - u_{k-2,m} = f_h l_m l_{m+1}.$$

Hence

$$|u_{k-1,m}| \leq |1 - a_m|\, |\varphi_h| + |1 + a_m|\, |\psi_h| + |\varphi_h| + |f_h|\, l_m l_{m+1}.$$

By $0 < a_m < 1,\ 0 < l_{m+1} < l_m$ we get

$$|u_{k-1,m}| \leq |\varphi_h| + 3\, |\psi_h| + |f_h|\, l_m^2.$$

Write equation (13) for the point $(k-1, m+1)$:

$$u_{k-1,m+1} = (1 + a_m)\, u_{k-1,m} + (1 + a_m)\, u_{k-2,m+2} - u_{k-2,m+1} = f_h l_{m+1} l_{m+2},$$

then we get:

$$|u_{k-1,m+1}| \leq \left(|\varphi_h| + 3|\psi_h| + |f_h| l_m^2\right) |1 - a_m| |1 + a_m| |\varphi_h| + |\psi_h| + |f_h| l_{m+1} l_{m+2}$$

By $0 < a_m < 1,\ 0 < l_{m+2} < l_{m+1} < l_m$ we have

$$|u_{k-1,m+1}| \leq |\varphi_h| + 6|\psi_h| + 2l_m^2 |f_h| .$$

Assume that for the point $(N, M-1)$ the following condition is fulfilled

$$|u_{N,M-1}| \leq M_1 |\varphi_h| + M_2 |\psi_h| + M_3 |f_h| .$$

By the abovesaid ones, for the point (N, M) it holds the following respresentation

$$u_{N,M} = (1 - a_m) u_{N,M-1} + (1 + a_m) u_{N-1,M-1} - -U_{N-1,M} - f_h l_M l_{M+1}$$

Hence we have:

$$|u_{N,M}| \leq M_4 |\varphi_h| + M_5 |\psi_h| + M_6 |f_h| .$$

Denote

$$\overline{M} = \max(M_4, M_5)$$

then

$$\max_{\overline{\omega_h}} |u_{N,M}| \leq \overline{M} \max\left[\max_{k,m} |\varphi_h|, \max_{k,m} |\psi_h|\right] + M_6 \max_{\overline{\omega_h}} |f_h|$$

that proves the stability of the difference scheme.

Consequently, by the mathematical induction method it is proved that at all the nodes of the grid the following estimation is valid:

$$\max_{\overline{\omega_h}} |u_h| \leq \overline{M} \max\left[\max_{k,m} |\varphi_h|, \max_{k,m} |\psi_h|\right] + \overline{\overline{M}} \max_{\overline{\omega_h}} |f_h|$$

Hence it is seen that at small change of the initial value φ_h, ψ_h and the right hand side of the difference equation u_h changes little i.e. the solution of the difference scheme [4]-[6] converge to the solution of the difference problem (1)-(3). Cheek the efficiency of the worked out difference scheme. Take solution of equation (1) in the from:

$$u(x, y) = \left(x^2 + y^2\right)^{1/4} \sin\left(\frac{1}{2} arctg \frac{y}{x}\right) + \left(x^2 + y^2\right)^3 .$$

Then by (1), (2),(3) the model problem takes the from:

$$yu_{xx} + u_{yy} = \left(x^2 + y^2\right)^{-3/4} \sin\left(\frac{1}{2} arctg \frac{y}{x}\right) \left(\frac{y}{2} - \frac{1}{2} - \frac{x^2}{4}\right) +$$

$$+ \left(x^2 + y^2\right)^{-3/4} \sin\left(\frac{1}{2} arctg \frac{y}{x}\right) \left(-\frac{3}{4} x^2 y - \frac{y^3}{4} - \frac{3y^2}{4} + \frac{xy}{4}\right) +$$

$$+ \left(x^2 + y^2\right)^{-3/4} \cos\left(\frac{1}{2} arctg \frac{y}{x}\right) \left(-\frac{3}{2} xy^2 - \frac{3}{4} xy\right) + 36\left(x^2 + y^2\right)^2 ,$$

$$u|_{OA} = x^6,$$

nodal points	exact solution	Apoximate solutions obtained on the grid $\overline{\omega}_h$	on the grid $\overline{\omega}_{h_1}$	on the grid $\overline{\omega}_{h2}$	$u_{\tilde{h}}$	$\lvert u_h - u_{\tilde{h}}\rvert$
C(2,2)	0,0541	0,0537	0,0539	0,0544	0,0553	0,0012
C(3,2)	0,0941	0,0926	0,0927	0,0927	0,0935	0,0006
C(4,2)	0,1541	0,1538	0,1540	0,1544	0,1556	0,0015
C(5,2)	0,2341	0,2294	0,2298	0,2302	0,2311	0,0031
C(6,2)	0,3341	0,3339	0,3342	0,3347	0,3362	0,0021
C(7,2)	0,4541	0,4573	0,4575	0,4579	0,4591	0,0050
C(2,3)	0,1197	0,1152	0,1154	0,1155	0,1156	0,0041
C(3,3)	0,1697	0,1679	0,1683	0,1684	0,1681	0,0016
C(5,3)	0,3297	0,3235	0,3237	0,3237	0,3221	0,0076
C(6,3)	0,4397	0,4372	0,4373	0,4373	0,4372	0,0025
C(2,4)	0,1994	0,1981	0,1983	0,1984	0,1985	0,0009
C(3,4)	0,2594	0,2550	0,2552	0,2553	0,2555	0,0039
C(4,4)	0,3394	0,3352	0,3356	0,3357	0,3357	0,0037

$$u\,|_{OC} = \left[x^2 + \left(\frac{3}{2}x\right)^{-4/3}\right]^{1/4} \sin\left(\frac{1}{2}arctg\frac{\left(\frac{3}{2}x\right)^{-2/3}}{x}\right) + \left[x^2 + \left(\frac{3}{2}x\right)^{-4/3}\right]^{3}$$

The numerical experiment is performed on a computer and the obtained result is characterized by the following table:

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