Transactions of NAS of Azerbaijan, 2014, vol. XXXIV, No 4, pp. 107-120. ______ 107

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ASYMPTOTICS OF THE NUMBER OF EIGEN VALUES OF 2n-th ORDER OPERATOR-DIFFERENTIAL EQUATIONS ON A SEMI –AXIS

Abstract

In the paper the Green function and the spectrum of even higher order operator-differential equations are studied, and an asymptotic formula for the number of eigen values are obtained. As first the Green function of the principal part of the equation with frozen coefficients is constructed. By using the levy method, the integral equation is obtained for the Green function of the principal part of the equation with variable coefficients. In the Banach spaces of operator valued functions the solution of the obtained integral equation is studied. The uniform estimation of the Greeen function from which in particular the discreteness of the spectrum is derived, is obtained. Using the Titchmarch's Tauberian theorems, the asymptotic formula for the function of distribution of eigen values of the given operator is obtained.

Let H be a separable Hilbert space. Denote by H_1 a Hilbert space of strongly measurable on the interval $[0,\infty)$ functions $f(x)$ with the values from H for which

$$\int_0^\infty \|f(x)\|_H^2\,dx<\infty$$

The scalar product of the elements $f(x)$, $g(x)\in H_1$ is determined by the equality

$$(f,g)_{H_1}=\int_0^\infty (f(x),g(x))_H\,dx$$

In the space $H_1 =_2 [H; 0\le x<\infty]$ consider the operator L generated by the differential expression

$$l(y)=(-1)^n y^{(2n)}+\sum_{j=2}^{2n}Q_j(x)\,y^{(2n-j)},\quad 0\le x<\infty \tag{1}$$

and the boundary conditions

$$B_j y\,|_{x=0}=y^{l_j}(0)+\sum_{m=1}^{l_j}\alpha_m^{(l_j)}y^{(l_j-m)}(0)=0 \tag{2}$$

Here $0\le l_1<l_2<...<l_n\le 2n-1$, $j=1,2,...,n$, $y\in H_1$ and the derivatives are understood in the strong sense. Everywhere by $Q(x)$ we'll denote $Q_{2n}(x)$.

Let D' be a totality of all the functions of the form $\sum\limits_{m=1}^{l_j}\varphi_k(x)\,f_k$, where $\varphi_k(x)$ are finite, $2n-$ times continuously- differentiable scalar functions and $f_k\in D(Q)$.

Determine the operator L' generated by the expression (1) and boundary conditions (2) with domain of definition D'. Under some conditions on the coefficients of expressions (1) and boundary conditions the operator L' is a positive, symmetric operator in the space H_1.

We'll assume that the closure L of the operator L' is a self-adjoint and lower semi-bounded operator in H_1.

In this paper we study the Green function and discreteness of the spectrum of the operator L. Notice that the Green function of the Sturm-Liouville equation with a self- adjoint operator coefficient was first studied by B.M. Levitan [1], while the asymptotic distribution of the eigen values of the operator L was studied in the paper of B.M.Levitan and A.G.Kostychenko [2]. The Green function and asymptotic behavior of the eigen values of the operator L generated by the expression $l(y) = -(P(x)y')' + Q(x)y$ in the self-adjoint case was studied by E. Abdukadyrov [3]. The Green function and asymptotic behavior of eigen values of higher even order operator given on the whole axis was studied by M. Bairamoglu [4] .The case of a semi-axis was considered in the paper of G.I.Aslanov [5], A.A. Abudov, and G.I.Aslanov [6] B.I. Aliyev, M.Bairamoglu [7], G.I. Aslanov and G.I.Kasumova [8] and others. Note that some spectral and boundary value problems for abstract polyharmonic operators were considered by G.D.Orujov [9]

We'll assume that the coefficients of the operator L satisfy the following conditions:

1.The operators $Q(x)$ for almost all $x \in [0,\infty)$ are self-adjoint in H and exist commonly for all x of the set $D\{Q(x)\}$ on which the operators are determined and symmetric (this, we admit that the operators $Q(x)$ may be unbounded in H)

2.The operators $Q(x)$ are uniformly lower bounded, i.e. for all $f \in D$ the inequality $(Q(x)f,f) > c(f,f), c > 0$ is fulfilled.

3.For $|x-\xi| < 1$ the inequalities $\|[Q(\xi) - Q(x)]Q^{-\alpha}(x)\| < A|x-\xi|$, where $0 < \alpha < \frac{2n+1}{2n}$, $A > 0$, $\left\|Q^{-\frac{1}{2n}}(x)Q^{-\frac{1}{2n}}(\xi)\right\| < c_1$,

$\left\|Q^{\frac{1}{2n}}(x)Q^{-\frac{1}{2n}}(\xi)\right\| < c_1$, c_1, c_2 are positive constants, are fulfilled.

4. For $|x-\xi| > 1$ it is fulfilled the inequality

$$\left\|Q(\xi)\exp\left[-\frac{Jm\omega_1}{2}|x-\xi|Q^{\frac{1}{2n}}(x)\right]\right\| < B,$$

where $Jm\omega_1 = \min\limits_i \left\{Jm\omega_i > 0, \omega_i^{2n} = -1\right\}, B > 0$

5.$\left\|Q_j(x)Q^{\frac{1-j}{2n}+\varepsilon}(x)\right\| < c, \quad j = 1,2,...,2n-1, \ c > 0.$

6. Suppose that $Q(x)$ for almost all $x \in [0,\infty)$ is inverse to the completely continuous operator. Denote by $\alpha_1(x) \le \alpha_2(x) \le ... \le \alpha_n(x) \le ...$the eigen values of the operator $Q(x)$, for which we'll assume that they are measurable functions. In what follows, for almost all x the series $\sum\limits_{i=1}^{\infty} \alpha_i^{\frac{1-4n}{2n}}(x)$ converges and its sum $F(x) \in L_1[0,\infty)$.

One of the main results of the paper is that the operator $R_\mu = (L+\mu E)^{-1}$, $\mu > 0$ is an integral operator with an operator kernel $G(x,\eta;\mu)$, that is called the Green operator function of the operator L. By definition of the Green function $G(x,\eta;\mu)$ is an operator function in H, dependent on two variables x and η $(0 \leq x, \eta < \infty)$, the parameter μ and satisfy the conditions:

a) $\frac{\partial^k G(x,\eta;\mu)}{\partial \eta^k}$ $(k = 0, 1, 2, ..., 2n-2)$ is a strongly continuous operator valued function with respect to variables (x, η);

b) $\frac{\partial^{2n-1} G(x,x+0,\mu)}{\partial \eta^{2n-1}} - \frac{\partial^{2n-1} G(x,x-0,\mu)}{\partial \eta^{2n-1}} = (-1)^n E;$

v) $(-1)^n G_\eta^{(2n)} + \sum\limits_{j=2}^{2n-1} G_\eta^{(2n-j)} Q_j(\eta) + \mu G = 0$

q) $B_j G\,|_{\eta=0} = G_\eta^{(l_j)}(x, 0; \mu) + \sum\limits_{m=1}^{l_j} \alpha_m^{(l_j)} G_\eta^{(l_j - m)}(x, 0, \mu) = 0, j = 1, 2, ..., n$

d) $G^*(x, \eta, \mu) = G(\eta, x, \mu)$

e) $\int\limits_0^\infty \|G(\eta, x, \mu)\|_H^2 \, d\eta < \infty.$

Note that condition d) provides symmetry H_1 of the integral operator

$$Af = \int\limits_0^\infty G(\eta, x, \mu) f(\eta) \, d\eta.$$

The Green function of the operator L is studied in three stages.

In the first stage the Green function of the operator L_1 generated by the differential expression

$$l_1(y) = (-1)^n y^{(2n)} + Q(\xi) y + \mu y \tag{3}$$

and boundary conditions (2) is constructed. Here "ξ" is a fixed point.

In the second stage we construct and study some properties of the Green function of the operator L_0 generated by the differential expression

$$l_0(y) = (-1)^n y^{(2n)} + Q(\xi) y + \mu y \tag{4}$$

and boundary conditions (2).

In the third stage the Green function of the operator L,generated by the differential expression (1) and boundary conditions (2) is studied

Construction of the Green function of the operator L_1

The Green function $G_1(x, \eta, \xi, \mu)$ of the operator L_1 we'll seek in the form:

$$G_1(x, \eta, \xi, \mu) = g(x, \eta, \xi, \mu) + V(x, \eta, \xi, \mu) \tag{5}$$

where $g(\eta, x, \xi, \mu)$ is the Green function of the equation $l_1(y) = 0$ on the whole axis. As is known [4], it is of the form:

$$g(x, \eta, \xi, \mu) = \frac{1}{2ni} K^{1-2n} + \sum_{k=1}^{n} \omega_k \exp(i\omega_k |x - \eta| K) \tag{6}$$

Here the roots from (-1) of degree $2n$ lying in the upper half-plane are denoted by ω_k and $K_\xi = [Q(\xi) + \mu E]^{\frac{1}{2n}}$.

As $x \to \infty$ the function $V(x,\eta,\xi,\mu)$ is the bounded solution of the following problem :

$$l_1(V) = 0, \tag{7}$$

$$B_j V|_{x=0} = -B_j g|_{x=0} \tag{8}$$

For the general solution of equation (7) we get:

$$V(x,\eta,\xi,\mu) = \frac{K_\xi^{1-2n}}{2ni}\sum_{k=1}^{n} A_k(\eta,\xi,\mu)\, e^{i\omega_k K_\xi x} \tag{9}$$

The coefficients A_k are determined from boundary conditions (8). Write the obtained system of equations:

$$\left[V^{(l_j)} + \sum_{m=1}^{l_j}\alpha_m^{(l_j)} V^{(l_j-m)}\right]_{x=0} = -\left[g^{(l_j)} + \sum_{m=1}^{l_j}\alpha_m^{(l_j)} g^{(l_j-m)}\right]_{x=0}, j = 1,2,...,n$$

or in the expanded form

$$(iK_\xi)^{l_j}\sum_{k=1}^{n} A_k\omega_k^{l_j} + \sum_{m=1}^{l_j}\alpha_m^{(l_j)}(iK_\xi)^{l_j-m}\sum_{k=1}^{n} A_k\omega_k^{l_j-m} =$$

$$= -\left[(iK_\xi)^{l_j}\sum_{k=1}^{n}\omega_k^{l_j+1}e^{iK_\xi\omega_k\eta} + \sum_{m=1}^{l_j}\alpha_m^{(l_j)}\sum_{k=1}^{n}\omega_k^{l_j-m+1}e^{iK_\xi\omega_k\eta}\right], j = 1,2,...n$$

Making regrouping of the addends in this system, we get:

$$\sum_{k=1}^{n} A_k\left[\omega_k^{l_j} + \sum_{m=1}^{l_j}\alpha_m^{(l_j)}\frac{\omega_k^{l_j-m+1}}{(iK_\xi)^m}\right] =$$

$$= -\sum_{k=1}^{n}\left[\omega_k^{l_j+1} + \sum_{m=1}^{l_j}\frac{\alpha_m^{(l_j)}\omega_k^{l_j-m+1}}{(iK_\xi)^m}\right]e^{iK_\xi\omega_k\eta}\ j = 1,2,...,n. \tag{10}$$

$$\Delta_0 = \begin{vmatrix} \omega_1^{l_1} + \sum\limits_{m=1}^{l_n}\frac{\alpha_m^{(l_n)}\omega_1^{l_1-m}}{(iK_\xi)^m} & \omega_2^{l_1} + \sum\limits_{m=1}^{l_1}\frac{\alpha_m^{(l_1)}\omega_2^{l_1-m}}{(iK_\xi)^m} & ... & ...\ \omega_n^{l_1} + \sum\limits_{m=1}^{l_1}\frac{\alpha_m^{(l_1)}\omega_n^{l_1-m}}{(iK_\xi)^m} \\ .... & .... & .... & ....\ .... \\ \omega_1^{l_n} + \sum\limits_{m=1}^{l_n}\frac{\alpha_m^{(l_n)}\omega_1^{l_n-m}}{(iK_\xi)^m} & \omega_2^{l_1} + \sum\limits_{m=1}^{l_n}\frac{\alpha_m^{(l_n)}\omega_2^{l_n-m}}{(iK_\xi)^m} & ... & ...\omega_2^{l_1} + \sum\limits_{m=1}^{l_n}\frac{\alpha_m^{(l_n)}\omega_n^{l_n-m}}{(iK_\xi)^m} \end{vmatrix}$$

$$\Delta_k = \left| \begin{array}{ccc} \omega_1^{l_1} + \sum\limits_{m=1}^{l_1} \dfrac{\alpha_m^{(l_1)}\omega_1^{l_1-m}}{(iK_\xi)^m} \; ... & -\sum\limits_{k=1}^{n}\left[\omega_k^{l_1+1} + \sum\limits_{m=1}^{l_1}\dfrac{\alpha_m^{(l_1)}\omega_k^{l_1-m+1}}{(iK_\xi)^m}\right] e^{i\omega_k K_\xi \eta} \; ... & ... \; \omega_n^{l_1} + \sum\limits_{m=1}^{l_1}\dfrac{\alpha_m^{(l_1)}\omega_n^{l_1-m}}{(iK_\xi)^m} \\ .... & .... \; ... \; .... & ... \; .... \; ... \; .... \\ \omega_1^{l_n} + \sum\limits_{m=1}^{ln} \dfrac{\alpha_m^{(ln)}\omega_1^{ln-m}}{(iK_\xi)^m} & -\sum\limits_{k=1}^{n}\left[\omega_k^{l_1+1} + \sum\limits_{m=1}^{l_1}\dfrac{\alpha_m^{(l_1)}\omega_k^{l_1-m+1}}{(iK_\xi)^m}\right] e^{i\omega_k K_\xi \eta} .. & ...\omega_n^{l_1} + \sum\limits_{m=1}^{l_n}\dfrac{\alpha_m^{(l_n)}\omega_n^{l_n-m}}{(iK_\xi)^m} \end{array} \right|$$

If we expand the determinant Δ_k with respect to the elements of the k-th column, it is easy to see that

$$\Delta_k = -\omega_k e^{i\omega_k K_\xi \eta}\Delta_0$$

Then

$$A_k = \frac{\Delta_k}{\Delta_0} = -\omega_k e^{i\omega_k K_\xi \eta}$$

Substituting the expression of A_k in (10), we get

$$V(x,\eta,\xi,\mu) = \frac{K_\xi^{1-2n}}{2ni}\sum_{k=1}^{n}\omega_k e^{i\omega_k K_\xi (x+\eta)} \tag{11}$$

Then the Green function of the operator L_1 will be of the form

$$G_1(x,\eta,\xi,\mu) = \frac{K_\xi^{1-2n}}{2ni}\sum_{k=1}^{n}\omega_k e^{i\omega_k |x+\eta|} - \frac{K_\xi^{1-2n}}{2ni}\sum_{k=1}^{n}\omega_k e^{i\omega_k K_\xi (x+\eta)} \tag{12}$$

We can write the function $G_1(x,\eta,\xi,\mu)$ in the form

$$G_1(x,\eta,\xi,\mu) = \begin{cases} \frac{K_\xi^{1-2n}}{2ni}\sum\limits_{k=1}^{n}\omega_k e^{i\omega_k (x+\eta)}\left\{E - e^{2i\omega_k K_\xi \eta}\right\}, \; x \geq \eta \\ \frac{K_\xi^{1-2n}}{2ni}\sum\limits_{k=1}^{n}\omega_k e^{i\omega_k (x-\eta)}\left\{E - e^{2i\omega_k K_\xi \eta}\right\}, \; x \leq \eta \end{cases} \tag{13}$$

Hence we get that as $\mu \to \infty$ for the function $G_1(\eta,x,\xi,\mu)$ it holds the asymptotic equality

$$G_1(x,\eta,\xi,\mu) = \frac{K_\xi^{1-2n}}{2ni}\sum_{k=1}^{n}\omega_k e^{i\omega_k K_\xi |x+\eta|}\left[E + r(x,\eta,\xi,\mu)\right]$$

Here as $\mu \to \infty$ it holds $\|r(x,\eta,\xi,\mu)\|_H = O(1)$ uniformly with respect to (x,η).

Construction of the Green function of the operator L_0

As is known [4] the Green function $G_0(x,\eta,\mu)$ of the operator L_0 satisfies the following integral equation:

$$G_0(x,\eta;\mu) = G_1(x,\eta;\mu) - \int_0^\infty G_1(x,\xi,\mu)\left[Q(\xi) - Q(x)\right]G_0(x,\eta;\mu)\,d\xi \tag{14}$$

where $G_1(x,\eta;\mu)$ is the Green function of the operator L_1

For studying the solution of integral equation (14), following the paper [1], we introduce the Banach spaces X_1,X_2,$X_3^{(p)}$,$X_2^{(s)}$,$X_4^{(s)}$ and $X_2^{(5)}$ $(p\geq 1, s\geq 0)$, whose elements are the operator functions in $A(x,\eta)$ in the space H $(0\leq x,\eta<\infty)$, and the norms are determined in the following form:

$$\|A(x,\eta)\|_{X_1}^2=\int\limits_0^\infty\left\{\int\limits_0^\infty\|A(x,\eta)\|_H^2\,d\eta\right\}dx,$$

$$\|A(x,\eta)\|_{X_2}^2=\int\limits_0^\infty\left\{\int\limits_0^\infty\|A(x,\eta)\|_2^2\,d\eta\right\}dx.$$

Here by $\|A(x,\eta)\|_2^2$ we denote the Hilbert- Schmidt norm (absolute- norm) of the vector-function $A(x,\eta)$ in H.

$$\|A(x,\eta)\|_{X_3^{(p)}}=\left[\sup_{0\leq x<\infty}\int\limits_0^\infty\|A(x,\eta)\|_H^p\,d\eta\right]^{\frac{1}{p}},$$

$$\|A(x,\eta)\|_{X_2^{(s)}}=\int\limits_0^\infty\left[\int\limits_0^\infty\left\|A(x,\eta)\,Q^{(s)}(\eta)\right\|_2^2d\eta\right],$$

$$\|A(x,\eta)\|_{X_3^{(p)}}=\sup_{0\leq x<\infty}\int\limits_0^\infty\|A(x,\eta)\,Q^s(\eta)\|_H\,d\eta,$$

$$\|A(x,\eta)\|_{X_5}=\sup_{0\leq x<\infty}\sup_{0\leq\eta<\infty}\int\limits_0^\infty\|A(x,\eta)\|_H\,.$$

Definition and proof of their completeness in the case $-\infty<x,\eta<\infty$ were given by B.M.Levitan in [1].

Determine the following integral operator:

$$NA(x,\eta)=\int\limits_0^\infty G_1(x,\xi;\mu)\,[Q(\xi)-Q(x)]\,A(\xi,\eta)\,d\xi \tag{15}$$

The kernel $G_1(x,\xi;\mu)\,[Q(\xi)-Q(x)]$ is a bounded operator in H with respect to variables $(x,\xi,)\,,0\leq x,\xi<\infty$ for $\mu>0$. Indeed, for $|x-\xi|\leq 1$

$$\|G_1(x,\xi;\mu)\,[Q(\xi)-Q(x)]\|_H=$$

$$=\frac{1}{2n}\left\|[Q(\xi)+\mu E]^{\frac{1-2n}{2n}}\sum_{k=1}^n\omega_k e^{i\omega_k[Q(\xi)+\mu E]\frac{1}{2n}|x+\xi|}\times\right.$$

$$\times[E+r(x,\eta,\xi,\nu)]\,[Q(\xi)-Q(x)]\|_H\leq$$

$$\leq \frac{1+O(1)}{2}\sum_{k=1}^{n}\left\|[Q(\xi)+\mu E]^{\frac{1-2n}{2n}}\, e^{i\omega_k [Q(\xi)+\mu E]\frac{1}{2n}|x-\xi|}\,[Q(\xi)-Q(x)]\right\| \leq$$

$$\leq \frac{1+O(1)}{2}\left\|[Q(\xi)+\mu E]^{\frac{1-2n}{2n}}\,[Q(\xi)-Q(x)]\right\| \leq C$$

For $|x-\xi|>1$:

$$\|G_1(x,\xi;\mu)\,[Q(\xi)-Q(x)]\| \leq$$

$$\leq \frac{1+O(1)}{2}\left\|\exp\left\{-Jm\omega_1 (Q(\xi)+\mu E)^{\frac{1}{2n}}\,|x-\xi|\,\{Q(\xi)-Q(x)\}\right\}\right\| \leq$$

$$\leq c\left\|\exp\left(-\frac{Jm\omega_1}{2}(Q(\xi)+\mu E)^{\frac{1}{2n}}\,|x-\xi|\right)Q(\xi)\right\| +$$

$$+c\left\|\exp\left(-\frac{Jm\omega_1}{2}(Q(\xi)+\mu E)^{\frac{1}{2n}}\right)|x-\xi|\right\| \leq c$$

Therefore it makes sense to consider the operator T generated by the kernel $G_1(x,\xi;\mu)\,[Q(\xi)-Q(x)]$.

Theorem. *It the operator-valued function $Q(x)$ satisfies conditions 1)-4), then for sufficiently large $\mu>0$ the operator N is a contractive operator in the spaces X_1,X_2,$X_3^{(p)}$,$X_2^{(s)}$,$X_4^{(s)}$ X_5 and the following estimation is valid*

$$\|NA(x,\eta)\|_x \leq \frac{1}{\mu^{\beta}}\|A(x,\eta)\|_x\,,\ \beta>0,\ \mu\to\infty. \tag{16}$$

In all above considered Banach spaces, equation (13) has a unique solution that may be obtained with the help of iterations method if only the operator- function $G_1(x,\eta;\mu)$ belongs to the appropriate space. From estimation (16) it follows that the norm of the operator N as $\mu\to\infty$ tends to zero.

Therefore, as $\mu\to\infty$ we get the following asymptotic equality

$$G_0(x,\eta;\mu)=G_1(x,\eta;\mu)\,[E+\beta(x,\eta,\mu)] \tag{17}$$

where $\|\beta(x,\eta,\mu)\|_H=O(1)$ as $\mu\to\infty$.

In the space H estimate the norm of the operator- function $G_1(x,\eta;\mu)$:

$$\|G_1(x,\eta;\mu)\|_H \leq \frac{1+O(1)}{2}\left\|[Q(\xi)+\mu E]^{\frac{1-2n}{2n}}\right\| \times$$

$$\times\max_k\left\|e^{i\omega_k[Q(\xi)+\mu E]\frac{1}{2n}}\,|x-\eta|\right\| = \frac{1+O(1)}{2}\left\|\int_0^{\infty}(\lambda+\mu)^{\frac{1-2n}{2n}}\,dE_\lambda(x)\right\| \times$$

$$\times\max_k\left\|\int_1^{\infty} e^{i\omega_k}(\lambda+\mu)^{\frac{1}{2n}|x-\eta|}\,dE_\lambda(x)\right\| \leq$$

$$\leq \frac{1+O(1)}{2}(1+\mu)^{\frac{1-2n}{2n}}\exp\left[-Jm\omega_1(1+\mu)\,|x-\eta|\right] \tag{18}$$

Here ω_1 is the nearest point among $\omega_1,\omega_2,...\omega_n$ to the real axis.

Hence we have:

$$\int_0^\infty \|G_1(x,\eta;\mu)\|_H^2\,d\eta \le \frac{(1+O(1))^2}{4}(1+\mu)^{\frac{1-2n}{n}}\times$$

$$\times\int_0^\infty e^{2Jm\omega_1(1+\mu)^{\frac{1-2n}{n}}|x-\eta|}d\eta \le \frac{(1+O(1))^2}{4}(1+\mu)^{\frac{1-2n}{n}}\times$$

$$\times\frac{1}{2Jm\omega_1(1+\mu)^{\frac{1}{2n}}} \le \frac{(1+O(1))}{8Jm\omega_1}(1+\mu)^{\frac{1-4n}{2n}} \tag{19}$$

For $\|G_1(x,\eta,\xi,\mu)\|_2^2$ we have (denote by $\|\cdot\|$ the Hilbert- Schmidt norm in H)

$$\|G_1(x,\eta,\xi,\mu)\|_2^2 = \frac{[1+O(1)]^2}{4n^2}\sum_{j=1}^\infty\times$$

$$\times\left|\sum_{k=1}^n\left(\beta_j(x)+\mu\right)^{\frac{1-2n}{2n}}\omega_k e^{i\omega_k\left(\beta_j(x)+\mu\right)^2|x-\eta|}\right|^2\le$$

$$\le\frac{[1+O(1)]^2}{4n^2}\sum_{j=1}^\infty\left\{\left(\beta_j(x)+\mu\right)^{\frac{2-4n}{2n}}\times\right.$$

$$\left.\times\left|\sum_{k=1}^n\omega_k e^{i\omega_k\left(\beta_j(x)+\mu\right)^{\frac{1}{2n}}|x-\eta|}\right|^2\right\}\le\frac{[1+O(1)]^2}{4n^2}\times$$

$$\times\sum_{j=1}^\infty\left(\beta_j(x)+\mu\right)^{\frac{1-2n}{2}}e^{-2Jm\omega_1\left(\beta_j(x)+\mu\right)^{\frac{1}{2n}}|x-\eta|}$$

Hence

$$\int_0^\infty\|G_1(x,\eta,\xi,\mu)\|_2^2\,d\eta\le\frac{[1+O(1)]^2}{4n^2}\sum_{j=1}^\infty\left\{\left(\beta_j(x)+\mu\right)^{\frac{1-2n}{2}}\times\right.$$

$$\left.\times\int_0^\infty e^{-2Jm\omega_1\left(\beta_j(x)+\mu\right)^{\frac{1}{2n}}|x-\eta|}d\eta\le\right.$$

$$\le\frac{(1+O(1))^2}{8nJm\omega_1}\sum_{j=1}^\infty\left(\beta_j(x)+\mu\right)^{\frac{1-4n}{2n}}=\frac{(1+O(1))^2}{8Jm\omega_1}F(x)$$

Integrating on the integral $[0,\infty)$ with respect to x, we get:

$$\int_0^\infty\left\{\|G_1(x,\eta,\xi,\mu)\|_2^2\,d\eta\right\}dx\le\frac{(1+O(1))^2}{8nJm\omega_1}\int_0^\infty F(x)\,dx<\infty \tag{20}$$

From estimations (19) and (20) we get that the function $G_1(x,\eta;\mu)$ belongs to the spaces X_3^2 and X_2 if only the operator function $Q(x)$ satisfies conditions 1) -6).

Therefore the function $G_0(x,\eta;\mu)$ for sufficiently large $\mu > 0$ is an element of the spaces X_3^2 and X_2 as well . It is proved that the solution of integral equation (14) is the Green function of the operator L_0 i.e. it satisfies all basic properties of the Green function.

Construction and asymptotics of the Green function as $\mu \to \infty$ of the operator L.

The Green function $G(x,\eta;\mu)$ of the operator L generated by expression (1) and boundary conditions (2) is sought in the form

$$G(x,\eta;\mu) = G_0(x,\eta;\mu) - \int_0^\infty G_0(x,\xi;\mu)\,\rho(\xi,\eta)\,d\xi \tag{21}$$

Using the basic properties of the Green function $G_0(x,\eta;\mu)$ for defining $\rho(\xi,\eta)$ we get the following integral equation

$$\rho(\xi,\eta) + \sum_{j=1}^{2n} Q_j(x)\,\frac{\partial^{2n-j} G_0(x,\eta;\mu)}{\partial x^{2n-j}} -$$

$$-\sum_{j=1}^{2n} Q_j(x)\int_0^\infty \frac{\partial G_0}{\partial x^{2n-j}}\,\rho(\xi,\eta)\,d\eta = 0$$

If we denote $F(x,\eta;\mu) = -\sum\limits_{j=1}^{2n} Q_j(x)\,\frac{\partial^{2n-j} G_0}{\partial x^{2n-j}}$ we get the equation

$$\rho(\xi,\eta) = F(x,\eta;\mu) - \int_0^\infty F(x,\xi;\mu)\,\rho(\xi,\eta)\,d\xi \tag{22}$$

Using the explicit form (12) of the Green function $G_0(x,\eta;\mu)$ it is easy to get the following estimation for the norm of the operator function $F(x,\eta;\mu)$:

$$\|F(x,\eta;\mu)\|_H \le c\mu^{-\gamma} e^{-Jm\omega_1\sqrt{\mu}|x-2|}$$

Hence

$$\sup_{0\le x>\infty} \int_0^\infty \|F(x,\eta;\mu)\|_H^2\,d\eta \le c\mu^{-2\gamma}$$

i.e. the function $F(x,\eta;\mu)$ is an element of the space and $X_3^{(2)}$ and as $\mu \to \infty$ tends (with respect to the norm of the space $X_3^{(2)}$) to zero. Therefore equation (22) in the space $X_3^{(2)}$ has a solution and this solution is unique. Hence in particular it follows the fact that for sufficiently large $\mu > 0$ the solution $\rho(\xi,\eta)$ of equation (22) behaves in the same way as $R(x,\eta;\mu)$.

For sufficiently large $\mu > 0$ the integral operator contained in equation (21) is contractive (and as $\mu \to \infty$ convergent to zero) and therefore as $\mu \to \infty$ we have:

$$\underset{as\ \mu\to\infty}{G(x,\eta;\mu)} = C_0(x,\eta;\mu)\,[E + q(x,\eta;\mu)]\,,\ \ where\ \ \|q(x,\eta;\mu)\|_H = 0\,(1) \tag{23}$$

From this asymptotics and from (17) it follows that as $\mu \to \infty$

$$G(x,\eta;\mu) \underset{as\ \mu\to\infty}{=} G_1(x,\eta;\mu)\left[E + p(x,\eta;\mu)\right], \ \ where \ \ \|p(x,\eta;\mu)\|_H = 0(1) \tag{24}$$

Using the expression for the function $G_1(x,\eta;\mu)$ for the function $G(x,\eta;\mu)$ we finally get:

$$G(x,\eta;\mu) = \frac{\left[Q(x)+\mu E\right]^{\frac{1-2n}{2}}}{2ni}\sum_{k=1}^{n}\omega_k e^{\left[Q(x)+\mu E\right]^{\frac{1}{2}}|x-\eta|}\left(E + \Omega(x,\eta;\mu)\right), \tag{25}$$

where $\|\Omega(x,\eta;\mu)\|_H = 0(1)$ as $\mu \to \infty$,

From asymptotics (24) and belonging of $G_1(x,\eta;\mu)$ to the space X_2 it follows that the integral operator generated by the kernel $G(x,\eta;\mu)$ is a Hilbert- Schmidt type operator, i.e.

$$\int_0^\infty\int_0^\infty \|G(x,\eta;\mu)\|_2^2\, dxd\eta < \infty$$

Since $G(x,\eta;\mu)$ is the kernel of the operator $(L+\mu E)^{-1}$ in H_1 we get that the operator L has a pure discrete spectrum $\lambda_1,\lambda_2,...,\lambda_n...$ with a unique limit point at infinity.

Asymptotics of the number of eigen values of the operator L

Denote by $\varphi_1(x)$, $\varphi_2(x),...\varphi_n(x),...$ the orthonormed eigen functions of the operator L corresponding to eigen values$\lambda_1,\lambda_2,...,\lambda_n....$ As the function $G(x,\eta;\mu)$ is the Green function of the operator L, i.e. the kernel of the resolvent $R_\mu = (L+\mu E)^{-1}$, we can write

$$\varphi_n(x) = (\lambda_n+\mu)\int_0^\infty G(x,\eta;\mu)\,\varphi_n(\eta)\,d\eta \tag{26}$$

Since as $\mu\to\infty$ $G(x,\eta;\mu) = G_1(x,\eta;\mu)\left[E+p(x,\eta;\mu)\right]$, where

$$G_1(x,\eta;\mu) = \frac{\left[Q(x)+\mu E\right]^{\frac{1-2n}{2}}}{2ni}\sum_{k=1}^{n}\omega_k e^{\omega_k\left[Q(x)+\mu E\right]^{\frac{1}{2}}|x-\eta|}\left(1+0(1)\right)$$

from (26) we get

$$\varphi_n \sim (\lambda_n+\mu)\int_0^\infty G_1(x,\eta;\mu)\,\varphi_n(\eta)\,d\eta \ \ \text{as } \mu\to\infty$$

or

$$\frac{\varphi_n(x)}{\lambda_n+\mu} \sim \int_0^\infty G_1(x,\eta;\mu)\,\varphi_n(\eta)\,d\eta \tag{27}$$

Denote $a_n = \int\limits_0^\infty G_1(x,\eta;\mu)\,\varphi_n(\eta)\,d\eta$. Then we have

$$\frac{\|\varphi_n(x)\|_H^2}{(\lambda_n+\mu)^2} \sim \|a_n\|_H^2$$

Hence

$$\sum_{n=1}^N \frac{1}{(\lambda_n+\mu)^2} \sim \int\limits_0^\infty \left(\sum_{n=1}^N \|a_n\|_H^2\right) dx \tag{28}$$

The expression for a_n is the Fourier coefficient for the operator valued function $G_1(x,\eta;\mu)$. Then from the Parseval equality we have

$$\sum_{n=1}^N \|a_n\|_H^2 = \int\limits_0^\infty \sum_{m=1}^\infty r_{mm}^2(x,\eta;\mu)\,d\eta \tag{29}$$

where $r_{ii}(x,\eta;\mu)$ are diagonal elements of the matrix corresponding to the operator $G_0(x,\eta;\mu)$ in the orthonormed basis composed of eigen vectors $\beta_m(x)$ of the operator $Q(x)$ i.e.

$$r_{mm} = \frac{\{\beta_{mm}(x)+\mu\}^{\frac{1-2n}{2}}}{2ni} \sum_{a=1}^n \omega_a e^{i\omega_a\{\beta_{mm}(x)+\mu\}^{\frac{1}{2n}}|x-\eta|} (1+0(1))$$

Then from (29) we get:

$$\sum_{n=1}^\infty \|a_n\|^2 =$$

$$= \int\limits_0^\infty \sum_{m=1}^\infty \left\{ \frac{\{\beta_m(x)+\mu\}^{\frac{1-2n}{2}}}{2ni} \sum_{\alpha=1}^n \omega_a e^{i\omega_a\{\beta_m(x)+\mu\}^{\frac{1}{2n}}|x-\eta|} \right\}^2 d\eta =$$

$$= \sum_{m=1}^\infty \frac{(\beta_m(x)+\mu)^{\frac{1-2n}{2}}}{-4n^2} \left\{ \int\limits_0^\infty \left[\sum_{\alpha=1}^\infty \omega_a^2 e^{2i\omega_a[\beta_m(x)+\mu]^{\frac{1}{2n}}|x-\eta|} + \right.\right.$$

$$\left.\left. + 2\sum_{\alpha_1,\alpha_2=1}^n \omega_{\alpha_1}\omega_{\alpha_2} e^{i\left(\omega_{\alpha_1}+\omega_{\alpha_2}\right)\{\beta_m(x)+\mu\}^{\frac{1}{2n}}|x-\eta|} \right\} d\eta =$$

$$= \sum_{m=1}^\infty \frac{\{\beta_m(x)+\mu\}^{\frac{1-2n}{2}}}{-4n^2} \left\{ \sum_{\alpha=1}^n \omega_\alpha^2 \int\limits_0^\infty e^{2i\omega_a[\beta_m(x)+\mu]^{\frac{1}{2n}}|x-\eta|} d\eta + \right.$$

$$\left. + 2\sum_{\substack{\alpha_1,\alpha_2=1\\ \alpha_1\neq\alpha_2}}^n \omega_{\alpha_1}\omega_{\alpha_2} \int\limits_0^\infty e^{i\left(\omega_{\alpha_1}+\omega_{\alpha_2}\right)[\beta_m(x)+\mu]^{\frac{1}{2n}}|x-\eta|} d\eta \right\} =$$

$$= \sum_{m=1}^\infty \frac{\{\beta_m(x)+\mu\}^{\frac{1-2n}{2}}}{-4n^2} \left\{ \sum_{\alpha=1}^n \frac{\omega_\alpha^2}{2i\omega_\alpha[\beta_m+\mu]^{\frac{1}{2n}}} + \right.$$

$$+2\sum_{\substack{\alpha_1,\alpha_2=1\\ \alpha_1\neq\alpha_2}}^{n}\frac{\omega_{\alpha_1}\omega_{\alpha_2}}{(\omega_{\alpha_1}+\omega_{\alpha_2})\left[\beta_m(x)+\mu\right]^{\frac{1}{2n}}}\Bigg\}=$$

$$=\frac{1}{8}\sum_{m=1}^{\infty}\frac{\{\beta_m(x)+\mu\}^{\frac{1-2n}{2}}}{\{\beta_m(x)+\mu\}^{\frac{1}{2n}}}\left\{\frac{i}{n^2}\left[\sum_{\alpha=1}^{n}\omega_\alpha+4\sum_{\substack{\alpha_1,\alpha_2=1\\ \alpha_1\neq\alpha_2}}^{n}\frac{\omega_{\alpha_1}\omega_{\alpha_2}}{\omega_{\alpha_1}+\omega_{\alpha_2}}\right]\right\}=$$

$$=\frac{C_n}{8}\sum_{m=1}^{\infty}\left(\beta_m(x)+\mu\right)^{\frac{1-4n}{2n}},$$

where

$$C_n=\frac{i}{n^2}\left(\sum_{\substack{\alpha_1,\alpha_2=1\\ \alpha_1\neq\alpha_2}}^{n}\frac{\omega_{\alpha_1}\omega_{\alpha_2}}{\omega_{\alpha_1}+\omega_{\alpha_2}}\right),$$

so,

$$\sum_{n=1}^{\infty}\|a_n\|^2=\frac{C_n}{8}\sum_{m=1}^{\infty}\left[\beta_m(x)+\mu\right]^{\frac{1-4n}{2n}}.$$

Integrating with respect to x in the interval $(0,\infty)$ (taking into account summability of the function $F(x)=\sum\limits_{m=1}^{\infty}\left(\beta_m(x)+\mu\right)^{\frac{1-4n}{2n}}$), we get

$$\int_0^\infty\sum_{n=1}^{\infty}\|a_n\|^2\,dx=\frac{C_n}{8}\sum_{m=1}^{\infty}\frac{dx}{\left[\beta_m(x)+\mu\right]^{\frac{4n-1}{2n}}}\tag{30}$$

Taking into attention (28), we get

$$\sum_{n=1}^{\infty}\frac{1}{(\lambda_n+\mu)^2}\sim\frac{C_n}{8}\sum_{m=1}^{\infty}\frac{dx}{\left[\beta_m(x)+\mu\right]^{\frac{4n-1}{2n}}}\tag{31}$$

The known identity holds ([ ],p. 209)

$$\int_0^\infty\frac{N(\lambda)\,d\lambda}{(\lambda+\mu)^3}\sim\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{(\lambda_n+\mu)^2}.\tag{32}$$

Then from (31) and (32) we can write the relation

$$\int_0^\infty\frac{N(\lambda)\,d\lambda}{(\lambda+\mu)^3}\sim\frac{C_n}{16}\sum_{m=1}^{\infty}\int_0^\infty\frac{dx}{\left[\beta_m(x)+\mu\right]^{\frac{4n-1}{2n}}}.\tag{33}$$

For obtaining the asymptotics $N(\lambda)$, we use Titchmarsh's following tauberian theorem [9] p. 422)

Theorem. *Let $f(x)$ be a non-negative and non-decreasing function, and let as* $x \to \infty$

$$\int_0^{\infty} \frac{f(y)\,dy}{(x+y)^{\alpha}} \sim \int_{-\infty}^{\infty} \frac{d\xi}{\{q(\xi)+x\}^{\beta}} \quad where \quad \beta > 0, \alpha - \beta \geq 1$$

If $q(x)$ satisfies the condition

$$\frac{C_2}{x^e} \int_{q(\xi)<x} d\xi \leq \int_{q(\xi)<x} \frac{d\xi}{\{q(\xi)\}^{\beta}} \leq \frac{C_1}{x^{\beta}} \int_{q(\xi)<x} d\xi, \; C_1, C_2 = const,$$

then

$$f(x) \sim \frac{C\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)} \int_{q(\xi)<x} \{x - q(\xi)\}^{\alpha-\beta-1} d\xi.$$

In order to get from formula (33) the asymptotic formula $N(\lambda)$ by means of the Titchmarsh theorem, the following condition should be fulfilled:

B).There exist positive constants C_1 and C_2 such that the following inequality is fulfilled:

$$\frac{C_1}{t^{\frac{4n-1}{2n}}} \sum_{m=1}^{\infty} \int_{\beta_m(x)\leq t} dx \leq \sum_{m=1}^{\infty} \int_{\beta_m(x)>t} \frac{C_1}{\beta_m^{\frac{4n-1}{2n}}} \leq \frac{C_2}{t^{\frac{4n-1}{2n}}} \sum_{m=1}^{\infty} \int_{\beta_m(x)\leq t} dx.$$

Thus we get

Theorem *Let conditions 1)- 6) and condition B) be fulfilled. Then for the function $N(\lambda)$ the following asymptotic formula holds:*

$$N(\lambda) \sim \frac{C_n n^2}{2(2n-1)\Gamma\left(\frac{1}{2n}\right)\Gamma\left(1-\frac{1}{2n}\right)} \sum_m \int_{\beta_m(x)<\lambda} \{\lambda - \beta_m(x)\}^{\frac{1}{2n}} dx.$$

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Received February 05, 2014; Revised April 09, 2014.