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APPROXIMATE SOLUTION OF AN INVERSE PROBLEM FOR A SEMILINEAR PARABOLIC EQUATION

Abstract

The goal of the paper is the approximate solution of the inverse problem on definition of the coefficient for the minor term of a semi-linear parabolic equation in the case of a problem with a nonlinear boundary condition. For approximate solution of the inverse problem under consideration, the method of sequential approximations is used. A theorem on convergence of sequential approximations to exact solution is proved.

Let R^n be a real n - dimensional Euclidean space, $x = (x_1, ..., x_n)$ be an arbitrary point of a bounded domain $D \subset R^n$ with rather smooth boundary ∂D, $\Omega = D \times (0, T]$, $S = \partial D \times [0, T]$, $0 < T$ be a fixed number.

The space $C^l(\cdot)$, $C^{l+\alpha}(\cdot)$, $C^{l,l/2}(\cdot)$, $C^{l+\alpha,(l+\alpha)/2}(\cdot)$, $l = 0, 1, 2$, $\alpha \in (0,1)$ and the norms in these spaces were determined, for instance, in [1, p. 12-20] $\|u\|_l = \|u\|_{C^l}$, $\|g(x,t,u)\|_0 = \sup\limits_{\Omega} |g(x,t,u(x,t))|$, $u_t = \frac{\partial u}{\partial t}$, $u_{x_i} = \frac{\partial u}{\partial x_i}$, $i = \overline{1,n}$, $\Delta = \sum\limits_{i=1}^{n} \dfrac{\partial^2 u}{\partial x_i^2}$ is the Laplace operator, $\dfrac{\partial u}{\partial \nu}$ is the internal conormal derivative.

We consider an inverse problem on definition of a pair of functions $\{c(x), u(x,t)\}$ from the conditions:

$$u_t - \Delta u + c(x) u = f(x,t,u), \quad (x,t) \in \Omega, \tag{1}$$

$$u(x,0) = \varphi(x), x \in \overline{D} = D \cup \partial D, \tag{2}$$

$$\frac{\partial u}{\partial \nu} = \psi(x,t,u), \quad (x,t) \in S, \tag{3}$$

$$\int\limits_0^T u(x,t)\, dt = h(x), \quad x \in \overline{D}, \tag{4}$$

here $f(x,t,p)$, $\varphi(x)$, $\psi(x,t,p)$, $h(x)$ are the given functions.

We can cite examples that the stated problem (1)-(4) is ill-posed in the sense of Hadamard. Similar problems were studied in [2-5] (see also references in these papers).

For the input data of problem (1)-(4) we make the following suppositions:

1^0. The function $f(x,t,p)$ is determined and is continuous in
$A = \{(x,t,p) \,|\, (x,t) \in \overline{\Omega}, \quad p \in R^1\}$;

- for $m_1 > 0$ and $|p| < m_1$ the function $f(x,t,p)$ is Holder continuous with respect, for x and t with the exponents α and $\alpha/2$, respectively, for $(x,t) \in \overline{\Omega}$;

- there exists a constant $m_2 > 0$ such that for all p_1, $p_2 \in R^1$ and $(x,t) \in \overline{\Omega}$

$$|f(x,t,p_1) - f(x,t,p_2)| \leq m_2 |p_1 - p_2|$$

2^0. $\varphi(x) \in C^{2+\alpha}\left(\overline{D}\right)$, $h(x) \in C^{2+\alpha}\left(\overline{D}\right)$;

3^0. The function $\psi = \psi(x,t,p)$ satisfies the following conditions:

-the function $\psi(x,t,p)$ is determined and continuous in totality of variables in $B = \left\{(x,t,p) \,|\, (x,t) \in S, p \in R^1\right\}$;

- for $m_3 > 0$ and $|p| < m_3$ the function $\psi(x,t,p)$ is Holder discontinuous with respect to x and t with the exponents α and $\alpha/2$ respectively, for $(x,t) \in S$,

- there exists a constant $m_4 > 0$ such that for all $p_1, p_2 \in R^1$ and $(x,t) \in S$

$$|\psi(x,t,p_1) - \psi(x,t,p_2)| \le m_4 |p_1 - p_2|.$$

Definition 1. *The pair of functions $\{c(x), u(x,t)\}$ is called the solution of problem (1)-(4), if:*

1) $c(x) \in C\left(\overline{D}\right)$;

2) $u(x,t) \in C^{2,1}(\Omega) \cap C^{1,0}\left(\overline{\Omega}\right)$;

3) for these functions the relations (1)-(4) are fulfilled, and the conditions (3) are determined as follows:

$$\frac{\partial u(x,t)}{\partial \nu(x,t)} = \lim_{\substack{y \to x \\ y \in \sigma}} \frac{\partial u(y,t)}{\partial \nu(x,t)},$$

where σ is any closed cone on the vertex x that is contained in $D \cup \{x\}$.

Determine the so called correctness set K_α:

$$K_\alpha = \Big\{(u,c) \,|u(x,t) \in C^{2+\alpha,1+\alpha/2}\left(\overline{\Omega}\right), \quad c(x) \in C^{\alpha}\left(\overline{D}\right), |u(x,t)|,$$

$$|u_i(x,t)|, \quad \left|u_{x_i x_j}(x,t)\right| \le m_5, i,j = \overline{1,n},$$

$$(x,t) \in \overline{\Omega}, \quad |c(x)| \le m_6, x \in \overline{D}\Big\}.$$

In the paper [4], it is proved that if the input data of problem (1)-(4) satisfy conditions 1^0, 2^0, 3^0 respectively, then the solution of problem (1)-(4) on the set K_α is unique and stable.

In conformity to problem (1)-(4), the method of sequential approximations consists of the following: let $\left\{c^{(s)}(x), u^{(s)}(x,t)\right\}$ be already found, and $c^{(s)}(x) \in C^{\alpha}\left(\overline{D}\right)$, $u^{(s)}(x,t) \in C^{2+\alpha,1+\alpha/2}\left(\overline{\Omega}\right)$.

Let's consider a problem on definition of the function $u^{(s+1)}(x,t)$ from the conditions:

$$u_t^{(s+1)} - \Delta u^{(s+1)} + c^{(s)}(x)\, u^{(s+1)} = f\left(x,t,u^{(s)}\right), (x,t) \in \Omega, \tag{5}$$

$$u_t^{(s+1)}(x,0) = \varphi(x), \quad x \in \overline{D}, \tag{6}$$

$$\frac{\partial u^{(s+1)}(x,t)}{\partial \nu(x,t)} = \varphi\left(x,t,u^{(s)}\right), \quad (x,t) \in S. \tag{7}$$

If the input data of problem (5), (6), (7) satisfy conditions 1^0, 2^0, 3^0 respectively, this problem has a unique classic solution belonging to $C^{2+\alpha,1+\alpha/2}\left(\overline{\Omega}\right)$. Further, by the functions $u^{(s+1)}(x,t)$ from the formula

$$c^{(s+1)}(x) = \left[\varphi(x) + \Delta h(x) + \int_0^T f\left(x,t,u^{(s+1)}(x,t)\right) dt - u^{(s+1)}(x,T)\right] (h(x))^{-1}, \tag{8}$$

$c^{(s+1)}(x) \in C^{\alpha}\left(\overline{D}\right)$, are determined and the functions $\left\{c^{(s+1)}(x), u^{(s+1)}(x,t)\right\}$ are used for conducting the next interation step.

Theorem. *Let*

1. conditions 1^0, 2^0, 3^0 *be fulfilled;*

2. the solution of problem (1)-(4) exist and belong to the set K_{α}.

Then the functions $\left\{c^{(s)}(x), u^{(s)}(x,t)\right\}$ *obtained from (5), (6), (7), (8), uniformly converge to the solution of problem (1)-(4) with velocity of geometric progression.*

Proof. Integrating equation (1) with respect to t within $(0,T)$ for $c(x)$ we get:

$$c(x) = \left[\varphi(x) + \Delta h(x) + \int_0^T f(x,t,u(x,t))\,dt - u(x,T)\right](h(x))^{-1}, x \in \overline{D}. \quad (9)$$

Subtracting from relations of the system (1), (2), (3), (9) the appropriate relations of the system (5), (6), (7), (8) we get that the functions

$$z^{(s)}(x,t) = u(x,t) - u^{(s)}(x,t), \quad \lambda^{(s)}(x) = c(x) - c^{(s)(x)}$$

satisfy the conditions of the system:

$$z_t^{(s+1)} - \Delta z^{(s+1)} + c(x)\,z^{(s+1)} = -\lambda^{(s)}(x)\,u^{(s+1)} +$$

$$+f(x,t,u) - f\left(x,t,u^{(s)}\right), \quad (x,t) \in \Omega, \quad (10)$$

$$z^{(s+1)}(x,0) = 0, \quad x \in \overline{D}, \quad \frac{\partial z^{(s+1)}}{\partial \nu} = \psi(x,t,u) - \psi\left(x,t,u^{(s)}\right), \quad (x,t) \in S, \quad (11)$$

$$\lambda^{(s+1)}(x) = \left\{\int_0^T \left[f(x,t,u) - f\left(x,t,u^{(s+1)}\right)\right]dt - z^{(s+1)}(x,T)\right\} \times$$

$$\times (h(x))^{-1}, \; x \in \overline{D}. \quad (12)$$

It is easy to check that if we choose $c^{(0)}(x) \in C^{\alpha}\left(\overline{D}\right)$, $u^{(0)}(x,t) \in C^{2+\alpha,1+\alpha/2}(\Omega) \cap C^{1+\alpha,(1+\alpha)/2}\left(\overline{\Omega}\right)$, then from the assumption on the smoothness of input data and the theorem proved in [1, p. 364], it follows that for any $s = 1,2,...$ $c^{(s)}(x) \in C^{\alpha}\left(\overline{D}\right)$, $u^{(s)}(x,t) \in C^{2+\alpha,1+\alpha/2}(\Omega) \cap C^{1+\alpha,(1+\alpha)/2}\left(\overline{\Omega}\right)$. Furthermore, as it was made in [2], we can prove uniform (with respect to sup norm) boundedness of the sequences $\left\{c^{(s)}(x)\right\}$, $\left\{u^{(s)}(x,t)\right\}$. Therefore, under the supposition of the theorem, there exists a classical solution of the problem on determination of $z^{(s+1)}(x,t)$ from conditions (10), (11) and it may be represented in the form [6, p.182]

$$z^{(s+1)}(x,t) =$$

$$= \int_0^t \int_D \Gamma(x,t;\xi,\tau)\left[f(\xi,\tau,u) - f\left(\xi,\tau,u^{(s)}\right) - \lambda^{(s)}(\xi)\,u^{(s+1)}\right]d\xi d\tau +$$

$$+ \int_0^t \int_{\partial D} \Gamma(x,t;\xi,\tau)\,\rho^{(s+1)}(\xi,\tau)\,d\xi_0 d\tau, \quad (13)$$

where $\Gamma(x,t;\xi,\tau)$ is the fundamental solution of the equation $z_t^{(s+1)} - \Delta z^{(s+1)} + c(x) z^{(s+1)} = 0$, $d\xi = d\xi_1 ... d\xi_n$, $d\xi_0$ is the element of the surface ∂D, $\rho^{(s+1)}(x,t)$, $s = 0,1,...$ is the bounded, continuous solution of the following integral equation [6, p. 182]:

$$\rho^{(s+1)}(x,t) =$$

$$= 2\int_0^t \int_D \frac{\Gamma(x,t;\xi,\tau)}{\partial\nu(x,t)} \left[f(\xi,\tau,u) - f\left(\xi,\tau,u^{(s)}\right) - \lambda^{(s)}(\xi) u^{(s+1)} \right] d\xi d\tau +$$

$$+2\int_0^t \int_D \frac{\Gamma(x,t;\xi,\tau)}{\partial\nu(x,t)} \rho^{(s+1)}(\xi,\tau) d\xi_0 d\tau - 2\left[\psi(x,t,u) - \psi\left(x,t,u^{(s)}\right) \right]. \tag{14}$$

Estimate the function $\left| z^{(s+1)}(x,t) \right|$. From (13) we have:

$$\left| z^{(s+1)}(x,t) \right| \leq$$

$$\leq \int_0^t \int_D |\Gamma(x,t;\xi,\tau)| \left| f(\xi,\tau,u) - f\left(\xi,\tau,u^{(s)}\right) - \lambda^{(s)}(\xi) u^{(s+1)} \right| d\xi d\tau +$$

$$+\int_0^t \int_D |\Gamma(x,t;\xi,\tau)| \left| \rho^{(s+1)}(\xi,\tau) \right| d\xi_0 d\tau. \tag{15}$$

The following estimates are valid for the fundamental solution of $\Gamma(x,t;\xi,\tau)$ and its derivatives

$$\int_{R^n} |\Gamma(x,t;\xi,\tau)| d\xi \leq m_7,$$

$$\int_{R^n} \left| D_x^l \Gamma(x,t;\xi,\tau) \right| d\xi \leq m_8 (t-\tau)^{-\frac{l-\alpha}{2}}, l = 1,2. \tag{16}$$

The integrand expression in the first addend of the right side of (15) is estimated with regard to the theorem assumptions:

$$\left| f(x,t,u) - f\left(x,t,u^{(s)}\right) - \lambda^{(s)}(x) u^{(s+1)} \right| \leq$$

$$\leq m_9 \left| u - u^{(s)} \right| + \left| \lambda^{(s)}(x) \right| \left| u^{(s+1)}(x,t) \right| \leq m_{10} \gamma^{(s)}, \tag{17}$$

where $m_9,\ m_{10} > 0$ depend on the data of problem (1)-(4) and the set K_α, $\gamma^{(s)} = \left\| u - u^{(s)} \right\|_0 + \left\| c - c^{(s)} \right\|_0$.

Taking into account the Gauss-Ostrogradsky formula and estimation (16), for $l = 1$ we get

$$\int_{\partial D} |\Gamma(x,t;\xi,\tau)| d\xi_0 \leq m_{11} (t-\tau)^{-\frac{1-\alpha}{2}}, \tag{18}$$

$\left|\rho^{(s+1)}(x,t)\right|$ is the integrand function in the second addend of the right side of (15) and is estimated from (13) subject to the theorem conditions, estimations (16), (17):

$$\left|\rho^{(s+1)}(x,t)\right| \leq 2\int_0^t \int_D \left|\frac{\partial\Gamma(x,t;\xi,\tau)}{\partial\nu(x,t)}\right| \times$$

$$\times \left|f(\xi,\tau,u) - f\left(\xi,\tau,u^{(s)}\right) - \lambda^{(s)}(\xi)\, u^{(s+1)}\right| d\xi d\tau +$$

$$+2\int_0^t \int_{\partial D} \left|\frac{\partial\Gamma(x,t;\xi,\tau)}{\partial\nu(x,t)}\right| \left|\rho^{(s+1)}(\xi,\tau)\right| d\xi_0 d\tau + 2\left|\psi(x,t,u) - \psi\left(x,t,u^{(s)}\right)\right| \leq$$

$$\leq 2m_{10}\gamma^{(s)} t^{\frac{1+\alpha}{2}} + m_{12}\left\|\rho^{(s+1)}\right\|_0 t^{\frac{\alpha}{2}} + m_{13}\left\|z^{(s)}\right\|_0,$$

or

$$\left|\rho^{(s+1)}(x,t)\right| \leq m_{14}\gamma^{(s)} + m_{12}\left\|\rho^{(s+1)}\right\|_0 t^{\frac{\alpha}{2}}, \quad (x,t)\in\overline{\Omega}.$$

The last inequality is fulfilled for all $(x,t)\in\overline{\Omega}$. Therefore it should be fulfilled for maximal values of the left side:

$$\left\|\rho^{(s+1)}\right\|_0 \leq m_{14}\gamma^{(s)} + m_{12}\left\|\rho^{(s+1)}\right\|_0 t^{\frac{\alpha}{2}}.$$

Let T_1 $(0 < T_1 \leq T)$ be so small that $m_{12}T_1^{\frac{\alpha}{2}} < 1$. Then from the last inequality we get

$$\left\|\rho^{(s+1)}\right\|_0 \leq m_{15}\gamma^{(s)}, \quad s = 0,1,... \tag{19}$$

where $m_{15} > 0$ depends on the data of problem (1)-(4) and the set K_α.

Taking into account estimations (16), inequalities (17), (18), (19), from (15) we get:

$$\left|z^{(s+1)}(x,t)\right| \leq m_{16}\gamma^{(s)} t + m_{17}\gamma^{(s)} t^{\frac{1+\alpha}{2}}$$

or

$$\left|z^{(s+1)}(x,t)\right| \leq m_{18}\gamma^{(s)} t^{\frac{1+\alpha}{2}}, \quad (x,t)\in\overline{\Omega},$$

where $m_{18} > 0$ depends on the data of problem (1)-(4) and the set K_α.

The last inequality is fulfilled for all $(x,t)\in\overline{\Omega}$. Therefore it should be fulfilled also for maximal values of the left side:

$$\left\|z^{(s+1)}\right\|_0 \leq m_{18}\gamma^{(s)} t^{\frac{1+\alpha}{2}}, \quad s = 0,1,... \tag{20}$$

Now estimate the function $\left|\lambda^{(s)}(x)\right|$. From equality (12) we get

$$\left|\lambda^{(s)}(x)\right| \leq \left[\int_0^T \left|f(x,t,u) - f\left(x,t,u^{(s+1)}\right)\right| dt + \left|z^{(s+1)}(x,T)\right|\right] |h(x)|^{-1}.$$

Taking the theorem conditions, inequality (20), from the last inequality we have:

$$\left|\lambda^{(s+1)}(x)\right| \leq m_{19}\gamma^{(s)} t^{\frac{1+\alpha}{2}}, \quad x\in\overline{D}.$$

The last inequality is fulfilled for all $x \in \overline{D}$. So, we can affirm that

$$\left\| \lambda^{(s+1)} \right\|_0 \leq m_{19} \gamma^{(s)} t^{\frac{1+\alpha}{2}}, \tag{21}$$

where $m_{19} > 0$ depends on the data of problem (1)-(4) and the set K_α.

From inequality (20) and (21) we have:

$$\gamma^{(s+1)} \leq m_{20} \gamma^{(s)} t^{\frac{1+\alpha}{2}}, \tag{22}$$

Successively applying the inequality (22), we get

$$\gamma^{(s+1)} \leq \sigma^s \gamma^{(0)}, \quad \sigma = m_{20} t^{\frac{1+\alpha}{2}}. \tag{23}$$

Let T_2 $(0 < T_2 \leq T)$ be a number such that $m_{20} T_2^{\frac{1+\alpha}{2}} < 1$. Consequently, the sequence $\left\{\gamma^{(s)}\right\}$ is majorized for $(x,t) \in \overline{D} \times [0, T^*]$, $T^* = \min(T_1, T_2)$ with decreasing geometric progression, i.e. $\gamma^{(s)} \to 0$ as $s \to 0$ no slower than geometric progression.

Thus, we get that the functions $c^{(s)}(x)$, $u^{(s)}(x,t)$ obtained from (10), (11), (12) uniformly converge to the solution of problem (1), (2), (3), (4) as $s \to \infty$ with convergence rate no slower than the convergence rate of geometric progression.

The theorems proved.

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