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COMMUTATORS OF VECTOR-VALUED INTRINSIC SQUARE FUNCTIONS ON VECTOR-VALUED GENERALIZED MORREY SPACES

Abstract

In this paper, we will obtain the strong type and weak type estimates for vector-valued analogues of intrinsic square functions in the generalized Morrey spaces $M^{\Phi,\varphi}(l^2)$. We study the boundedness of intrinsic square functions including the Lusin area integral, Littlewood-Paley g*-function and* g^*_λ *-function and their commutators on vector-valued generalized Morrey spaces* $M^{\Phi,\varphi}(l^2)$. *In all the cases the conditions for the boundedness are given either in terms of Zygmund-type integral inequalities on* $\varphi(x,r)$ *without assuming any monotonicity property of* $\varphi(x,r)$ *on* r.

1. Introduction

It is well-known that the commutator is an important integral operator and it plays a key role in harmonic analysis. In 1965, Calderon [2], [3] studied a kind of commutators, appearing in Cauchy integral problems of Lip-line. Let K be a Calderón-Zygmund singular integral operator and $b \in BMO(\mathbb{R}^n)$. A well known result of Coifman, Rochberg and Weiss [9] states that the commutator operator $[b,K]f = K(bf) - b\,Kf$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \imath$. The commutator of Calderón-Zygmund operators plays an important role in studying the regularity of solutions of elliptic partial differential equations of second order (see, for example, [6]-[8], [5], [10], [11]).

The classical Morrey spaces were originally introduced by Morrey in [25] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [10], [11], [16], [25].

For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r) = \{y \in \mathbb{R}^n \ : \ |x-y| < r\}$, denote the open ball centered at x of radius r. The intrinsic square functions were first introduced by Wilson in [29], [30]. They are defined as follows. For $0 < \alpha \le 1$, let C_α be the family of functions $\phi : \mathbb{R}^n \to \mathbb{R}$ such that ϕ's support is contained in $B(0,1)$, $\int_{\mathbb{R}^n} \phi(x)dx = 0$, and for $x, \ x' \in \mathbb{R}^n$, $|\phi(x) - \phi(x')| \le |x - x'|^\alpha$.

For $(y, \ t) \in \mathbb{R}^{n+1}_+$ and $f \in L^{1,loc}(\mathbb{R}^n)$, set $A_\alpha f(y,t) \equiv \sup\big\{|f * \phi_t(y)| : \phi \in C_\alpha\big\}$, where $\phi_t(y) = t^{-n}\phi(\frac{y}{t})$. Then we define the varying-aperture intrinsic square (intrinsic Lusin) function of f by the formula

$$G_{\alpha,\beta}(f)(x) = \left(\int\int_{\Gamma_\beta(x)} (A_\alpha f(y,t))^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}}$$

where $\Gamma_\beta(x) = \{(y,\ t) \in \mathbb{R}^{n+1}_+ : |x-y| < \beta t\}$. Denote $G_{\alpha,1}(f) = G_\alpha(f)$.

This function is independent of any particular kernel, such as Poisson kernel. It dominates pointwise the classical square function(Lusin area integral) and its real-variable generalizations. Although the function $G_{\alpha,\beta}(f)$ is depend of kernels with uniform compact support, there is pointwise relation between $G_{\alpha,\beta}(f)$ with different β: $G_{\alpha,\beta}(f)(x) \le \beta^{\frac{3n}{2}+\alpha} G_\alpha(f)(x)$. We can see details in [29].

The intrinsic Littlewood-Paley g-function and the intrinsic g^*_λ function are defined respectively by

$$\mathrm{g}_\alpha f(x) = \left(\int_0^1 (A_\alpha f(x,t))^2 \frac{dt}{t} \right)^{\frac{1}{2}}$$

$$\mathrm{g}^*_{\lambda,\alpha} f(x) = \left(\int\int_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|x-y|} \right)^{n\lambda} (A_\alpha f(y,t))^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}}$$

When we say that f maps into l^2, we mean that $\vec{f}(x) = \left(f_j\right)^{1}_{j=1}$, where each f_j is Lebesgue measurable and, for almost every $x \in \mathbb{R}^n$ $\|\vec{f}(x)\|_{l^2} = \big(\sum\limits_{j=1}^{1} |f_j(x)|^2 \big)^{1/2}$. Let $\vec{f} = (f_1,\ f_2,\ \ldots)$ be a sequence of locally integrable functions on $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$, Wilson [30] also defined the vector-valued intrinsic square functions of $\vec{f}$ by $\|G_\alpha \vec{f}(x)\|_{l^2}$ and proved the following result.

Theorem A. *Let $1 \le p < \imath$ and $0 < \alpha \le 1$. Then the operators G_α and $\mathrm{g}^*_{\lambda,\alpha}$ are bounded from $L^p(l^2)$ into itself for $p > 1$ and from $L^1(l^2)$ to $WL^1(l^2)$.*

Moreover, in [24], Lerner showed sharp L^p_w norm inequalities for the intrinsic square functions in terms of the A_p characteristic constant of w for all $1 < p < \imath$. Also Huang and Liu [12] studied the boundedness of intrinsic square functions on weighted Hardy spaces. Moreover, they characterized the weighted Hardy spaces by intrinsic square functions. In [27] and [28], Wang and Liu obtained some weak type estimates on weighted Hardy spaces. In [26], Wang considered intrinsic functions and the commutators generated with BMO functions on weighted Morrey spaces. Let b be a locally integrable function on $\mathbb{R}^n$ Setting

$$A_{\alpha,b} f(y,t) \equiv \sup_{\phi \in C_\alpha} \left| \int_{\mathbb{R}^n} [b(x) - b(z)] \phi_t(y-z) f(z) dz \right|,$$

the commutators are defined by

$$[b, G_\alpha] f(x) = \left(\int\int_{\Gamma(x)} (A_{\alpha,b} f(y,t))^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}}$$

$$[b, \mathrm{g}_\alpha] f(x) = \left(\int_0^1 (A_{\alpha,b} f(x,t))^2 \frac{dt}{t} \right)^{\frac{1}{2}}$$

and

$$[b, \mathrm{g}^*_{\lambda,\alpha}] f(x) = \left(\int\int_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} (A_{\alpha,b} f(y,t))^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}}$$

A function $b \in L_1^{loc}(\mathbb{R}^n)$ is said to be in $BMO(\mathbb{R}^n)$ if

$$\|b\|_* = \sup_{x\in\mathbb{R}^n,\, r>0} \frac{1}{|B(x,r)|}\int_{B(x,r)} |b(y) - b_{B(x,r)}|dy < \infty,$$

where $b_{B(x,r)} = \frac{1}{|B(x,r)|}\int_{B(x,r)} b(y)dy$.

In [26], Wang proved the following result.

Theorem B. *Let $1 < p < \infty$, $0 < \alpha \le 1$ and $b \in BMO(\mathbb{R}^n)$. Then the commutator operators $[b, G_\alpha]$ and $[b, \mathrm{g}^*_{\lambda,\alpha}]$ are bounded from $L^p(l^2)$ into itself.*

In this paper, we will consider the boundedness of the operators G_α, g_α, $\mathrm{g}^*_{\lambda,\alpha}$ and their commutators on vector-valued generalized Morrey spaces. Let $\varphi(x,\ r)$ be a positive measurable function on $\mathbb{R}^n \times \mathbb{R}_+$. For any $\vec{f} \in L^p_{loc}(l^2)$, we denote by $M^{p,\varphi}(l^2)$ the vector-valued generalized Morrey spaces, if

$$\|\vec{f}\|_{M^{p,\varphi}(l^2)} = \sup_{x\mathbb{R}^n,\, r>0} \varphi(x,\ r)^{-1}\, |B(x,r)|^{-\frac{1}{p}}\, \|\|\vec{f}(\cdot)\|_{l^2}\|_{L^p(B(x,r))} < \infty.$$

There are many papers discussed the conditions on $\varphi(x,r)$ to obtain the boundedness of operators on the generalized Morrey spaces. For example, in [15] (see, also [16]), by Guliyev the following condition was imposed on the pair (φ_1, φ_2) :

$$\int_r^\infty \varphi_1(x,t)\frac{dt}{t} \le C\varphi_2(x,r). \tag{1}$$

where $C > 0$ does not depend on x and r. Under the above condition, they obtained the boundedness of Calderón-Zygmund singular integral operators from $M^{p,\varphi_1}(\mathbb{R}^n)$ to $M^{p,\varphi_2}(\mathbb{R}^n)$. Also, in [1] and [18], Guliyev et. introduced a weaker condition: If $1 \le p < \infty$, there exits a constant $C > 0$, such that, for any $x \in \mathbb{R}^n$ and $r > 0$,

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s)s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}}dt \le C\,\varphi_2(x,r). \tag{2}$$

If the pair (φ_1, φ_2) satisfies condition (1), then (φ_1, φ_2) satisfied condition (2). But the opposite is not true. We can see remark 4.7 in [18] for details.

In this paper, we will obtain the boundedness of the vector-valued intrinsic function, the intrinsic Littlewood-Paley g function, the intrinsic g^*_λ function and their commutators on vector-valued generalized Morrey spaces when the pair (φ_1, φ_2) satisfies condition (2) or the following inequalities,

$$\int_r^\infty \left(1 + \ln\frac{t}{r}\right) \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s)s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}}dt \le C\,\varphi_2(x,r), \tag{3}$$

where C does not depend on x and r. Our main results in this paper are stated as follows.

Theorem 1 *Let $1 \le p < \infty$, $0 < \alpha \le 1$ and (φ_1, φ_2) satisfies condition (2). Then the operator G_α is bounded from $M^{p,\varphi_1}(l^2)$ to $M^{p,\varphi_2}(l^2)$ for $p > 1$ and from $M^{1,\varphi_1}(l^2)$ to $WM^{1,\varphi_2}(l^2)$.*

Theorem 2 *Let* $1 \le p < \imath$, $0 < \alpha \le 1$, $\lambda > 3 + \dfrac{\alpha}{n}$ *and* (φ_1, φ_2) *satisfies condition* (2)*. Then the operator* $\mathrm{g}^*_{\lambda,\alpha}$ *is bounded from* $M^{p,\varphi_1}(l^2)$ *to* $M^{p,\varphi_2}(l^2)$ *for* $p > 1$ *and from* $M^{1,\varphi_1}(l^2)$ *to* $WM^{1,\varphi_2}(l^2)$*.*

Theorem 3 *Let* $1 < p < \imath$, $0 < \alpha \le 1$, $b \in BMO$ *and* (φ_1, φ_2) *satisfies condition* (3)*. Then* $[b, G_\alpha]$ *is bounded from* $M^{p,\varphi_1}(l^2)$ *to* $M^{p,\varphi_2}(l^2)$ *.*

Theorem 4 *Let* $1 < p < \imath$, $0 < \alpha \le 1$, $b \in BMO$ *and* (φ_1, φ_2) *satisfies condition* (3)*, then for* $\lambda > 3 + \frac{\alpha}{n}$, $[b, \mathrm{g}^*_{\lambda,\alpha}]$ *is bounded from* $M^{p,\varphi_1}(l^2)$ *to* $M^{p,\varphi_2}(l^2)$*.*

In [29], the author proved that the functions $G_\alpha f$ and $\mathrm{g}_\alpha f$ are pointwise comparable. Thus, as a consequence of Theorem 1 and Theorem 3, we have the following results.

Corollary 5 *Let* $1 \le p < \imath$, $0 < \alpha \le 1$ *and* (φ_1, φ_2) *satisfies condition* (2)*, then* g_α *is bounded from* $M^{p,\varphi_1}(l^2)$ *to* $M^{p,\varphi_2}(l^2)$ *for* $p > 1$ *and from* $M^{1,\varphi_1}(l^2)$ *to* $WM^{1,\varphi_2}(l^2)$*.*

Corollary 6 *Let* $1 < p < \imath$, $0 < \alpha \le 1$, $b \in BMO$ *and* (φ_1, φ_2) *satisfies condition* (3)*, then* $[b, \mathrm{g}_\alpha]$ *is bounded from* $M^{p,\varphi_1}(l^2)$ *to* $M^{p,\varphi_2}(l^2)$*.*

Remark 7 Note that, in the scalar valued case the Theorems 1 - 4 and Corollaries 5 - 6 was proved in [19].

Throughout this paper, we use the notation $A \lesssim B$ to mean that there is a positive constant C independent of all essential variables such that $A \le CB$. Moreover, C may be different from place to place.

2. Vector-valued generalized Morrey spaces

The classical Morrey spaces $M^{p,\lambda}$ were originally introduced by Morrey in [25] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [13],[23].

We denote by $M^{p,\lambda}(l^2) \equiv M^{p,\lambda}(\mathbb{R}^n, l^2)$ the vector-valued Morrey space, the space of all vector-valued functions $\vec{f} \in L^p_{\rm loc}(l^2)$ with finite quasinorm

$$\left\| \vec{f} \right\|_{M^{p,\lambda}(l^2)} = \sup_{x \in \mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|\vec{f}\|_{L^p(B(x,r),l^2)},$$

where $1 \le p < \imath$ and $0 \le \lambda \le n$. Note that $M^{p,0}(l^2) = L^p(l^2)$ and $M^{p,n}(l^2) = L^\imath(l^2)$. If $\lambda < 0$ or $\lambda > n$, then $M^{p,\lambda}(l^2) = \Theta$, where Θ is the set of all vector-valued functions equivalent to 0 on $\mathbb{R}^n$.

We define the vector-valued generalized Morrey spaces as follows.

Definition 8 *Let $1 \leq p < \infty$ and φ be a positive measurable vector-valued function on $\mathbb{R}^n \times (0,\infty)$. We denote by $M^{p,\varphi}(l^2)$ the vector-valued generalized Morrey space, the space of all vector-valued functions $\vec{f} \in L^p_{\rm loc}(l^2)$ with finite norm*

$$\|\vec{f}\|_{M^{p,\varphi}(l^2)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \, |B(x,r)|^{-\frac{1}{p}} \, \|f\|_{L^p(B(x,r),l^2)},$$

where $L^p(B(x,r),l^2)$ denotes the vector-valued L^p-space of measurable functions f for which

$$\|\vec{f}\|_{L^p(B(x,r))} \equiv \|\vec{f}\chi_{B(x,r)}\|_{L^p(\mathbb{R}^n)} = \left(\int_{B(x,r)} \|\vec{f}(y)\|_{l^2}^p dy\right)^{\frac{1}{p}}.$$

Furthermore, by $WM^{p,\varphi}(l^2)$ we denote the vector-valued weak generalized Morrey space of all functions $f \in WL^p_{\rm loc}(l^2)$ for which

$$\|\vec{f}\|_{WM^{p,\varphi}(l^2)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \, |B(x,r)|^{-\frac{1}{p}} \, \|\vec{f}\|_{WL^p(B(x,r),l^2)} < \infty,$$

where $WL^p(B(x,r),l^2)$ denotes the vector-valued weak L^p-space of measurable functions f for which

$$\|\vec{f}\|_{WL^p(B(x,r),l^2)} \equiv \|\vec{f}\chi_{B(x,r)}\|_{WL^p(l^2)} = \sup_{t>0} t \left(\int_{\{y\in B(x,r):\, \|\vec{f}(y)\|_{l^2}>t\}} dy\right)^{\frac{1}{p}}.$$

3. Preliminaries

We are going to use the following result on the boundedness of the Hardy operator

$$(Hg)(t) := \frac{1}{t}\int_0^t g(r)d\mu(r), \;\; 0 < t < \infty,$$

where μ is a non-negative Borel measure on $(0,\infty)$.

Theorem 9 *([4]) The inequality*

$$\operatorname*{ess\,sup}_{t>0} \omega(t)Hg(t) \leq c \operatorname*{ess\,sup}_{t>0} v(t)g(t)$$

holds for all functions g non-negative and non-increasing on $(0,\infty)$ if and only if

$$A := \sup_{t>0} \frac{\omega(t)}{t} \int_0^t \frac{d\mu(r)}{\operatorname*{ess\,sup}_{0<s<r} v(s)} < \infty,$$

and $c \approx A$.

We also need the following statement on the boundedness of the Hardy type operator

$$(H_1 g)(t) := \frac{1}{t}\int_0^t \ln\left(e + \frac{t}{r}\right) g(r)d\mu(r), \;\; 0 < t < \infty,$$

where μ is a non-negative Borel measure on $(0,\infty)$.

Theorem 10 *The inequality*

$$\operatorname*{ess\,sup}_{t>0} \omega(t) H_1 g(t) \le c \operatorname*{ess\,sup}_{t>0} v(t) g(t)$$

holds for all functions g non-negative and non-increasing on $(0,\infty)$ if and only if

$$A_1 := \sup_{t>0} \frac{\omega(t)}{t} \int_0^t \ln\left(e + \frac{t}{r}\right) \frac{d\mu(r)}{\operatorname*{ess\,sup}_{0<s<r} v(s)} < \infty,$$

and $c \approx A_1$.

Note that, Theorem 10 can be proved analogously to Theorem 4.3 in [17].

Definition 11 *$BMO(\mathbb{R}^n)$ is the Banach space modulo constants with the norm $\|\cdot\|_*$ defined by*

$$\|b\|_* = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |b(y) - b_{B(x,r)}| dy < \infty,$$

where $b \in L^1_{\rm loc}(\mathbb{R}^n)$ and

$$b_{B(x,r)} = \frac{1}{|B(x,r)|} \int_{B(x,r)} b(y) dy.$$

Remark 12 (1) The John-Nirenberg inequality : there are constants C_1, $C_2 > 0$, such that for all $b \in BMO(\mathbb{R}^n)$ and $\beta > 0$

$$|\{x \in B : |b(x) - b_B| > \beta\}| \le C_1 |B| e^{-C_2\beta/\|b\|_*}, \quad \forall B \subset \mathbb{R}^n.$$

(2) For $1 \le p < \infty$ the John-Nirenberg inequality implies that

$$\|b\|_* \approx \sup_B \left(\frac{1}{|B|} \int_B |b(y) - b_B|^p dy\right)^{\frac{1}{p}}. \tag{4}$$

(3) Let $f \in BMO(\mathbb{R}^n)$. Then there is a constant $C > 0$ such that

$$\left|f_{B(x,r)} - f_{B(x,t)}\right| \le C \|f\|_* \ln\frac{t}{r} \quad \text{for} \quad 0 < 2r < t, \tag{5}$$

where C is independent of f, x, r and t (see, for example, [22], also [14]).

4. Proofs of main theorems

Before proving the main theorems, we need the following lemmas.

Lemma 13 *[26] For $j \in \mathbb{Z}_+$, denote*

$$G_{\alpha,2^j}(f)(x) = \left(\int_0^\infty \int_{|x-y|\le 2^j t} (A_\alpha f(y,t))^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}}$$

Let $0 < \alpha \le 1$ and $1 < p < \infty$. Then for any $j \in \mathbb{Z}_+$, we have

$$\|G_{\alpha,2^j}(f)\|_{L^p} \lesssim 2^{j\left(\frac{3n}{2}+\alpha\right)} \|G_\alpha(f)\|_{L^p}.$$

This lemma follows easily from the following inequality that was proved in [29].

$$G_{\alpha,\beta}(f)(x) \le \beta^{\frac{3n}{2}+\alpha} G_\alpha(f)(x).$$

By the similar argument as in [3], we can get the following lemma.

Lemma 14 *Let $1 < p < \imath$ and $0 < \alpha \le 1$, then the commutators $[b, G_\alpha]$ is bounded from $L^p(l^2)$ to itself whenever $b \in BMO$.*

Now we are in a position to prove theorems.

Lemma 15 *Let $1 \le p < \imath$ and $0 < \alpha \le 1$. Then, for $p > 1$ the inequality*

$$\|G_\alpha \vec{f}\|_{L^p(B,l^2)} \lesssim r^{\frac{n}{p}} \int_{2r}^{1} \|\vec{f}\|_{L^p\left(B(x_0,t),l^2\right)} t^{-\frac{n}{p}} \frac{dt}{t}$$

holds for any ball $B = B(x_0, r)$ and for all $\vec{f} \in L^p_{\rm loc}(l^2)$.

Moreover, for $p = 1$ the inequality

$$\|G_\alpha \vec{f}\|_{WL^1(B,l^2)} \lesssim r^n \int_{2r}^{1} \|\vec{f}\|_{L^1\left(B(x_0,t),l^2\right)} r^{-n} \frac{dt}{t},$$

holds for any ball $B = B(x_0, r)$ and for all $\vec{f} \in$ Ł1locl2.

Proof. The main ideas of these proofs come from [15]. For arbitrary $x \in \mathbb{R}^n$, set $B = B(x_0, r)$, $2B \equiv B(x_0, 2r)$. We decompose $\vec{f} = \vec{f_0} + \vec{f_1}$, where $\vec{f_0}(y) = \vec{f}(y)\chi_{2B}(y)$, $\vec{f_1}(y) = \vec{f}(y) - \vec{f_0}(y)$. Then,

$$\|G_\alpha \vec{f}\|_{L^p\left(B(x_0,r),l^2\right)} \le \|G_\alpha \vec{f_0}\|_{L^p\left(B(x_0,r),l^2\right)} + \|G_\alpha \vec{f_1}\|_{L^p\left(B(x_0,r),l^2\right)} := I + II.$$

First, let us estimate I. By Theorem A, we can obtain that

$$I \le \|G_\alpha \vec{f_0}\|_{L^p(l^2)} \lesssim \|\vec{f_0}\|_{L^p(l^2)} = \|\vec{f}\|_{L^p(2B,l^2)}. \tag{6}$$

On the other hand,

$$\lesssim r^{\frac{n}{p}} \int_{2r}^{1} \|\vec{f}\|_{L^p\left(B(x_0,t),l^2\right)} t^{-\frac{n}{p}-1} \, dt. \tag{7}$$

Therefore from (6) and (7) we get

$$I \lesssim r^{\frac{n}{p}} \int_{2r}^{1} \|\vec{f}\|_{L^p\left(B(x_0,t),l^2\right)} t^{-\frac{n}{p}} \, dt. \tag{8}$$

Then let us estimate II.

$$\|\vec{f} * \phi_t(y)\|_{l^2} = \left\| t^{-n} \int_{|y-z|\le t} \phi\left(\frac{y-z}{t}\right) \vec{f_1}(z) dz \right\|_{l^2} \le t^{-n} \int_{|y-z|\le t} \|\vec{f_1}(z)\|_{l^2} dz.$$

Since $x \in B(x_0,r)$, $(y,t) \in \Gamma(x)$, we have $|z-x| \le |z-y|+|y-x| \le 2t$, and

$$r \le |z-x_0| - |x_0-x| \le |x-z| \le |x-y|+|y-z| \le 2t.$$

So, we obtain

$$\|G_\alpha \vec{f_1}(x)\|_{l^2} \le \left(\int\int_{\Gamma(x)} \left(t^{-n}\int_{|y-z|\le t} \|\vec{f_1}(z)\|_{l^2} dz\right)^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}} \le$$

$$\le \left(\int_{t>r/2}\int_{|x-y|<t} \left(\int_{|x-z|\le 2t} \|\vec{f_1}(z)\|_{l^2} dz\right)^2 \frac{dydt}{t^{3n+1}}\right)^{\frac{1}{2}} \lesssim$$

$$\lesssim \left(\int_{t>r/2} \left(\int_{|z-x|\le 2t} \|\vec{f_1}(z)\|_{l^2} dz\right)^2 \frac{dt}{t^{2n+1}}\right)^{\frac{1}{2}}.$$

By Minkowski and Hölder's inequalities and $|z-x| \ge |z-x_0| - |x_0-x| \ge \dfrac{1}{2}|z-x_0|$, we have

$$\|G_\alpha \vec{f_1}(x)\|_{l^2} \lesssim \int_{\mathbb{R}^n} \left(\int_{t>\frac{|z-x|}{2}} \frac{dt}{t^{2n+1}}\right)^{\frac{1}{2}} \|\vec{f_1}(z)\|_{l^2} dz \lesssim \int_{|z-x_0|>2r} \frac{\|\vec{f}(z)\|_{l^2}}{|z-x|^n} dz \lesssim$$

$$\lesssim \int_{|z-x_0|>2r} \frac{\|\vec{f}(z)\|_{l^2}}{|z-x_0|^n} dz = \int_{|z-x_0|>2r} \|\vec{f}(z)\|_{l^2} \int_{|z-x_0|}^{+1} \frac{dt}{t^{n+1}} dz =$$

$$= \int_{2r}^{+1} \int_{2r<|z-x_0|<t} \|\vec{f}(z)\|_{l^2} dz \frac{dt}{t^{n+1}} \lesssim \int_{2r}^{1} \|\vec{f}\|_{L^p\left(B(x_0,t),l^2\right)} t^{-\frac{n}{p}-1} dt. \tag{9}$$

Thus,

$$\|G_\alpha \vec{f_1}\|_{L^p(B,l^2)} \lesssim r^{\frac{n}{p}} \int_{2r}^{1} \|\vec{f}\|_{L^p\left(B(x_0,t),l^2\right)} t^{-\frac{n}{p}-1} dt. \tag{10}$$

By combining (8) and (10), we have

$$\|G_\alpha \vec{f}\|_{L^p(B,l^2)} \lesssim r^{\frac{n}{p}} \int_{2r}^{1} \|\vec{f}\|_{L^p\left(B(x_0,t),l^2\right)} t^{-\frac{n}{p}-1} dt.$$

Proof of Theorem 1

By Lemma 15 and Theorem 9 we have for $p>1$

$$\|G_\alpha \vec{f}\|_{M^{p,\varphi_2}(l^2)} \lesssim \sup_{x_0\in\mathbb{R}^n, r>0} \varphi_2(x_0,\ r)^{-1} \int_r^1 \|\vec{f}\|_{L^p\left(B(x_0,t),l^2\right)} t^{-\frac{n}{p}-1} dt =$$

$$= \sup_{x_0\in\mathbb{R}^n, r>0} \varphi_1(x_0,r)^{-1} r^{-\frac{n}{p}} \|\vec{f}\|_{L^p\left(B(x_0,r),l^2\right)} = \|\vec{f}\|_{M^{p,\varphi_1}(l^2)}$$

and for $p=1$

$$\|G_\alpha \vec{f}\|_{WM^{1,\varphi_2}(l^2)} \lesssim \sup_{x_0\in\mathbb{R}^n, r>0} \varphi_2(x_0,r)^{-1} \int_r^1 \|\vec{f}\|_{L^1\left(B(x_0,t),l^2\right)} t^{-n} \frac{dt}{t} =$$

$$= \sup_{x_0\in\mathbb{R}^n, r>0} \varphi_1(x_0,r)^{-1} r^{-n} \|\vec{f}\|_{L^1\left(B(x_0,r),l^2\right)} = \|\vec{f}\|_{M^{1,\varphi_1}(l^2)}.$$

Lemma 16 *Let* $1 \le p < ı$, $0 < \alpha \le 1$ *and* $\lambda > 3 + \dfrac{\alpha}{n}$. *Then, for* $p > 1$ *the inequality*

$$\|g^*_{\lambda,\alpha}(\vec{f})\|_{L^p\left(B,l^2\right)} \lesssim r^{\frac{n}{p}} \int_{2r}^{1} \|\vec{f}\|_{L^p\left(B(x_0,t),l^2\right)} \, t^{-\frac{n}{p}-1} \, dt$$

holds for any ball $B = B(x_0, r)$ *and for all* $\vec{f} \in L^p_{\rm loc}(l^2)$.

Moreover, for $p = 1$ *the inequality*

$$\|g^*_{\lambda,\alpha}(\vec{f})\|_{WL^1\left(B,l^2\right)} \lesssim r^n \int_{2r}^{1} \|\vec{f}\|_{L^1\left(B(x_0,t),l^2\right)} \, t^{-n-1} \, dt$$

holds for any ball $B = B(x_0, r)$ *and for all* $\vec{f} \in Ł1locl2$.

Proof. From the definition of $g^*_{\lambda,\alpha}(f)$, we readily see that

$$\begin{aligned}
\|g^*_{\lambda,\alpha}(\vec{f})(x)\|_{l^2} &= \Big\|\Big(\int_0^{1}\int_{\mathbb{R}^n}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}\left(A_\alpha \vec{f}(y,t)\right)^2 \frac{dydt}{t^{n+1}}\Big)^{l/2}\Big\|_{l^2} \le \\
&\le \Big\|\Big(\int_0^{1}\int_{|x-y|<t}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}\left(A_\alpha \vec{f}(y,t)\right)^2 \frac{dydt}{t^{n+1}}\Big)^{l/2}\Big\|_{l^2} + \\
&+ \Big\|\Big(\int_0^{1}\int_{|x-y|\ge t}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}\left(A_\alpha \vec{f}(y,t)\right)^2 \frac{dydt}{t^{n+1}}\Big)^{l/2}\Big\|_{l^2} := \\
&:= III + IV.
\end{aligned}$$

First, let us estimate III.

$$III \le \Big\|\Big(\int_0^{1}\int_{|x-y|<t}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}\left(A_\alpha \vec{f}(y,t)\right)^2 \frac{dydt}{t^{n+1}}\Big)^{l/2}\Big\|_{l^2} \le \|G_\alpha \vec{f}(x)\|_{l^2}.$$

Now, let us estimate IV.

$$\begin{aligned}
IV &\le \Big\|\Big(\sum_{j=1}^{1}\int_0^{1}\int_{2^{j-1}t\le|x-y|\le 2^j t}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}\left(A_\alpha \vec{f}(y,t)\right)^2 \frac{dydt}{t^{n+1}}\Big)^{l/2}\Big\|_{l^2} \lesssim \\
&\lesssim \Big\|\Big(\sum_{j=1}^{1}\int_0^{1}\int_{2^{j-1}t\le|x-y|\le 2^j t} 2^{-jn\lambda}\left(A_\alpha \vec{f}(y,t)\right)^2 \frac{dydt}{t^{n+1}}\Big)^{l/2}\Big\|_{l^2} \lesssim \\
&\lesssim \sum_{j=1}^{1} 2^{-jn\lambda}\Big\|\Big(\int_0^{1}\int_{|x-y|\le 2^j t}\left(A_\alpha \vec{f}(y,t)\right)^2 \frac{dydt}{t^{n+1}}\Big)^{l/2}\Big\|_{l^2} := \sum_{j=1}^{1} 2^{-jn\lambda}\Big\|G_{\alpha,2^j}(\vec{f})(x)\Big\|_{l^2}.
\end{aligned}$$

Thus,

$$\|g^*_{\lambda,\alpha}(\vec{f})\|_{L^p(B,l^2)} \le \|G_\alpha \vec{f}\|_{L^p(B,l^2)} + \sum_{j=1}^{1} 2^{-\frac{jn\lambda}{2}} \|G_{\alpha,2^j}(\vec{f})\|_{L^p(B,l^2)}. \tag{11}$$

By Lemma 15, we have

$$\|G_\alpha \vec{f}\|_{L^p(B,l^2)} \lesssim r^{\frac{n}{p}} \int_{2r}^{1} \|\vec{f}\|_{L^p(B(x_0,t))} \, t^{-\frac{n}{p}-1} \, dt. \tag{12}$$

In the following, we will estimate $\|G_{\alpha,2^j}(\vec{f})\|_{L^p(B,l^2)}$. We divide $\|G_{\alpha,2^j}(\vec{f})\|_{L^p(B,l^2)}$ into two parts.

$$\|G_{\alpha,2^j}(\vec{f})\|_{L^p(B,l^2)} \leq \|G_{\alpha,2^j}(\vec{f_0})\|_{L^p(B,l^2)} + \|G_{\alpha,2^j}(\vec{f_1})\|_{L^p(B,l^2)}, \tag{13}$$

where $\vec{f_0}(y) = \vec{f}(y)\chi_{2B}(y)$, $\vec{f_1}(y) = \vec{f}(y) - \vec{f_1}(y)$. For the first part, by Lemma 13,

$$\|G_{\alpha,2^j}(\vec{f_0})\|_{L^p(B,l^2)} \lesssim 2^{j(\frac{3n}{2}+\alpha)}\|G_\alpha(\vec{f_0})\|_{L^p(l^2)} \lesssim 2^{j(\frac{3n}{2}+\alpha)}\|f\|_{L^p(B,l^2)} \lesssim$$
$$\lesssim 2^{j(\frac{3n}{2}+\alpha)} r^{\frac{n}{p}} \int_{2r}^{1} \|\vec{f}\|_{L^p\left(B(x_0,t),l^2\right)} t^{-\frac{n}{p}-1}\,dt. \tag{14}$$

For the second part

$$\left\|G_{\alpha,2^j}(\vec{f_1})(x)\right\|_{l^2} = \left\| \left(\int_0^1 \int_{|x-y|\leq 2^j t} \left(\sup_{\phi\in C_\alpha} |\vec{f} * \phi_t(y)| \right)^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}} \right\|_{l^2} \leq$$
$$\leq \left(\int_0^1 \int_{|x-y|\leq 2^j t} \left(\int_{|z-y|\leq t} \|\vec{f_1}(z)\|_{l^2} dz \right)^2 \frac{dydt}{t^{3n+1}} \right)^{\frac{1}{2}}.$$

Since $|x-z| \leq |y-z| + |x-y| \leq 2^{j+1}t$, we get

$$\left\|G_{\alpha,2^j}(\vec{f_1})(x)\right\|_{l^2} \leq \left(\int_0^1 \int_{|x-y|\leq 2^j t} \left(\int_{|x-z|\leq 2^{j+1}t} \|\vec{f_1}(z)\|_{l^2} dz \right)^2 \frac{dydt}{t^{3n+1}} \right)^{\frac{1}{2}} \leq$$
$$\leq \left(\int_0^1 \left(\int_{|z-x|\leq 2^{j+1}t} \|\vec{f_1}(z)\|_{l^2} dz \right)^2 \frac{2^{jn}dt}{t^{2n+1}} \right)^{\frac{1}{2}} \leq$$
$$\leq 2^{\frac{jn}{2}} \int_{\mathbb{R}^n} \left(\int_{t\geq \frac{|x-z|}{2^{j+1}}} \|\vec{f_1}(z)\|_{l^2}^2 \frac{dt}{t^{2n+1}} \right)^{\frac{1}{2}} dz \leq 2^{\frac{3jn}{2}} \int_{|x_0-z|>2r} \frac{\|\vec{f}(z)\|_{l^2}}{|x-z|^n} dz.$$

For $|z-x| \geq |x_0-z| - |x-x_0| \geq |x_0-z| - \dfrac{1}{2}|x_0-z| = \dfrac{1}{2}|x_0-z|$, so by Fubini's theorem and Hölder's inequality, we obtain

$$\left\|G_{\alpha,2^j}(\vec{f_1})(x)\right\|_{l^2} \leq 2^{\frac{3jn}{2}} \int_{|x_0-z|>2r} \frac{\|\vec{f}(z)\|_{l^2}}{|x_0-z|^n} dz = 2^{\frac{3jn}{2}} \int_{|x_0-z|>2r} \|\vec{f}(z)\|_{l^2} \int_{|x_0-z|}^{1} \frac{dt}{t^{n+1}} dz \leq$$
$$\leq 2^{\frac{3jn}{2}} \int_{2r}^{1} \int_{|x_0-z|<t} \|\vec{f}(z)\|_{l^2} dz \frac{dt}{t^{n+1}} \leq 2^{\frac{3jn}{2}} \int_{2r}^{1} \|\vec{f}\|_{L^p\left(B(x_0,t),l^2\right)} t^{-\frac{n}{p}-1}\,dt.$$

So,

$$\|G_{\alpha,2^j}(\vec{f_1})\|_{L^p(B,l^2)} \leq 2^{\frac{3jn}{2}} r^{\frac{n}{p}} \int_{2r}^{1} \|\vec{f}\|_{L^p\left(B(x_0,t),l^2\right)} t^{-\frac{n}{p}-1}\,dt. \tag{15}$$

Combining (13), (14) and (15), we have

$$\|G_{\alpha,2^j}(\vec{f})\|_{L^p(B,l^2)} \lesssim 2^{j(\frac{3n}{2}+\alpha)} r^{\frac{n}{p}} \int_{2r}^{1} \|\vec{f}\|_{L^p\left(B(x_0,t),l^2\right)} t^{-\frac{n}{p}-1}\,dt. \tag{16}$$

Thus,

$$\|\mathrm{g}^*_{\lambda,\alpha}(\vec{f})\|_{L^p(B,l^2)} \leq \|G_\alpha \vec{f}\|_{L^p(B,l^2)} + \sum_{j=1}^{\infty} 2^{-\frac{jn\lambda}{2}} \|G_{\alpha,2^j}(\vec{f})\|_{L^p(B,l^2)}. \tag{17}$$

Since $\lambda > 3 + \dfrac{\alpha}{n}$, by (12), (16) and (17), we have the desired lemma.

Proof of Theorem 2

From inequality (18) we have

$$\|\mathrm{g}^*_{\lambda,\alpha}(\vec{f})\|_{M^{p,\varphi_2}(l^2)} \leq \|G_\alpha \vec{f}\|_{M^{p,\varphi_2}(l^2)} + \sum_{j=1}^{\infty} 2^{-\frac{jn\lambda}{2}} \|G_{\alpha,2^j}(\vec{f})\|_{M^{p,\varphi_2}(l^2)}. \tag{18}$$

By Theorem 1, we have

$$\|G_\alpha \vec{f}\|_{M^{p,\varphi_2}(l^2)} \lesssim \|\vec{f}\|_{M^{p,\varphi_1}(l^2)}. \tag{19}$$

In the following, we will estimate $\|G_{\alpha,2^j}(\vec{f})\|_{M^{p,\varphi_2}(l^2)}$. Thus, by substitution of variables and Theorem 9, we get

$$\|G_{\alpha,2^j}(\vec{f})\|_{M^{p,\varphi_2}(l^2)} \lesssim 2^{j(\frac{3n}{2}+\alpha)} \sup_{x_0\in\mathbb{R}^n, r>0} \varphi_2(x_0,r)^{-1} \int_r^{\infty} \|\vec{f}\|_{L^p\left(B(x_0,t),l^2\right)} t^{-\frac{n}{p}-1}\, dt \lesssim$$
$$\lesssim 2^{j(\frac{3n}{2}+\alpha)} \sup_{x_0\in\mathbb{R}^n, r>0} \varphi_1(x_0,r^{-1})^{-1}\, r^{\frac{n}{p}} \|\vec{f}\|_{L^p\left(B(x_0,r^{-1}),l^2\right)} = 2^{j(\frac{3n}{2}+\alpha)} \|\vec{f}\|_{M^{p,\varphi_1}(l^2)}. \tag{20}$$

Since $\lambda > 3 + \dfrac{\alpha}{n}$, by (18), (19) and (20), we have the desired theorem.

Lemma 17 *Let $1 < p < \infty$, $0 < \alpha \leq 1$ and $b \in BMO$.*

Then the inequality

$$\|[b, G_\alpha]\vec{f}\|_{L^p(B,l^2)} \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \ln\left(e + \frac{t}{r}\right) \|\vec{f}\|_{L^p\left(B(x_0,t),l^2\right)} t^{-\frac{n}{p}-1}\, dt$$

holds for any ball $B = B(x_0, r)$ and for all $f \in L^p_{\mathrm{loc}}(l^2)$.

Proof. We decompose $\vec{f} = \vec{f_0} + \vec{f_1}$, where $\vec{f_0} = \vec{f}\chi_{2B}$ and $\vec{f_1} = \vec{f} - \vec{f_0}$. Then

$$\|[b, G_\alpha]\vec{f}\|_{L^p(B,l^2)} \leq \|[b, G_\alpha]\vec{f_0}\|_{L^p(B,l^2)} + \|[b, G_\alpha]\vec{f_1}\|_{L^p(B,l^2)}.$$

By Lemma 14, we have that

$$\|[b,\ G_\alpha]\vec{f_0}\|_{L^p(B,l^2)} \lesssim \|b\|_* \|\vec{f_0}\|_{L^p(l^2)} = \|b\|_* \|\vec{f}\|_{L^p(2B,l^2)} \lesssim$$
$$\lesssim \|b\|_* \, r^{\frac{n}{p}} \int_{2r}^{\infty} \|\vec{f}\|_{L^p\left(B(x_0,t),l^2\right)} t^{-\frac{n}{p}-1}\, dt.$$

For the second part, we divide it into two parts.

$$\Big\|[b,G_\alpha]\vec{f}_1(x)\Big\|_{l^2}=\Big\|\Big(\int\int_{\Gamma(x)}\sup_{\phi\in C_\alpha}\Big|\int_{\mathbb{R}^n}[b(x)-b(z)]\phi_t(y-z)\vec{f}_1(z)dz\Big|^2\frac{dydt}{t^{n+1}}\Big)^{\frac12}\Big\|_{l^2}\leq$$

$$\leq A(x)+B(x):=\Big\|\Big(\int\int_{\Gamma(x)}\sup_{\phi\in C_\alpha}\Big|\int_{\mathbb{R}^n}[b(x)-b_B]\phi_t(y-z)\vec{f}_1(z)dz\Big|^2\frac{dydt}{t^{n+1}}\Big)^{\frac12}\Big\|_{l^2}+$$

$$+\Big\|\Big(\int\int_{\Gamma(x)}\sup_{\phi\in C_\alpha}\Big|\int_{\mathbb{R}^n}[b(z)-b_B]\phi_t(y-z)\vec{f}_1(z)dz\Big|^2\frac{dydt}{t^{n+1}}\Big)^{\frac12}\Big\|_{l^2}.$$

Therefore

$$\|[b,G_\alpha]\vec{f}_1\|_{L^p(B,l^2)}\leq\|A(\cdot)\|_{L^p(B)}+\|B(\cdot)\|_{L^p(B)}.$$

First, for $A(x)$, we find that

$$A(x)=|b(x)-b_B|\,\Big\|\Big(\iint_{\Gamma(x)}\sup_{\phi\in C_\alpha}\Big|\int_{\mathbb{R}^n}\phi_t(y-z)\vec{f}_1(z)dz\Big|^2\frac{dydt}{t^{n+1}}\Big)^{\frac12}\Big\|_{l^2}=$$
$$=\big|b(x)-b_B\big|\,\big\|G_\alpha\vec{f}_1(x)\big\|_{l^2}.$$

From the inequality (9), we can get

$$\|A(\cdot)\|_{L^p(B)}=\left(\int_B|b(x)-b_B|^p\left(\big\|G_\alpha\vec{f}_1(x)\big\|_{l^2}\right)^p w(x)dx\right)^{\frac1p}\leq$$
$$\leq\left(\int_B|b(x)-b_B|^p\,dx\right)^{\frac1p}\int_{2r}^1\|\vec{f}\|_{L^p\left(B(x_0,t),l^2\right)}\,t^{-\frac{n}{p}-1}\,dt\leq$$
$$\leq\|b\|_* r^{\frac{n}{p}}\int_{2r}^1\|\vec{f}\|_{L^p\left(B(x_0,t),l^2\right)}\,t^{-\frac{n}{p}-1}\,dt.$$

For $B(x)$, since $|y-x|<t$, we get $|x-z|<2t$. Thus, by Minkowski's inequality,

$$B(x)\leq\Big\|\Big(\int\int_{\Gamma(x)}\Big|\int_{|x-z|<2t}|b_B-b(z)|\,\vec{f}_1(z)dz\Big|^2\frac{dydt}{t^{3n+1}}\Big)^{\frac12}\Big\|_{l^2}\lesssim$$
$$\lesssim\Big(\int_0^1\Big|\int_{|x-z|<2t}|b_B-b(z)|\,\big\|\vec{f}_1(z)\big\|_{l^2}dz\Big|^2\frac{dt}{t^{2n+1}}\Big)^{\frac12}\leq$$
$$\leq\int_{|x_0-z|>2r}|b_B-b(z)|\,\big\|\vec{f}(z)\big\|_{l^2}\,\frac{dz}{|x-z|^n}.$$

For $B(x)$, using the inequality $|z-x|\geq\frac12|z-x_0|$, we have

$$B(x)\lesssim\int_{|x_0-z|>2r}|b(z)-b_B|\,\big\|\vec{f}(z)\big\|_{l^2}\,\frac{dz}{|x_0-z|^n}\lesssim$$
$$\lesssim\int_{|x_0-z|>2r}|b(z)-b_B|\,\big\|\vec{f}(z)\big\|_{l^2}\int_{|x_0-z|}^1\frac{dt}{t^{n+1}}\lesssim$$
$$\lesssim\int_{2r}^1\int_{2r\leq|x_0-z|\leq t}|b(z)-b_B|\,\big\|\vec{f}(z)\big\|_{l^2}\,dz\frac{dt}{t^{n+1}}.$$

Applying Hölder's inequality, we get

$$\|B(\cdot)\|_{L^p(B)} \lesssim r^{\frac{n}{p}} \int_{2r}^{1} \left(\int_{B(x_0,t)} |b(z)-b_B|^{p'} dz \right)^{\frac{1}{p'}} \| \|\vec{f}(\cdot)\|_{l^2} \|_{L^p(B(x_0,t))} \frac{dt}{t^{n+1}} \lesssim$$
$$\lesssim \|b\|_* r^{\frac{n}{p}} \int_{2r}^{1} \ln\left(e+\frac{t}{r}\right) \|\vec{f}\|_{L^p\left(B(x_0,t),l^2\right)} t^{-\frac{n}{p}-1}\, dt.$$

Thus,

$$\left\|[b,\ G_\alpha]\vec{f}\right\|_{L^p(B,l^2)} \lesssim \|b\|_*\, r^{\frac{n}{p}} \int_{2r}^{1} \ln\left(e+\frac{t}{r}\right) \|\vec{f}\|_{L^p\left(B(x_0,t),l^2\right)} t^{-\frac{n}{p}-1}\, dt.$$

Proof of Theorem 3

By substitution of variables, we obtain

$$\|[b,G_\alpha]\vec{f}\|_{M^{p,\varphi_2}(l^2)} \lesssim$$
$$\lesssim \|b\|_* \sup_{x_0\in\mathbb{R}^n, r>0} \varphi_2(x_0,r)^{-1} \int_{2r}^{1} \ln\left(e+\frac{t}{r}\right) \|\vec{f}\|_{L^p\left(B(x_0,t),l^2\right)} t^{-\frac{n}{p}-1}\, dt \lesssim$$
$$\lesssim \|b\|_* \sup_{x_0\in\mathbb{R}^n, r>0} \varphi_2(x_0,r)^{-1} \int_{0}^{r^{-1}} \ln\left(e+\frac{1}{tr}\right) \|\vec{f}\|_{L^p\left(B(x_0,t^{-1}),l^2\right)} t^{\frac{n}{p}-1}\, dt =$$
$$= \sup_{x\in\mathbb{R}^n,\, r>0} \|b\|_* \,\varphi_2(x_0,r^{-1})^{-1}\, r\, \frac{1}{r} \int_{0}^{r} \ln\left(e+\frac{r}{t}\right) \|\vec{f}\|_{L^p\left(B(x_0,t^{-1}),l^2\right)} t^{\frac{n}{p}-1}\, dt \lesssim$$
$$\lesssim \|b\|_* \sup_{x_0\in\mathbb{R}^n, r>0} \varphi_1(x_0,r^{-1})^{-1} r^{\frac{n}{p}} \|\vec{f}\|_{L^p\left(B(x_0,r^{-1}),l^2\right)} =$$
$$= \|b\|_* \sup_{x_0\in\mathbb{R}^n, r>0} \varphi_1(x_0,r)^{-1} r^{-\frac{n}{p}} \left\|\vec{f}\right\|_{L^p(B(x_0,r),l^2)} = \|b\|_* \left\|\vec{f}\right\|_{M^{p,\varphi_1}(l^2)}.$$

By using the argument as similar as the above proofs and that of Theorem 2, we can also show the boundedness of $[b, \mathrm{g}^*_{\lambda,\alpha}]$.

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