Transactions of NAS of Azerbaijan, 2014, vol. XXXIV, No 1, pp. 21-28. 21

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APPROXIMATION OF PERIODIC FUNCTIONS OF TWO VARIABLES BY TRIGONOMETRIC POLYNOMIALS

Abstract

Vallee –Poussin type theorem for approximation of a continuous function of two variables by trigonometric polynomials is proved. Such a theorem for a function of one variable was proved by Vallee- Poussin.

Vallee –Poussin [2] has studied the convergence of a singular integral for a function of one variable. The showed that this singular integral uniformly convergence to the given 2π periodic function on the axis. By means of this result, one can prove the Weierstrass second theorem on density of trigonometric polynomials of one variable in the space of continuous functions. Our goal is to generalize the Vallee –Poussin result for the case of functions of two variables. In special case, we obtained one more new proof of the Weirstrass second theorem for continuous functions of two variables. As first give the definition of Vallee-Poussin singular integral and its generalization for the case of functions of two variables .

Definition 1. [1]. *Let $f(x) \in C_{2\pi}$.The integral*

$$V_n(x) = \frac{(2n)!!}{(2n-1)!!} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \cos^{2n} \frac{t-x}{2} dt$$

is said to be the Vallee-Poussin singular integral .

The denotation $C_{2\pi}$ denotes the set of all real functions given and continuous on the real axis $(-\infty, +\infty)$ and of period 2π.

Definition 2. *Let $f(x) \in C_{2\pi,2\pi}$. The following integral is said to be the Vallee-Poussin singular integral for the functions of two variables*

$$V_{m,n}(x,y) = \frac{(2m)!!}{(2m-1)!!} \frac{(2n)!!}{(2n-1)!!} \frac{1}{(2\pi)^2} \times$$

$$\times \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(t,\tau) \cos^{2m} \frac{t-x}{2} dt \cos^{2n} \frac{\tau-x}{2} d\tau. \qquad (1)$$

The following Vallee-Poussin theorem is known .

Theorem 1 [1]. *Uniformly for all real x*

$$\lim_{n\to\infty} V_n(x) = f(x).$$

Prove the following theorem that is the generalization of the Vallee-Poussin theorem.

Theorem 2.

$$\lim_{\substack{m\to\infty \\ n\to\infty}} V_{m,n}(x,y) = f(x,y)$$

is fulfilled uniformly for all arbitrary real x and y.

For the proof we need the following auxiliary lemmas.

Lemma 1. *Let $\varphi(x,y) \in C_{2\pi,2\pi}$. Then for any a,b it is valid*

$$\int\limits_a^{a+2\pi}\int\limits_b^{b+2\pi} \varphi(x,y)\,dxdy = \int\limits_0^{2\pi}\int\limits_0^{2\pi} \varphi(x,y)\,dxdy.$$

The lemma is obtained by double application of the appropriate result.

Lemma 2. *The following identity is valid*

$$\int\limits_0^{\pi/2}\int\limits_0^{\pi/2} \cos^{2m} x \cos^{2n} y dxdy = \frac{(2m-1)!!}{(2m)!!}\frac{(2n-1)!!}{(2n)!!}\cdot\left(\frac{\pi}{2}\right)^2.$$

The proof is conducted similar to the Vallee-Possin appropriate result for the functions of one variable.

Now prove the theorem itself.

Proof. Assume in Vallee –Poussin's integral (1) $t = x+u,\ \tau = y+v$. By virtue of lemma 1, it is possible not to change the integration boundary

$$V_{m,n}(x,y) = \frac{(2m)!!}{(2m-1)!!}\frac{(2n)!!}{(2n-1)!!}\frac{1}{(2\pi)^2}\int\limits_{-\pi}^{\pi}\int\limits_{-\pi}^{\pi} f(x+u,y+\nu)\cos^{2m}\frac{u}{2}du\cos^{2n}\frac{\nu}{2}d\nu.$$

Changing u by $2t$ and ν by 2τ we get

$$V_{m,n}(x,y) = \frac{(2m)!!}{(2m-1)!!}\frac{(2n)!!}{(2n-1)!!}\frac{1}{\pi^2}\int\limits_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\int\limits_{\frac{-\pi}{2}}^{\frac{\pi}{2}} f(x+2t,y+2\tau)\cos^{2m} t\cos^{2n}\tau dtd\tau.$$

Partition the first integral into two integrals distributed on the segments $\left[-\frac{\pi}{2},0\right]$ and $\left[0,\frac{\pi}{2}\right]$, and obtain

$$V_{m,n}(x,y) = \frac{(2m)!!}{(2m-1)!!}\frac{(2n)!!}{(2n-1)!!}\frac{1}{\pi^2}\int\limits_0^{\frac{\pi}{2}}\int\limits_0^{\frac{\pi}{2}} \{f(x+2t,y+2\tau)+f(x-2t,y+2\tau)+$$

$$+f(x+2t,y-2\tau)+f(x-2t,y-2\tau)\}\cos^{2m} t\cos^{2n}\tau dtd\tau.$$

(the other cases are considered similarly) .

For any $\varepsilon > 0$ we find $\delta > 0$, that for

$$\left|x'' - x'\right| < 2\delta \quad \text{and} \quad \left|y'' - y'\right| < 2\delta$$

there will be

$$\left|f\left(x'',y''\right) - f\left(x',y'\right)\right| < \frac{\varepsilon}{4}.$$

From lemma 2, for any real x and y

$$f(x,y) = \frac{(2m)!!}{(2m-1)!!}\frac{(2n)!!}{(2n-1)!!}\frac{1}{\pi^2}\int\limits_0^{\frac{\pi}{2}}\int\limits_0^{\frac{\pi}{2}} 4f(x,y)\cos^{2m} t\cos^{2n}\tau dtd\tau.$$

Hence we have

$$V_{m,n}(x,y)-f(x,y)=\frac{(2m)!!}{(2m-1)!!}\frac{(2n)!!}{(2n-1)!!}\frac{1}{\pi^2}\times$$

$$\times\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\{f(x+2t,y+2\tau)+f(x-2t,y+2\tau)+$$

$$+f(x+2t,y-2\tau)+f(x-2t,y-2\tau)-4f(x,y)\cos^{2m}t\cos^{2n}\tau dtd\tau.$$

Partition this integral into 4 integrals distributed on squares and rectangles

$$[0,\delta;0,\delta],\left[0,\delta;\delta,\frac{\pi}{2}\right],\ \left[\delta,\frac{\pi}{2};0,\delta\right]\ \text{and}\ \left[\delta,\frac{\pi}{2};\delta,\frac{\pi}{2}\right].$$

Note that in the first

$$|f(x+2t,y+2\tau)+f(x-2t,y+2\tau)+f(x+2t,y-2\tau)+$$

$$+f(x-2t,y-2\tau)-4f(x,y)|\le|f(x+2t,y+2\tau)-f(x,y)|+$$

$$+|f(x-2t,y+2\tau)-f(x,y)|+|f(x+2t,y-2\tau)-f(x,y)|+$$

$$+|f(x-2t,y-2\tau)-f(x,y)|<\varepsilon.$$

in the second

$$|f(x+2t,y+2\tau)+f(x-2t,y+2\tau)-f(x+2t,y-2\tau)+$$

$$+f(x-2t,y-2\tau)-4f(x,y)|\le|f(x+2t,y+2\tau)|+$$

$$+|f(x-2t,y+2\tau)|+|4f(x,y)|+|f(x+2t,y-2\tau)|+$$

$$+|f(x-2t,y-2\tau)|\le 8M.$$

in the fourth

$$|f(x+2t,y+2\tau)+f(x-2t,y+2\tau)+f(x+2t,y-2\tau)+$$

$$+f(x-2t,y-2\tau)-4f(x,y)|\le 8M,$$

where $M=\max|f(x,y)|$.

Hence

$$|V_{m,n}(x,y)-f(x,y)|<\frac{(2m)!!}{(2m-1)!!}\frac{(2n)!!}{(2n-1)!!}\frac{1}{\pi^2}\times$$

$$\times\left\{\varepsilon\int_0^{\delta}\int_0^{\delta}\cos^{2m}t\cos^{2n}dtd\tau+8M\int_0^{\delta}\int_{\delta}^{\frac{\pi}{2}}\cos^{2m}t\cos^{2n}\tau dtd\tau+\right.$$

$$\left.+8M\int_{\delta}^{\frac{\pi}{2}}\int_0^{\delta}\cos^{2m}t\cos^{2n}\tau dtd\tau+8M\int_{\delta}^{\frac{\pi}{2}}\int_{\delta}^{\frac{\pi}{2}}\cos^{2m}t\cos^{2n}\tau dtd\tau\right\}.$$

But

$$\int_0^{\delta}\int_0^{\delta}\cos^{2m}t\cos^{2n}dtd\tau<\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\cos^{2m}t\cos^{2n}\tau dtd\tau=\frac{(2m-1)!!}{(2m)!!}\frac{(2n-1)!!}{(2n)!!}\cdot\left(\frac{\pi}{2}\right)^2.$$

Similarly,

$$\int_0^{\delta}\int_{\delta}^{\frac{\pi}{2}}\cos^{2m}t\cos^{2n}\tau dtd\tau<\int_0^{\frac{\pi}{2}}\cos^{2m}\tau dt\int_{\delta}^{\frac{\pi}{2}}\cos^{2n}\tau d\tau<\frac{(2m-1)!!}{(2m)!!}\left(\cos^{2n}\delta\right)\cdot\left(\frac{\pi}{2}\right)^2;$$

$$\int_{\delta}^{\frac{\pi}{2}}\int_0^{\delta}\cos^{2m}t\cos^{2n}\tau dtd\tau<\int_{\delta}^{\frac{\pi}{2}}\cos^{2m}\tau dt\int_0^{\frac{\pi}{2}}\cos^{2n}\tau d\tau<\frac{(2n-1)!!}{(2n)!!}\left(\cos^{2m}\delta\right)\cdot\left(\frac{\pi}{2}\right)^2.$$

$$\int_{\delta}^{\frac{\pi}{2}}\int_{\delta}^{\frac{\pi}{2}}\cos^{2m}t\cos^{2n}\tau dtd\tau<\left(\cos^{2m}\delta\right)\left(\cos^{2n}\delta\right)\left(\frac{\pi}{2}\right)^2.$$

On the other hand, the cosine function decreases by $\left[0,\frac{\pi}{2}\right]$, and denoting $\cos^2\delta$ by q we find

$$\int_{\delta}^{\frac{\pi}{2}}\int_0^{\delta}\cos^{2m}t\cos^{2n}\tau dtd\tau<\left(\frac{\pi}{2}\right)^2q^n\frac{(2m-1)!!}{(2m)!!},$$

$$\int_0^{\delta}\int_{\delta}^{\frac{\pi}{2}}\cos^{2m}t\cos^{2n}\tau dtd\tau<\left(\frac{\pi}{2}\right)^2q^m\frac{(2n-1)!!}{(2n)!!}.$$

$$\int_{\delta}^{\frac{\pi}{2}}\int_{\delta}^{\frac{\pi}{2}}\cos^{2m}t\cos^{2n}\tau dtd\tau<q^{m+n}\left(\frac{\pi}{2}\right)^2.$$

Comparing the all mentioned ones we find

$$|V_{m,n}(x,y)-f(x,y)|<$$

$$<\frac{\varepsilon}{4}+\left(8M\frac{(2m-1)!!}{(2M)!!}q^n+8M\frac{(2n-1)!!}{(2n)!!}q^m+8M\frac{\pi^2}{4}q^{m+n}\right)\frac{(2m)!!}{(2m-1)!!}\frac{(2n)!!}{(2n-1)!!}\frac{1}{\pi^2}.$$

It remains to keep in mind that

$$\frac{(2n)!!}{(2n-1)!!}=\frac{2}{3}\cdot\frac{4}{5}...\frac{2n-2}{2n-1}2n<2n;\ \frac{(2n)!!}{(2n-1)!!}<2n$$

$$\frac{(2m)!!}{(2m-1)!!}=\frac{2}{3}\cdot\frac{4}{5}...\frac{2m-2}{2m-1}2m<2m;\ \frac{(2m)!!}{(2m-1)!!}<2m.$$

Thus,

$$|V_{m,n}(x,y)-f(x,y)|<\frac{\varepsilon}{2}+\left(8Mq^n+8Mq^m+8Mq^{m+n}mn\right).$$

We know that for $0 < q < 1$

$$\lim_{\substack{m\to\infty\\ n\to\infty}} mnq^{m+n} = 0,$$

$$\lim_{n\to\infty} nq^n = 0,$$

$$\lim_{m\to\infty} mq^n = 0,$$

therefore as $n \to \infty,\ m \to \infty$ there will be

$$8Mnmq^{m+n} < \frac{\varepsilon}{6},$$

$$8Mmq^m < \frac{\varepsilon}{6},$$

$$8Mnq^n < \frac{\varepsilon}{6}.$$

Finally,

$$\left|V_{m,n}(x,y) - f(x,y)\right| < \varepsilon.$$

The theorem is proved.

By means of theorem 2 we can get a new proof of the Weierstrass second theorem for the functions of two variables.

As first give the Weierstrass theorem for the case of one variable and its generalizations for the functions of two variables.

Theorem 3 [1]. *Let $f(x) \in C_{2\pi}$. For any $\varepsilon > 0$ there exists a trigonometric polynomial $T(x)$ such that for all real x*

$$\left|T(x) - f(x)\right| < \varepsilon.$$

Theorem 4. *Let $f(x) \in C_{2\pi,2\pi}$ For any $\varepsilon > 0$ there exists a trigonometric polynomial $T(x,y)$ that for all x and y*

$$\left|T(x,y) - f(x,y)\right| < \varepsilon.$$

Give the Vallee –Poussin type proof that was given in [2] for a function of one variable. In order to get the Weierstrass second theorem, it suffices to show that $V_{m,n}(x,y)$ is a trigonometric polynomial.

Lemma 3. *The product of two trigonometric polynomials of two variables is also the trigonometric polynomial of two variables whose order equals the sum of order of cofactors.*

Proof. Remultiply the two polynomials

$$T_{\mu\nu}(x,y) = \sum_{k=0}^{\mu}\sum_{l=0}^{\nu}(a_{kl}\cos kx\cos ly + b_{kl}\cos kx\sin ly +$$

$$+c_{kl}\sin kx\cos ly + d_{kl}\sin kx\sin ly)$$

and

$$U_{\rho\theta}(x,y) = \sum_{k=0}^{\rho}\sum_{l=0}^{\theta}(a_{kl}\cos kx\cos ly + b_{kl}\cos kx\sin ly + c_{kl}\sin kx\cos ly + d_{kl}\sin kx\sin ly)$$

and use the formulas

$$\left.\begin{array}{r}\cos\alpha\cos\beta=\frac{1}{2}\left[\cos\left(\alpha-\beta\right)+\cos\left(\alpha+\beta\right)\right],\\ \cos\alpha\sin\beta=\frac{1}{2}\left[\sin\left(\alpha+\beta\right)+\sin\left(\beta-\alpha\right)\right]\\ \sin\alpha\cos\beta=\frac{1}{2}\left[\sin\left(\alpha+\beta\right)+\sin\left(\alpha-\beta\right)\right]\\ \sin\alpha\cos\beta=\frac{1}{2}\left[\cos(\alpha-\beta)-\cos\left(\alpha+\beta\right)\right]\end{array}\right\}. \tag{2}$$

Remultiplying the addends of both polynomials, notice that each of the obtained products is a trigonometric polynomial of two variables. Hence it follows that the sum of these polynomials is a trigonometric polynomial with respect to x and y. From formula (2) it is seen that the order of cofactors is no more and no less than the sum $\mu+\nu+\rho+\theta$. The product of leading terms $T_{\mu\nu}\left(x,y\right)$ and $U_{\rho\theta}\left(x,y\right)$ is

$$(a_{\mu\nu}\cos\mu x\cos vy+b_{\mu\nu}\cos\mu x\sin vy+c_{\mu\nu}\sin\mu x\cos vy+d_{\mu\nu}\sin\mu x\sin vy)\times$$

$$(l_{\rho\theta}\cos\rho x\cos\theta y+h_{\rho\theta}\cos\rho x\sin\theta y+z_{\rho\theta}\sin\rho x\cos\theta y+\omega_{\rho\theta}\sin\rho x\sin\theta y)=$$

$$=\frac{1}{4}\left(a_{\mu\nu}l_{\rho\theta}-a_{\mu\nu}\omega_{\rho\theta}-d_{\mu\nu}l_{\rho\theta}+d_{\mu\nu}\omega_{\rho\theta}\right)\cos\left(\mu+\nu\right)\cos\left(\rho+\theta\right)+$$

$$+\frac{1}{4}\left(a_{\mu\nu}h_{\rho\theta}+a_{\mu\nu}z_{\rho\theta}-d_{\mu\nu}h_{\rho\theta}-d_{\mu\nu}z_{\rho\theta}\right)\cos\left(\mu+\nu\right)\sin\left(\rho+\theta\right)+$$

$$+\frac{1}{4}\left(b_{\mu\nu}l_{\rho\theta}-b_{\mu\nu}\omega_{\rho\theta}+c_{\mu\nu}l_{\rho\theta}-c_{\mu\nu}\omega_{\rho\theta}\right)\sin(\mu+\nu)\cos\left(\rho+\theta\right)+$$

$$+\frac{1}{4}\left(b_{\mu\nu}h_{\rho\theta}+b_{\mu\nu}z_{\rho\theta}+c_{\mu\nu}h_{\rho\theta}+c_{\mu\nu}z_{\rho\theta}\right)\sin\left(\mu+\nu\right)\sin\left(\rho+\theta\right)+\lambda,$$

where λ consists of lower order terms .The numbers $a_{\mu\nu}$,$b_{\mu\nu}$,$c_{\mu\nu}$,$d_{\mu\nu}$, $l_{\rho\theta}$,$h_{\rho\theta}$,$z_{\rho\theta}$,$\omega_{\rho\theta}$ are real, this means that the coefficients for
$\cos(\mu+\nu)\cos\left(\rho+\theta\right)$, $\cos(\mu+\nu)\sin\left(\rho+\theta\right)$, $\sin(\mu+\nu)\cos\left(\rho+\theta\right)$ and $\sin(\mu+\nu)\sin\left(\rho+\theta\right)$ are also real. Now it is necessary to define that there coefficients don't disappear simultaneously. For that it is necessary to find the sum of their squares

$$\left(a_{\mu\nu}l_{\rho\theta}-a_{\mu\nu}\omega_{\rho\theta}-d_{\mu\nu}l_{\rho\theta}+d_{\mu\nu}\omega_{\rho\theta}\right)^2+\left(a_{\mu\nu}h_{\rho\theta}+a_{\mu\nu}z_{\rho\theta}-d_{\mu\nu}h_{\rho\theta}-d_{\mu\nu}z_{\rho\theta}\right)^2+$$

$$+\left(b_{\mu\nu}l_{\rho\theta}-b_{\mu\nu}\omega_{\rho\theta}+c_{\mu\nu}l_{\rho\theta}-c_{\mu\nu}\omega_{\rho\theta}\right)^2+\left(b_{\mu\nu}h_{\rho\theta}+b_{\mu\nu}z_{\rho\theta}+c_{\mu\nu}h_{\rho\theta}+c_{\mu\nu}z_{\rho\theta}\right)^2=$$

$$=\left(\left(a_{\mu\nu}-d_{\mu\nu}\right)^2+\left(b_{\mu\nu}+c_{\mu\nu}\right)^2\right)\left(\left(l_{\rho\theta}-\omega_{\rho\theta}\right)^2+\left(h_{\rho\theta}+z_{\rho\theta}\right)^2\right)>0.$$

Lemma 4. It *the trigonometric polynomial $T\left(x,y\right)$ is an even function, i.e.*

$$T\left(-x,-y\right)=T\left(x,y\right),T\left(-x,y\right)=T\left(x,y\right),T\left(x,-y\right)=T\left(x,y\right),$$

then it may be represented in the form

$$T\left(x,y\right)=A+\sum_{k=1}^{m}\sum_{l=1}^{n}(a_{kl}\cos kx\cos ly+b_{kl}\cos kx+c_{kl}\cos ly),$$

not containing the sines of multiple arcs.

In order to prove this fact, it is necessary to put together

$$T\left(x,y\right)=\sum_{k=1}^{\mu}\sum_{l=1}^{\nu}(a_{kl}\cos kx\cos ly+b_{kl}\cos kx\sin ly+$$

$$+c_{kl}\sin kx\cos ly+d_{kl}\sin kx\sin ly)$$

$$T\left(-x,-y\right)=\sum_{k=1}^{\mu}\sum_{l=1}^{\nu}(a_{kl}\cos kx\cos ly-b_{kl}\cos kx\sin ly-$$

$$-c_{kl}\sin kx\cos ly+d_{kl}\sin kx\sin ly)$$

$$T\left(x,y\right)=\sum_{k=1}^{\mu}\sum_{l=1}^{\nu}(a_{kl}\cos kx\cos ly+b_{kl}\cos kx\sin ly-$$

$$-c_{kl}\sin kx\cos ly-d_{kl}\sin kx\sin ly)$$

$$T\left(x,-y\right)=\sum_{k=1}^{\mu}\sum_{l=1}^{\nu}(a_{kl}\cos kx\cos ly-b_{kl}\cos kx\sin ly+$$

$$+c_{kl}\sin kx\cos ly-d_{kl}\sin kx\sin ly)$$

i.e.

$$T\left(x,y\right)+T\left(-x,-y\right)+T\left(-x,y\right),T\left(x,y\right)+T\left(x,-y\right)$$

and divide the obtained results into two. Now show that $V_{m,n}\left(x,y\right)$ is a trigonometric polynomial. It is known that

$$\cos^2\frac{u}{2}\cos^2\frac{v}{2}=\frac{1+\cos u}{2}\frac{1+\cos v}{2}$$

is a polynomial of first order. Then $\cos^{2m}\frac{u}{2}\cos^{2n}\frac{v}{2}$ is a polynomial of $(m+n)$ order, and since it is an even function, it may be represented in the form

$$\cos^{2m}\frac{u}{2}\cos^{2n}\frac{v}{2}=\sum_{k=1}^{m}\sum_{l=1}^{n}\cos kx\cos ly.$$

Hence

$$V_{m,n}\left(x,y\right)=\frac{(2m)!!}{(2m-1)!!}\frac{(2n)!!}{(2n-1)!!}\frac{1}{(2\pi)^2}\times$$

$$\times\int\limits_{-\pi}^{\pi}\int\limits_{-\pi}^{\pi}f\left(t,\tau\right)\left[\sum_{k=1}^{m}\sum_{l=1}^{n}a_{kl}\cos k\left(t-x\right)\cos l\left(\tau-y\right)\right]dtd\tau.$$

So,

$$V_{m,n}\left(x,y\right)=\frac{(2m)!!}{(2m-1)!!}\frac{(2n)!!}{(2n-1)!!}\frac{1}{(2\pi)^2}\times$$

$$\times\int\limits_{-\pi}^{\pi}\int\limits_{-\pi}^{\pi}f\left(t,\tau\right)\left[\sum_{k=1}^{m}\sum_{l=1}^{n}a_{kl}(\cos kt\cos kx+\sin kt\sin kx)\times\right.$$

$$\left.\times\left(\cos l\tau\cos ly+\sin l\tau\sin ly\right)\right]dtd\tau$$

and therefore,

$$V_{m,n}\left(x,y\right)=\sum_{k=1}^{\mu}\sum_{l=1}^{\nu}(a_{kl}\cos kx\cos ly+b_{kl}\cos kx\sin ly+$$

$$+c_{kl}\sin kx\cos ly+d_{kl}\sin kx\sin ly),$$

where for brevity

$$a_{kl}=\frac{(2m)!!}{(2m-1)!!}\frac{(2n)!!}{(2n-1)!!}\frac{g_{kl}^2}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f\left(t,\tau\right)\cos kt\cos l\tau dtd\tau.$$

$$b_{kl}=\frac{(2m)!!}{(2m-1)!!}\frac{(2n)!!}{(2n-1)!!}\frac{g_{kl}^2}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f\left(t,\tau\right)\cos kt\sin l\tau dtd\tau.$$

$$c_{kl}=\frac{(2m)!!}{(2m-1)!!}\frac{(2n)!!}{(2n-1)!!}\frac{g_{kl}^2}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f\left(t,\tau\right)\cos l\tau\sin ktdtd\tau.$$

$$d_{kl}=\frac{(2m)!!}{(2m-1)!!}\frac{(2n)!!}{(2n-1)!!}\frac{g_{kl}^2}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f\left(t,\tau\right)\sin kt\sin l\tau dtd\tau.$$

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Received June 11, 2013 ; Revised September 09, 2013