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THE BOUNDEDNESS OF B-RIESZ POTENTIAL IN WEIGHTED B-MORREY SPACES

Abstract

We consider the generalized shift operator, associated with the Bessel (Hankel) differential operator $B = \frac{\partial^2}{\partial x^2} + \frac{\gamma}{x}\frac{\partial}{\partial x}$, $\gamma > 0$. *The fractional maximal operator* $M_{\alpha,\gamma}$ *(fractional B-maximal operator) and the Riesz potential* $I_{\alpha,\gamma}$ *(B-Riesz potential), associated with the generalized shift operator are investigated. At first, we prove that the fractional B-maximal operator* $M_{\alpha,\gamma}$ *is bounded from the weight B-Morrey space* $\mathcal{L}_{p,\lambda,|\cdot|^\beta,\gamma}$ *to* $\mathcal{L}_{q,\lambda,|\cdot|^\beta,\gamma}$, *where* $1/p - 1/q = \alpha/(1+\gamma-\lambda)$, $1 < p < (1+\gamma)/\alpha$, *for all* $1 \le p < \infty$ *and* $0 \le \lambda < 1+\gamma$.

We study the B-Riesz potential in the weight B-Morrey space. We prove that B-Riesz potential $I_{\alpha,\gamma}$, $0 < \alpha < 1+\gamma$ *is bounded from the weight B-Morrey space* $\mathcal{L}_{p,\lambda,|\cdot|^\beta,\gamma}$ *to* $\mathcal{L}_{q,\lambda,|\cdot|^\beta,\gamma}$, *where* $1/p - 1/q = \alpha/(1+\gamma-\lambda)$, $1 < p < (1+\gamma)/\alpha$, *for all* $1 \le p < \infty$ *and* $0 \le \lambda < 1+\gamma$.

Introduction

Assume $\gamma > 0$ and define as the class of all measurable functions f defined on $(0,\infty)$

$$L_{p,\gamma} \equiv L_{p,\gamma}(0,\infty) \equiv L_p((0,\infty),\ x^\gamma dx),\ (1 \le p < \infty),$$

which

$$\|f\|_{L_{p,\gamma}} \equiv \left(\int_0^\infty |f(x)|^p x^\gamma dx\right)^{1/p} < \infty,$$

and by $L_{\infty,\gamma}(0,\infty) = L_\infty(0,\infty)$ the space of the essentially bounded measurable functions with respect to the Lebesgue measure on $(0,\infty)$.

We recall some basic results in harmonic analysis related to the Hankel (Fourier-Bessel) transform. More details can be found in [31], [32]. The Hankel transform appears taking different forms in the literature (see, for instance, [26], [28], [31], [32]). Here we define the Hankel transformation h_γ through [31], [32]

$$h_\gamma(f)(x) = \int_0^\infty j_\gamma(xy) f(y) y^\gamma dy, \quad x \in (0,\infty),$$

where $j_\gamma(s) = 2^\gamma \Gamma(\gamma+1) s^{-\gamma} J_\gamma(s)$, with J_γ the Bessel function of the first kind and index γ.

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Remark 1. As is well known (see [31]), the function j_γ is just the solution of the differential equation

$$L_\gamma u = -u,\ u(0) = 1,\ u'(0) = 0,$$

where L_γ is the Bessel differential operator given by

$$L_\gamma = \frac{d^2}{dx^2} + \frac{\gamma}{x}\frac{d}{dx},\quad \gamma > 0.$$

Definition 1. *1) The generalized translation operator T^y, $y \geq 0$, are defined for smooth functions on $(0,\infty)$ by*

$$T^y f(x) = C_\gamma \int_0^\pi f(\sqrt{x^2+y^2-2xycos\theta}\,)sin^{\gamma-1}\theta d\theta,$$

where $C_\gamma = \pi^{-\frac{1}{2}}\,\frac{\Gamma(\frac{\gamma+1}{2})}{\Gamma(\frac{\gamma}{2})}$.

2) The generalized convolution operator of two smooth functions f, g on $(0,\infty)$ is defined by

$$(f\#g)(x) = \int_0^\infty T^y f(x)g(y)y^\gamma dy,\quad x \in (0,\infty).$$

For $0 \leq \beta < 1+\gamma$, the fractional maximal function $M_{\beta,\gamma}$ associated with Hankel transform is defined at $f \in L^1_{loc,\gamma}(0,\infty)$ by

$$M_{\beta,\gamma} f(x) = \sup_{r>0} \frac{1}{r^{1+\gamma-\beta}} \int_0^r T^y\,|f(x)|\,y^\gamma dy,\ x \in (0,\infty),$$

for $\beta = 0$ we get the maximal function $M_{0,\gamma}f$ associated with the Hankel transform (see [29]).

1. Preliminaries

It is well known that the operator T^y is the solution of the

$$(L_\gamma)_x u = (L_\gamma)_y u,\ u(x,0) = f(x),\ u_y(x,0) = 0$$

differential equation. Further, the following properties are satisfied (see [27], [32]):

1. For $\lambda \in \mathbf{C}$ and $x, y \in (0,\infty)$,

$$T^y(j_\gamma(\lambda\ .))(x) = j_\gamma(\lambda x)j_\gamma(\lambda y).$$

2. If f belongs to $L_{p,\gamma}$, $1 \leq p \leq \infty$, then for all $y \geq 0$, the function $T^y f$ belongs to $L_{p,\gamma}$, and

$$\|T^y f\|_{L_{p,\gamma}} \leq \|f\|_{L_{p,\gamma}}.$$

3. For $f \in L_{1,\gamma}$ and $g \in L_{p,\gamma}$,

$$\|f\#g\|_{L_{p,\gamma}} \leq \|f\|_{L_{1,\gamma}}\,\|g\|_{L_{p,\gamma}}.$$

4. For $f, g \in L_{p,\gamma}$, and $p = 1$ or 2,

i). $h_\gamma(T^y f)(\lambda) = j_\gamma(\lambda y) h_\gamma(f)(\lambda)$,

ii). $h_\gamma(f\#g) = h_\gamma(f) h_\gamma(g)$.

Lemma 1. *For all $x \in (0, \infty)$ the following equality is valid*

$$\int_0^t T^y g(x) y^\gamma dy = \int_{E((x,0),t)} g\left(\sqrt{z^2 + \overline{z}^2}\right) \overline{z}^{\gamma-1} dz d\overline{z},$$

where $E((x,0),t) = \{(z,\overline{z}) \in (0,\infty) \times (0,\infty) \ : \ |(x - z, \overline{z})| < t\}$.

Lemma 2. *For all $x \in (0, \infty)$ the following equality is valid*

$$\int_0^\infty [T^x g(y)] M_\gamma \chi_{E_r}(y) y^\gamma dy = \int_{\mathbb{R}\times(0,\infty)} g\left(\sqrt{z^2 + \overline{z}^2}\right) M_\nu \chi_{E((x,0),r)}(z) \overline{z}^{\gamma-1} dz d\overline{z},$$

where $E((x,0),t) = \{(z,\overline{z}) \in (0,\infty) \times (0,\infty) \ : \ |(x - z, \overline{z})| < t\}$, $E_r = (0,r)$.

The proof of Lemmas 1 and 2 is straightforward via the following substitutions

$$z = y\cos\theta, \quad \overline{z} = y \sin\theta, \quad 0 \le \theta < \pi,$$
$$y \in (0,\infty), \ (z,\overline{z}) \in (0,\infty) \times (0,\infty).$$

Definition 2. *Let $1 \le p < \infty$. By $WL_{p,\gamma}(0,\infty)$ we denote the weak $L_{p,\gamma}$ space defined as the set of locally integrable functions $f(x)$, $x \in (0,\infty)$ with the finite norms*

$$\|f\|_{WL_{p,\gamma}} = \sup_{r>0} r \left|\{x \in (0,\infty) \ : \ |f(x)| > r\}\right|_\gamma^{1/p}.$$

Definition 3. *[9] Let $1 \le p < \infty$, $0 \le \lambda \le 1 + \gamma$, $[t]_1 = \min\{1, t\}$ and $\omega(x)$ positive weight function on $(0,\infty)$. We denote by $\mathcal{L}_{p,\lambda,\gamma}(0,\infty)$ the Morrey space ($\equiv$ B-Morrey space) and $\mathcal{L}_{p,\lambda,\omega,\gamma}(0,\infty)$ the weighted Morrey space, associated with the Bessel differential operator as the set of locally integrable functions $f(x)$, $x \in (0,\infty)$, with the finite norms*

$$\|f\|_{\mathcal{L}_{p,\lambda,\gamma}} = \sup_{t>0,\, x\in(0,\infty)} \left(t^{-\lambda} \int_0^t T^y |f(x)|^p y^\gamma dy\right)^{1/p},$$

$$\|f\|_{\mathcal{L}_{p,\lambda,\omega,\gamma}} = \sup_{t>0,\, x\in(0,\infty)} \left(t^{-\lambda} \int_0^t T^y (|f(x)|^p \omega(x)) y^\gamma dy\right)^{1/p}$$

respectively.

Note that

$$\mathcal{L}_{p,0,\gamma}(0,\infty) = L_{p,\gamma}(0,\infty),$$

$$\mathcal{L}_{p,1+\gamma,\gamma}(0,\infty) = L_\infty(0,\infty),$$

and if $\lambda < 0$ or $\lambda > 1 + \gamma$, then $\mathcal{L}_{p,\lambda,\gamma}(0,\infty) = \Theta$, where Θ is the set of all functions equivalent to 0 on $(0,\infty)$.

Definition 4. *[11] Let* $1 \le p < \infty$,$0 \le \lambda \le 1+\gamma$ *and* $\omega(x)$ *positive weight function on* $(0,\infty)$. *We denote by* $W\mathcal{L}_{p,\lambda,\gamma}(0,\infty)$ *the weak B-Morrey space as the set of locally integrable functions* $f(x)$,$x \in (0,\infty)$ *with finite norms*

$$\|f\|_{W\mathcal{L}_{p,\lambda,\gamma}} = \sup_{r>0} r \sup_{t>0,\, x\in(0,\infty)} \left(t^{-\lambda} \int_{\{y\in(0,t):\, T^y|f(x)|>r\}} y^\gamma dy \right)^{1/p},$$

$$\|f\|_{W\mathcal{L}_{p,\lambda,\omega,\gamma}} = \sup_{r>0} r \sup_{t>0,\, x\in(0,\infty)} \left(t^{-\lambda} \int_{\{y\in(0,t):\, T^y|f(x)|>r\}} \omega(y) y^\gamma dy \right)^{1/p},$$

respectively.

Note that

$$WL_{p,\gamma}(0,\infty) = W\mathcal{L}_{p,0,\gamma}(0,\infty).$$

2. $\mathcal{L}_{p,\lambda,|\cdot|^\beta,\gamma}$-boundedness of the B-maximal operator

In this section we study the $\mathcal{L}_{p,\lambda,|\cdot|^\beta,\gamma}$-boundedness of the fractional B-maximal operator

$$M_{\alpha,\gamma} f(x) = \sup_{r>0} r^{\alpha-1-\gamma} \int_0^r T^y|f(x)| y^\gamma dy, \quad 0 \le \alpha < 1+\gamma.$$

We write $M_{0,\gamma} f(x) = M_\gamma f(x)$ in the case where $\alpha = 0$.

Theorem 1. *Let* $0 \le \alpha < 1+\gamma$, $0 \le \lambda < 1+\gamma$ *and* $1 < p < \frac{1+\gamma-\lambda}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{1+\gamma-\lambda}$, $-\frac{p'(1+\gamma)}{p'+q} < \beta < \frac{q(1+\gamma)}{p'+q}$. *Then*

$$\|M_{\alpha,\gamma}(f|\cdot|^{\beta\alpha})\|_{\mathcal{L}_{q,\lambda,|\cdot|^\beta,\gamma}} \le C\|f\|_{\mathcal{L}_{p,\lambda,|\cdot|^\beta,\gamma}}$$

holds.

Proof. We need to introduce the maximal operator defined on a space of homogeneous type (Y,d,ν). By this we mean a topological space $Y = \mathbb{R}\times(0,\infty)$ equipped with a continuous pseudometric d and a positive measure ν satisfying

$$\nu(E((x,\overline{x}),2t)) \le C_1 \nu(E((x,\overline{x}),t)) \qquad (1)$$

with a constant C_1 independent of $(x,\overline{x})$ and $t>0$. Here $E((x,\overline{x}),t) = \{(y,\overline{y}) \in Y : d(((x,\overline{x}),(y,\overline{y})) < t\}$, $d\nu(y,\overline{y}) = (\overline{y})^{\gamma-1} dy\, d\overline{y}$, $d((x,\overline{x}),(y,\overline{y})) = |(x,\overline{x}) - (y,\overline{y})| \equiv (|x-y|^2 + (\overline{x}-\overline{y})^2)^{\frac{1}{2}}$.

Let (Y,d,ν) be a space of homogeneous type. Define

$$M_{\alpha,\nu}\overline{f}(x,\overline{x}) = \sup_{r>0} \nu(E((x,\overline{x}),r))^{\alpha-1} \int_{E((x,\overline{x}),r)} \left|\overline{f}(y,\overline{y})\right| d\nu(y),$$

where $\overline{f}(x,\overline{x}) = f\left(\sqrt{x^2+\overline{x}^2}\right)$.

It is well known that the fractional maximal function $M_{\alpha,\nu}$ is weak type $(1,q)$, $1-\frac{1}{q}=\frac{\alpha}{1+\gamma}$ and is bounded from $L_{p,\varphi}(Y,d\nu)$ to $L_{q,\varphi}(Y,d\nu)$ for $1\le p<\frac{1+\gamma-\lambda}{\alpha}$, $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{1+\gamma-\lambda}$, $\varphi(x)=|x|^{\beta}\in A_{1+\frac{q}{p'}}$ (see [?]). Here we are concerned with the maximal operator defined by $d\nu(y,\overline{y})=(\overline{y})^{\gamma-1}dy\ d\overline{y}$. It is clear that this measure satisfies the doubling condition (1).

It can be proved that

$$M_{\alpha,\gamma}(f|\cdot|^{\beta\alpha})\left(\sqrt{z^2+\overline{z}^2}\right)=M_{\alpha,\nu}(\overline{f}|\cdot|^{\beta\alpha})\left(\sqrt{z^2+\overline{z}^2},0\right) \tag{2}$$

and

$$M_{\alpha,\gamma}(f|\cdot|^{\beta\alpha})(x)=M_{\alpha,\nu}(\overline{f}|\cdot|^{\beta\alpha})(x,0). \tag{3}$$

Indeed, Lemma 2

$$\int_0^\infty T^y(|f(x)|^p\,\varphi(x))M_\gamma\chi_{E_r}(y)y^\gamma\ dy=$$
$$=\int_Y\left|\overline{f}\left(\sqrt{y^2+\overline{y}^2},0\right)\right|^p\varphi\left(\sqrt{y^2+\overline{y}^2},0\right)M_\nu\chi_{E((x,0),r)}(y,\overline{y})\ d\nu(y,\overline{y'})$$

and

$$|E_r|_\gamma=\nu E\left(\left(\sqrt{z^2+\overline{z}^2},0\right),r\right)$$

imply (2). Furthermore, taking $\overline{z}=0$ in (2) we get (3).

Using Lemma 2 and equality (2) we have

$$\int_0^r T^y\left(\left(M_{\alpha,\gamma}(f|\cdot|^{\beta\alpha})(x)\right)^q x^\beta\right)y^\gamma\ dy\le$$
$$\le\int_0^\infty T^y\left(\left(M_{\alpha,\gamma}(f|\cdot|^{\beta\alpha})(x)\right)^q x^\beta\right)M_\gamma\chi_{E_r}(y)y^\gamma\ dy=$$
$$=\int_{(0,\infty)\times(0,\infty)}\left(M_{\alpha,\gamma}(f|\cdot|^{\beta\alpha})\left(\sqrt{z^2+\overline{z}^2}\right)\right)^q\left(\sqrt{z^2+\overline{z}^2}\right)^\beta\times$$
$$\times M_\gamma\chi_{E((x,0),r)}(z,\overline{z})\ d\nu(z,\overline{z})=$$
$$=\int_Y\left(M_{\alpha,\nu}(\overline{f}|\cdot|^{\beta\alpha})\left(\sqrt{z^2+\overline{z}^2},0\right)\right)^q\left(\sqrt{z^2+\overline{z}^2},0\right)^\beta M_\nu\chi_{E((x,0),r)}(z,\overline{z})\ d\nu(z,\overline{z}).$$

Also, in the work [20], [22] it was proved:

Proposition 1. *Let* $0<\alpha<1$, $1<p<\frac{1}{\alpha}$, $\frac{1}{p}-\frac{1}{q}=\alpha$. *Then the following two conditions are equivalent:*

1) There is a constant $C>0$ *such that for any* $f\in L_{p,\varphi}(Y)$ *the inequality*

$$\|M_{\alpha,\nu}(f\varphi^\alpha)\|_{L_{q,\varphi}}\le C\|f\|_{L_{p,\varphi}}$$

holds.

2) $\varphi\in A_{1+\frac{q}{p'}}(Y)$, $\frac{1}{p}+\frac{1}{p'}=1$.

By the Proposition 1 we have

$$\left(\int_0^r\left(\left(M_{\alpha,\gamma}(f|\cdot|^{\beta\alpha})(x)\right)^q x^\beta\right)y^\gamma dy\right)^{\frac{1}{q}}\le$$

$$\le \left(\int_Y \left(M_{\alpha,\nu}(\overline{f}|\cdot|^{\beta\alpha}) \left(\sqrt{y^2+\overline{y}^2},0\right)\right)^q \times \right.$$

$$\left. \times \left(\sqrt{y^2+\overline{y}^2},0\right)^{\beta} M_\nu \chi_{E((x,0),r)}(y,\overline{y})\, d\nu(y,\overline{y})\right)^{\frac{1}{q}} \le$$

$$\le C_2 \left(\int_Y \left| \overline{f}\left(\sqrt{y^2+\overline{y}^2},0\right)\right|^p \left(\sqrt{y^2+\overline{y}^2},0\right)^{\beta} M_\nu \chi_{E((x,0),r)}(y,\overline{y})\, d\nu(y,\overline{y})\right)^{\frac{1}{p}} =$$

$$= C_2 \left(\int_Y \left| f\left(\sqrt{y^2+\overline{y}^2}\right)\right|^p \left(\sqrt{y^2+\overline{y}^2}\right)^{\beta} M_\nu \chi_{E((x,0),r)}(y,\overline{y})\, d\nu(y,\overline{y})\right)^{\frac{1}{p}} =$$

$$= C_2 \left(\int_0^\infty T^y(|f(x)|^p x^\beta) M_\gamma \chi_{(0,r)}(y) y^\gamma dy \right)^{\frac{1}{p}} \le$$

$$\le C_2 \left(\int_0^r T^y(|f(x)|^p x^\beta) y^\gamma dy + C_2 \sum_{j=1}^\infty \int_{2^j r}^{2^{j+1} r} T^y(|f(x)|^p x^\beta) M_\gamma \chi_{E_r}(y) y^\gamma dy \right)^{\frac{1}{p}} \le$$

$$\le C_2 \left(\int_0^r T^y(|f(x)|^p x^\beta) y^\gamma dy + C_2 \sum_{j=1}^\infty \int_{2^j r}^{2^{j+1} r} T^y(|f(x)|^p x^\beta) \frac{r^{1+\gamma}}{(y+r)^{1+\gamma}} y^\gamma dy \right)^{\frac{1}{p}} \le$$

$$\le C_3 \, \|f\|_{\mathcal{L}_{p,\lambda,|\cdot|^\beta,\gamma}} \left(r^{\frac{\lambda}{p}} + \sum_{j=1}^\infty \frac{1}{(2^j+1)^{1+\gamma}} (2^{j+1} r)^{\frac{\lambda}{p}} \right) \le C_4 \, r^{\frac{\lambda}{p}} \|f\|_{\mathcal{L}_{p,\lambda,|\cdot|^\beta,\gamma}}.$$

3. The B-Riesz potential in the spaces $\mathcal{L}_{p,\lambda,|\cdot|^\beta,\gamma}(\mathbb{R}_+)$

In this we give a full description of measures for which weighted estimates for the fractional integral $I_{\alpha,\gamma}$ hold, using the method of G.Welland [33].

We consider the B-Riesz potential

$$I_{\alpha,\gamma} f(x) = \int_0^\infty T^y x^{\alpha-1-\gamma} f(y) y^\gamma dy, \quad 0 < \alpha < 1+\gamma$$

We start with a lemma.

Lemma 3. *[6] Let* $0 < \alpha < 1+\gamma$. *For any* ε, $0 < \varepsilon < \min(\alpha, 1+\gamma-\alpha)$, *there exists a constant* $c_\varepsilon > 0$ *such that for any nonnegative function* $f : (0,\infty) \to \mathbb{R}$ *and for any point* $x \in (0,\infty)$ *the following inequality holds:*

$$|I_{\alpha,\gamma} f(x)| \le C_\varepsilon \sqrt{M_{\alpha-\varepsilon,\gamma}|f(x)| M_{\alpha+\varepsilon,\gamma}|f(x)|}. \qquad (4)$$

Theorem 2. *Let* $0 < \alpha < 1+\gamma$, $0 \le \lambda < 1+\gamma$ *and* $1 < p < \frac{1+\gamma-\lambda}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{1+\gamma-\lambda}$, $-\frac{p'(1+\gamma)}{p'+q} < \beta < \frac{q(1+\gamma)}{p'+q}$. *Then*

$$\|I_{\alpha,\gamma}(f|\cdot|^{\beta\alpha})\|_{\mathcal{L}_{q,\lambda,|\cdot|^\beta,\gamma}} \le C \|f\|_{\mathcal{L}_{p,\lambda,|\cdot|^\beta,\gamma}}$$

holds.

Proof. Therefore it is possible to choose ε, $0<\varepsilon<\min(\alpha,1+\gamma-\alpha)$. If we now take

$$\frac{1}{q_\varepsilon}=\frac{1}{p}-\left(\frac{\alpha}{1+\gamma-\lambda}+\varepsilon\right),\qquad \frac{1}{\overline{q_\varepsilon}}=\frac{1}{p}-\left(\frac{\alpha}{1+\gamma-\lambda}-\varepsilon\right).$$

Denoting

$$p_1=\frac{2q_\varepsilon}{q}\quad and\quad p_2=\frac{2\overline{q_\varepsilon}}{q}$$

we have

$$\frac{1}{p_1}+\frac{1}{p_2}=1.$$

Put

$$F_1(x)=\left(M_{\alpha+\varepsilon,\gamma}|(f|\cdot|^{\beta\alpha})(x)|\right)^{\frac{q}{2}}x^{\frac{\beta}{p_1}}$$

and

$$F_2(x)=\left(M_{\alpha-\varepsilon,\gamma}|(f|\cdot|^{\beta\alpha})(x)|\right)^{\frac{q}{2}}x^{\frac{\beta}{p_2}}.$$

Further, (4) together with Holder's inequality implies the estimate

$$\int_0^r T^y(|I_{\alpha,\gamma}(f|\cdot|^{\beta\alpha})(x)|^q x^\beta)y^\gamma dy\le c_\varepsilon\int_0^r T^y(F_1(x)F_2(x))y^\gamma dy\le$$

$$\le c_\varepsilon\left(\int_0^r T^y\left(\left(M_{\alpha+\varepsilon,\gamma}|(f|\cdot|^{\beta\alpha})(x)|\right)^{\frac{qp_1}{2}}x^\beta\right)y^\gamma dy\right)^{\frac{1}{p_1}}\times$$

$$\times\left(\int_0^r T^y\left(\left(M_{\alpha-\varepsilon,\gamma}|(f|\cdot|^{\beta\alpha})(x)|\right)^{\frac{qp_2}{2}}x^\beta\right)y^\gamma dy\right)^{\frac{1}{p_2}}=$$

$$=c_\varepsilon\left(\int_0^r T^y\left(\left(M_{\alpha+\varepsilon,\gamma}|(f|\cdot|^{\beta\alpha})(x)|\right)^{q_\varepsilon}x^\beta\right)y^\gamma dy\right)^{\frac{1}{p_1}}\times$$

$$\times\left(\int_0^r T^y\left(\left(M_{\alpha-\varepsilon,\gamma}|(f|\cdot|^{\beta\alpha})(x)|\right)^{\overline{q_\varepsilon}}x^\beta\right)y^\gamma dy\right)^{\frac{1}{p_2}}.$$

Finally, using Theorem 2 we conclude that

$$\|I_{\alpha,\gamma}(f|\cdot|^{\beta\alpha})\|_{\mathcal{L}_{q,\lambda,|\cdot|^\beta,\gamma}}\le C\|f\|_{\mathcal{L}_{p,\lambda,|\cdot|^\beta,\gamma}}$$

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References

[1]. Adams D.R. *A note on Riesz potentials* . Duke Math., 1975, 42, pp.765-778.

[2]. Aliev I.A., Gadjiev A.D. *Weighted estimates for multidimensional singular integrals generated by a generalized shift operator,* Mat. Sb. 1992, 183 (9), pp.45-66 (in Russian); translation in Russian Acad. Sci. Sb. Math. 1994, 77 (1), pp. 37-55.

[3]. Coifman R.R., Weiss G. *Analyse harmonique non commutative sur certains expaces homogenes.* Lecture Notes in Math., **242**, Springer-Verlag. Berlin, 1971.

[4]. Chiarenza F., Frasca M. *Morrey spaces and Hardy–Littlewood maximal function.* Rend. Math. 1987, 7, pp. 273-279.

[5]. Gadjiev A.D., Aliev I.A. *On classes of operators of potential types, generated by a generalized shift.* Reports of enlarged Session of the Seminars of I.N.Vekua Inst. of Applied Mathematics, Tbilisi. 1988, **3**, 2, pp. 21-24 (in Russian).

[6]. Guliyev E.V. *Weighted inequality for fractional maximal functions and fractional integrals,associated with the laplace-bessel differential operator.* Transactions of NAS of Azerbaijan pp. 71-80

[7]. Guliyev V.S. *Function spaces, integral operators and two weighted inequalities on homogeneous groups. Some applications.* Baku. 1999, pp. 1-332. (in Russian)

[8]. Guliev V.S. *Sobolev theorems for the Riesz B-potentials.* Dokl. RAN, 1998, **358**, 4, pp. 450-451. (in Russian)

[9]. Guliev V.S. *Sobolev theorems for anisotropic Riesz–Bessel potentials on Morrey-Bessel spaces.* Doklady Academy Nauk Russia, 1999, **367**, 2, pp. 155-156.

[10]. Guliyev V.S. *On maximal function and fractional integral, associated with the Bessel differential operator.* Mathematical Inequalities and Applications, 2003, **6**, 2, pp. 317-330.

[11]. Guliyev V.S., Hasanov J.J. *The Sobolev-Morrey type inequality for Riesz potentials, associated with the Laplace-Bessel differential operator.* Fractional Calculus and Applied Analysis. 2006, **9** 1, pp.17-32.

[12]. Guliyev V.S., Hasanov J.J. *Necessary and sufficient conditions for the boundedness of B-Riesz potential in the B-Morrey spaces.* J. Math. Anal. Appl. 2008, **347**, pp.113-122.

[13]. Guliyev V.S., Hasanov J.J. *Necessary and sufficient conditions for the boundedness of Riesz potential in the modifed Morrey spaces.* (to appear)

[14]. Danielli D., *A Fefferman-Phong type inequality and applications to quasilinear subelliptic equatioins.* Potential Analysis, 1999, **11**, pp. 387-413.

[15]. Ekincioglu I., Serbetci A. *On Boundedness of Riesz potential generated by generalized shift operator on Ba spaces*, Czech. Math. J., 2004, **54**, 3, pp. 579-589.

[16]. Hasanov J.J. *A note on anisotropic potentials, associated with the Laplace-Bessel differential operator.* Operators and Matrices, 2008, **2** 4, pp.465-481.

[17]. Hasanov J.J., Zeren Yusuf, *On limiting case of the Sobolev theorem for B-Riesz potential in B-Morrey spaces.* Arab J. Math. Sci. 2007, **13**, pp.27-38.

[18]. Levitan B.M. *Expansion in Fourier series and integrals with Bessel functions,* (Russian) Uspehi Matem. Nauk (N.S.) 1951, 6, 2 (42), pp. 102-143.

[19]. Lyakhov L.N. *Multipliers of the mixed Fourier-Bessel transform,* (Russian) Proc. Steklov Inst. Math. 1997, 214 (3), pp. 234-249.

[20]. Kokilashvili V.M., Kufner A., *Fractional integrals on spaces of homogeneous type.* Comment. Math. Univ. Carolinae 1989, 3, **30**, pp.511-523.

[21]. Kipriyanov I.A. *Fourier-Bessel transformations and imbedding theorems,* Trudy Math. Inst. Steklov. 1967, 89, pp. 130-213.

[22]. Macias R.A., Segovia C. *A well behaved quasidistance for spaces of homogeneous type.* Trab. Mat. Inst. Argent. Mat., 1981, 32, 18p.

[23]. Muckenhoupt B., Stein E.M. *Classical expansions and their relation to conjugate harmonic functions.* Trans. Amer. Math. Soc., 1965, 118, pp. 17-92.

[24]. Morrey C.B. *On the solutions of quasi-linear elliptic partial differential equations.* Trans. Amer. Math. Soc. 1938, 43, pp.126-166.

[25]. Samko S.G., Kilbas A.A., Marichev O.I. *Fractional Integrals and Derivative. Theory and Applications.* Gordon and Breach Sci. Publishers, 1993.

[26]. Herz C.S. *On the mean inversion of Fourier and Hankel transforms*, Proc. Nat. Acad. Sci. USA, 1954, **40**, pp.996–999.

[27]. Mourou M.A., Trimeche K. *Inversion of the Weyl integral transform and the Radon transform on R^n using generalized wavelets*, C.R. Acad. Sci. Canada. 1996, **18**, No 2-3, pp.80-84.

[28]. Sneddon I.N. *The Use of Integral Transforms*, Tata McGraw–Hill, New Delhi, 1974.

[29]. Stein E.M. *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, NJ, 1970.

[30]. Stempak K. *Almost everywhere summability of Laguerre series.* Studia Math. 1991, 100 (2), pp. 129-147.

[31]. Trimeche K. *Transformation integrale de Weyl et theoreme de Paley-Wiener associerentiel sur* $(0,\infty)$, J. Math. Pures Appl., 1981, **60**, pp. 51-98.

[32]. Trimeche K. *Inversion of Lions transmutation operators using generalized wavelets*, Appl. Comput. Harmonic Anal., 1997, **4** pp. 1-16.

[33]. Welland G. *Weighted norm inequalities for fractional integrals.* Proc. Amer. Math. Soc. 1975, **51**, 1, pp. 143148.

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