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# ON A BOUNDARY VALUE PROBLEM FOR SECOND ORDER OPERATOR-DIFFERENTIAL EQUATIONS IN SPACE OF SMOOTH VECTOR-FUNCTIONS

**Abstract**

*In the paper the solvability conditions in abstract spaces of smooth vector-functions for some initial-boundary value problems for a second order equation with operator coefficients are found. All these conditions are expressed by the features of coefficients of an operator-differential equation.*

Let $H$ be a separable Hilbert space, $A$ a positive-definite self-adjoint operator in $H$, $H_\gamma\ (\gamma \geq 0)$ a scale of Hilbert spaces generated by the operator $A$, i.e.

$$H_\gamma = D\left(A^\gamma\right), \quad (x,y)_\gamma = \left(A^\gamma x, A^\gamma y\right), \quad x,y \in H_\gamma.$$

Let $L_2\left(R_+;H\right)$ be a Hilbert space of vector-functions $f\left(t\right)$ determined almost everywhere in $R_+$ with the values in $H$, for which

$$\|f\|_{L_2(R_+;H)} = \left(\int_0^\infty \|f\left(t\right)\|^2 dt\right)^{1/2}.$$

Following the monograph [1] define the following space for natural $m \geq 1$:

$$W_2^m\left(R_+;H\right) = \left\{u\left(t\right) : u^{(m)}\left(t\right) \in L_2\left(R_+;H\right), \quad A^m u\left(t\right) \in L_2\left(R_+;H\right)\right\},$$

with the norm

$$\|u\|_{W_2^m(R_+;H)} = \left(\left\|u^{(m)}\right\|^2_{L_2(R_+;H)} + \|A^m u\|^2_{L_2(R_+;H)}\right)^{1/2}.$$

For $m = 3$ we'll derive subspaces in $W_2^3\left(R_+;H\right)$

$$\overset{\circ}{W}{}_2^3\left(R_+;H\right) = \left\{u : u \in W_2^3\left(R_+;H\right), u\left(0\right) = u'\left(0\right) = 0\right\}.$$

The spaces $L_2\left(R;H\right)$ and $W_2^m\left(R;H\right)$ for $R = \left(-\infty;\infty\right)$ are determined similarly. Let $L\left(X,Y\right)$ be a space of linear bounded operators acting from $X$ to $Y$.

Consider in $H$ the following boundary value problem

$$\frac{d^2u}{dt^2} + \left(pA + A_1\right)\frac{du}{dt} + \left(qA^2 + A_2\right)u\left(t\right) = f\left(t\right), \qquad t \in R_+ \tag{1}$$

$$u\left(0\right) = u'\left(0\right) = 0 \tag{2}$$

where $f(t)$, $u(t \in H)$ for $t \in R_+$ almost everywhere, and the operator coefficients satisfy the following conditions:

1) $p > 0$, $q > 0$.

2) $A$ is a positive-definite self-adjoint operator;

3) The operators $A_1 \in L(H_1, H) \cap L(H_2, H_1)$, $A_2 \in L(H_2, H) \cap L(H_3, H_1)$,

**Definition 1.** *It for $f(t) \in W_2^1(R_+; H)$ there exists the vector-function $u(t) \in W_2^3(R_+; H)$ that satisfies equation (1) identically in $R_+ = (0, \infty)$, then $u(t)$ will be called a smooth solution of equation (1) from $W_2^3(R_+; H)$.*

**Definition 2.** *If for any $f(t) \in W_2^1(R_+; H)$ there exists a smooth solution $u(t)$ of equation (1) from $W_2^3(R_+; H)$ that satisfies boundary conditions in the sense of convergence $\lim\limits_{t \to +0} \|u(t)\|_{5/2} = 0$, $\lim\limits_{t \to +0} \|u'(t)\|_{3/2} = 0$ and it holds the estimation*

$$\|u\|_{W_2^3(R_+;H)} \leq const \, \|f\|_{W_2^1(R_+;H)},$$

*then problem (1), (2) is called regularly solvable in the space $W_2^3(R_+; H)$.*

In the given paper we find a condition on the coefficient of equation (1) that provides regular solvability of the problem in space $W_2^3(R_+; H)$. Notice that for the elliptic equation ($p = 0$, $q = -1$) such problems were investigated in [2,3,4].

Denote by

$$P_0 u = P_0(d/dt) u = u'' + pAu' + qA^2 u, \quad u \in \overset{\circ}{W}{}_2^3(R_+; H)$$

$$P_1 u = P_1(d/dt) u = A_1 \frac{du}{dt} + A_2 u, \quad u \in \overset{\circ}{W}{}_2^3(R_+; H)$$

$$Pu = P_0 u + P_1 u, \quad u \in \overset{\circ}{W}{}_2^3(R_+; H).$$

It is easy to see that subject to condition 1) the bundle

$$P_0(\lambda) = \lambda^2 + pA\lambda + qA^2$$

is of the form $P_0(\lambda) = (\lambda - \omega_1 A)(\lambda - \omega_2 A)$, where $\operatorname{Re}\omega_1 < 0$, $\operatorname{Re}\omega_2 < 0$.

Therefore equation (1) belongs to the parabolic type. For $f \in L_2(R_+; H)$, $u \in W_2^2(R_+; H)$ this problem was studied in [5].

At first consider the following equation

$$P_0 u = f, \quad u \in \overset{\circ}{W}{}_2^3(R_+; H), \quad f \in W_2^1(R_+; H).$$

It holds the following theorem.

**Theorem 1.** *Let conditions 1) and 2) be fulfilled. Then the operator $P_0$ isomorphically maps the space $\overset{\circ}{W}{}_2^3(R_+; H)$ onto $L_2(R_+; H)$.*

**Proof.** Show that the kernel of the operator $P_0$ consists of only a zero element.

Since the general solution of the equation $P_0(d/dt) u(t) = 0$ from the space $W_2^3(R_+; H)$ is of the form

$$u_0(t) = e^{\omega_1 t A} \varphi_0 + e^{\omega_2 t A} \varphi_1$$

for $\omega_1 \neq \omega_2$ or

$$u_0(t) = e^{\omega_1 tA}\varphi_0 + tAe^{\omega_2 tA}\varphi_1$$

for $\omega_1 = \omega_2$ where $\varphi_0, \varphi_1 \in H_{5/2}$, $\omega_1$ and $\omega_2$ are the roots of the polynomial

$$P_0(\lambda) = \lambda^2 + p\lambda + q.$$

In the first case $(\omega_1 \neq \omega_2)$ from boundary condition (2) it follows that $\varphi_0 + \varphi_1 = 0$ and $\omega_1 A\varphi_0 + \omega_2 A\varphi_1 = 0$ or $\varphi_1 = -\varphi_0$ and $(\omega_1 - \omega_2) A\varphi_0 = 0$. Hence it follows $A\varphi_0 = 0$ i.e. $\varphi_0 = 0$. Consequently $\varphi_1 = 0$, i.e. $u_0(t) = 0$. In the second case $(\omega_1 = \omega_2)$ from condition (2) it follows that $\varphi_0 = 0$ and $\varphi_1 = 0$, i.e. $u_0(t) = 0$. Thus, $KerP_0 = \{0\}$.

Now show that the image of the operator $P_0$ coincides with $\overset{\circ}{W}{}_2^3(R_+; H)$. Since $f \in W_2^1(R_+; H)$, we can continue it on the negative semi-axis so that [1] its continuation $f_1(t) \in W_2^1(R_+; H)$, and

$$\|f_1\|_{W_2^1(R_+;H)} \leq const\, \|f\|_{W_2^1(R_+;H)}.$$

Then from the equation $P_0(d/dt)\, u(t) = f_1(t)$, after Fourier transformation we get that the vector-function

$$u_1(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\xi^2 E + \xi pA + qA^2\right)^{-1} \widehat{f}_1(\xi)\, e^{\iota\xi t}, \quad t \in R$$

satisfies the equation $P_0(d/dt) = f(t)$ in $R_+$. Show that $u_1(t) \in W_2^3(R; H)$. From the Plancherel theorem it follows that it suffices to prove $A^3\widehat{u}_1(\xi) \in L_2(R; H)$ and $\xi^3 A^3\widehat{u}_1(\xi) \in W_2^3(R; H)$. Obviously,

$$\left\|A^3\widehat{u}_1(\xi)\right\|_{L_2(R;H)} = \left\|A^3\left(\xi^2 E + \xi pA + qA^2\right)^{-1}\widehat{f}_1(\xi)\right\|_{L_2(R;H)} \leq$$

$$\leq \sup_{\xi\in A}\left\|A^2\left(\xi^2 E + \xi pA + qA^2\right)^{-1}\right\| \cdot \left\|A\widehat{f}_1(\xi)\right\|_{L_2(R;H)} \leq$$

$$\leq const \sup_{\xi\in A}\left\|A^2\left(\xi^2 E + \xi pA + qA^2\right)^{-1}\right\| \cdot \|f\|_{W_2^1(R_+;H)}. \tag{3}$$

On the other hand, for any $\xi \in A$

$$\left\|A^2\left(\xi^2 E + \xi pA + qA^2\right)^{-1}\right\| = \sup_{\mu\in\sigma(A)}\left|\mu^2\left(\xi^2 + \xi p\mu + q\mu^2\right)^{-1}\right| \leq$$

$$\leq \sup_{\mu\geq\mu_0}\left|\left(\xi^2/\mu^2 + p\xi/\mu + q\right)^{-1}\right| \leq \sup_{\tau>0}\left|\tau^2 + p\tau + q\right|^{-1} \leq \beta_0^{-1/2},$$

where

$$\beta_0^{-1/2} = \begin{cases} \frac{2}{p(4q-p^2)^{1/2}}, & q \geq \frac{p^2}{2}, \\ \frac{1}{q}, & q \leq \frac{p^2}{2}. \end{cases}$$

From (3) it follows that $A^3u_1(\xi)\in L_2(R_+;H)$. It is similarly proved that $\xi^3u_1(\xi)\in L_2(R_+;H)$. Now denote by $\xi_1(t)=u_1(t)/_{[0,\infty)}$, i.e. $\xi_1(t)$ is the contraction of the vector-function $u_1(t)$ on $[0,\infty)$. Then $\xi_1(t)\in W_2^3(R_+;H)$ and $\xi_1(0)\in H_{5/2}$, $\xi_1'(0)\in H_{3/2}$. Now we'll look for the solution of the equation $P_0u=f$ in the form

$$u(t)=\xi_1(t)+e^{\omega_1tA}\varphi_0+e^{\omega_2tA}\varphi_1,\quad \varphi_0,\varphi_1\in H_{5/2}\quad for\quad \omega_1\neq\omega_2.$$

Hence, using boundary conditions (2), we get:

$$\varphi_0+\varphi_1=-\xi_1(0)\qquad and\qquad \omega_1A\varphi_1+\omega_2A\varphi_2=-\xi_1'(0).$$

Since $\xi_1(0)\in H_{5/2}$ and $\xi_1'(0)\in H_{3/2}$, then $\varphi_0=\frac{1}{\omega_2-\omega_1}\left[A^{-1}\xi_1'(0)+\omega_2\xi'(0)\right]\in H_{5/2}$ and $\varphi_1=-\varphi_0-\xi_1(0)\in H_{5/2}$. Consequently, $u\in \overset{\circ}{W}{}_2^3(R_+;H)$ i.e. the image of the operator $P_0$ coincides with the space $L_2(R_+;H)$. On the other hand, for $u\in W_2^3(R_+;H)$

$$\|P_0u\|^2_{W_2^1(R_+;H)}=\|A\cdot P_0u\|^2_{L(R_+;H)}+\left\|\frac{d}{dt}P_0u\right\|^2_{L_2(R_+;H)}=$$

$$=\left\|Au''+pA^2u'+A^3u\right\|^2_{L_2(R_+;H)}+\left\|u'''+pAu''+A^2u'\right\|^2_{L_2(R_+;H)}.$$

Using the theorem on intermediate derivatives, we get

$$\|P_0u\|_{W_2^1(R_+;H)}\leq const\,\|u\|_{W_2^3(R_+;H)}.$$

Finally, applying the Banach theorem on the inverse operator we get the statement of the theorem.

Consider in the space $H_6$ the following operator bundles of sixth order

$$R_0(\lambda;\beta;A)=$$
$$=\left((i\lambda)^4E+\left(p^2-2q\right)(i\lambda)^2A^2+q^2A^4-\beta A^4\right)\left(-\lambda^2E+A^2\right)\tag{4}$$

and

$$R_1(\lambda;\gamma;A)=$$
$$=\left((i\lambda)^4E+\left(p^2-2q\right)(i\lambda)^2A^2+q^2A^4-\gamma(i\lambda)^2A^2\right)\left(-\lambda^2E+A^2\right)\tag{5}$$

where $p,q$ are real numbers, $\beta$ and $\gamma$ real parameters.

It holds the following

**Theorem 2.** *Let $p>0$, $q>0$. Then for $\beta\in[0,\beta_0)$, and for $\gamma\in[0,\gamma_0)$ operator bundles (4) and (5) have no spectrum on the imaginary axis, where*

$$\beta_0=\begin{cases} q^2, & 0<q\leq\frac{p^2}{2}\\ \frac{p^2\left(4q-p^2\right)}{4}, & q\geq\frac{p^2}{2},\quad \gamma_0=p^2\end{cases}\tag{6}$$

*and they are represented in the form*

$$R_0(\lambda;\beta;A)=F_0(\lambda;\beta;A)F_0(-\lambda;\beta;A),\tag{7}$$

$$R_1(\lambda;\gamma;A)=F_1(\lambda;\gamma;A)\,F_1(-\lambda;\gamma;A)\,, \tag{8}$$

*where*

$$F_0(\lambda;\beta;A)=\prod_{j=1}^{3}(\lambda E-\omega_{j,0}(\beta)A)=$$

$$=\lambda^3E+c_{2,0}(\beta)\lambda^2A+c_{1,0}(\beta)\lambda A^2+c_{0,0}(\beta)A^3 \tag{9}$$

$$F_1(\lambda;\gamma;A)=\prod_{j=1}^{3}(\lambda E-\omega_{j,1}(\beta)A)=$$

$$=\lambda^3E+a_{1,0}(\gamma)\lambda^2A+a_{2,0}(\gamma)\lambda A^2+a_{0,0}(\gamma)A^3. \tag{10}$$

Here $\omega_{j,0}(\beta)=\omega_{j,0}(\alpha)=-1$, $\operatorname{Re}\omega_{j,0}(\beta)<0$, $\omega_{j,1}(\gamma)<0$, for $\beta\in[0,\beta_0)$ and $\gamma\in[0,\gamma_0)$, and the numbers

$$c_{2,0}=1+2\sqrt{\sqrt{q^2-\beta}+p^2-2q},\quad c_{1,0}=\sqrt{2\sqrt{q^2-\beta}+p^2-2q}+\sqrt{q^2-\beta}$$

$$c_{0,0}(\beta)=\sqrt{q^2-\beta}, \tag{11}$$

$$a_{2,1}(\gamma)=1+\sqrt{p^2-\gamma},\quad a_{1,1}=\sqrt{p^2-\gamma}+q,\quad a_{0,1}(\gamma)=q. \tag{12}$$

**Proof.** Prove the statement for the bundle $R_1(\lambda;\gamma;A)$, since it is proved for $R_0(\lambda;\beta;A)$ similarly.

Let $\mu\in\sigma(A)$ $(\mu\ge\mu_0>0)$. Then for $\lambda=i\xi$, $\xi\in R$

$$R_1(i\xi;\gamma;\mu)=\left(\xi^2+\mu^2\right)\left(\xi^4+\left(p^2-2q\right)\xi^2\mu^2+q^2\mu^4-\gamma\xi^2\mu^2\right)=$$

$$=\left(\xi^2+\mu^2\right)\left(1-\gamma\frac{\xi^2\mu^2}{\xi^4+(p^2-2q)\xi^2\mu^2+q^2\mu^4}\right)\left(\xi^4+\left(p^2-2q\right)\xi^2\mu^2+q^2\mu^4\right)=$$

$$=\left(\xi^2+\mu^2\right)\mu^4\left(\frac{\xi^4}{\mu^4}+\frac{\left(p^2-2q\right)\xi^2}{\mu^2}+q^2\right)\times$$

$$\times\left(1-\gamma\frac{\xi^2/\mu^2}{(\xi/\mu)^4+(p^2-2q)(\xi/\mu)^2+q^2}\right)\ge$$

$$\ge\left(\xi^2+\mu^2\right)\mu^4\left(\tau^4+\left(p^2-2q\right)\tau^2+q^2\right)\left(1-\gamma\sup_{\tau\ge0}\frac{\tau^2}{\tau^4+(p^2-2q)\tau^2+q^2}\right).$$

Since, $\tau^4+\left(p^2-2q\right)\tau^2+q^2>0$, that

$$R_1(i\xi;\gamma;\mu)\ge\left(\xi^2+\mu^2\right)\mu^4\left(1-\gamma\frac{1}{p^2}\right)>0.$$

Thus, for $\gamma\in[0,\gamma_0)$ $\left(\gamma_0=p^2\right)$ the polynomial $R_1(\lambda;\gamma;\mu)$ has no roots on the imaginary axis. Then its roots are symmetric with respect to the real axis and origin of coordinates. Therefore

$$R_1(\lambda;\gamma;\mu)=F_1(\lambda;\gamma;\mu)\cdot F_1(-\lambda;\gamma;\mu) \tag{13}$$

where

$$F_1(\lambda;\gamma;\mu) = (\lambda+\mu)\cdot(\lambda-\omega_{1,1}(\gamma)\mu)(\lambda-\omega_{1,2}(\gamma)\mu), \tag{14}$$

where

$$\operatorname{Re}\omega_{1,1}(\gamma) < 0, \quad \operatorname{Re}\omega_{1,2}(\gamma) < 0.$$

Assuming

$$F_1(\lambda;\gamma;\mu) = \lambda^3 + a_{2,1}(\gamma)\lambda^2\mu + a_{1,1}(\gamma)\lambda\mu + a_{0,1}(\gamma)\mu^3,$$

by comparing the coefficients in equality (13), we find the coefficients $a_{21}(\gamma)$, $a_{1,1}(\gamma)$, $a_{1,0}(\gamma)$.

The proof is obtained from equality (13) by using the spectral expansion of the operator $A$. The theorem is proved.

**Lemma 1.** *Let* $u \in W_2^3(R_+;H)$. *Then*

$$\|P_0u\|^2_{W_2^1(R_+;H)} = \left\|\frac{d^3u}{dt^3}\right\|^2_{L_2(R_+;H)} + \left(p^2+1-2q\right)\left\|A\frac{d^2u}{dt^2}\right\|^2_{L_2(R_+;H)} +$$

$$+\left(q^2+p^2-2q\right)\left\|A\frac{du}{dt}\right\|^2_{L_2(R_+;H)} + q^2\left\|A^3u\right\|^2_{L_2(R_+;H)} - p\|\varphi\|^2,$$

*where* $\varphi = A^{1/2}u''(0)$.

**Proof.** It is obvious that

$$\|P_0u\|^2_{W_2^1(R_+;H)} = \left\|\frac{d^3u}{dt^3}pA\frac{d^2u}{dt^2} + qA^2\frac{du}{dt}\right\|^2_{L_2(R_+;H)} +$$

$$+\left\|A\frac{d^2u}{dt^2} + pA^2\frac{du}{dt} + qA^3u\right\|^2_{L_2(R_+;H)} =$$

$$= \left\|\frac{d^3u}{dt^3}\right\|^2_{L_2(R_+;H)} + p^2\left\|A\frac{d^2u}{dt^2}\right\|^2_{L_2(R_+;H)} + q^2\left\|A^2\frac{du}{dt}\right\|^2_{L_2(R_+;H)} +$$

$$+2p\operatorname{Re}\left(\frac{d^3u}{dt^3}, A\frac{d^2u}{dt^2}\right)_{L_2(R_+;H)} + q^2 2\operatorname{Re}\left(\frac{d^3u}{dt^3}, A^2\frac{du}{dt}\right)_{L_2(R_+;H)} +$$

$$+pq2\operatorname{Re}\left(A\frac{d^2u}{dt^2}, qA^2\frac{du}{dt}\right)_{L_2(R_+;H)} + \left\|A\frac{d^2u}{dt^2}\right\|^2_{L_2(R_+;H)} +$$

$$+p^2\left\|A^2\frac{du}{dt}\right\|^2_{L_2(R_+;H)} + q^2\left\|A^3u\right\|^2_{L_2(R_+;H)} + 2p\operatorname{Re}\left(A\frac{d^2u}{dt^2}, A^2\frac{du}{dt}\right)_{L_2(R_+;H)} +$$

$$+q2\operatorname{Re}\left(A\frac{d^2u}{dt^2}, A^3u\right)_{L_2(R_+;H)} + q2\operatorname{Re}\left(A^2\frac{du}{dt}, A^3u\right)_{L_2(R_+;H)}. \tag{15}$$

For $u \in \overset{\circ}{W}{}_2^3(R_+;H)$ $(u(0) = u'(0) = 0)$

$$\left(\frac{d^3u}{dt^3}, A\frac{d^2u}{dt^2}\right)_{L_2(R_+;H)} = -\left(A^{1/2}u''(0), A^{1/2}u'(0)\right) - \left(A\frac{d^2u}{dt^2}, \frac{d^3u}{dt^3}\right)_{L_2(R_+;H)}$$

i.e.

$$2\operatorname{Re}\left(\frac{d^3u}{dt^3}, A\frac{d^2u}{dt^2}\right) = -\left\|u''(0)\right\|_{1/2}^2 = -\left\|\varphi_2\right\|_{1/2}. \tag{16}$$

Similarly we have

$$\left(\frac{d^3u}{dt^3}, A^2\frac{du}{dt}\right)_{L_2(R_+;H)} = \left(A^{1/2}u''(0), A^{3/2}u'(0)\right) - \left\|A\frac{d^2u}{dt^2}\right\|_{L_2(R_+;H)}^2 \tag{17}$$

and

$$\left(A\frac{d^2u}{dt^2}, A^2\frac{du}{dt}\right)_{L_2(R_+;H)} = -\left(A^{3/2}u'(0), A^{3/2}u'(0)\right) - \left(A^2\frac{du}{dt}, \frac{d^2u}{dt^2}\right)_{L_2(R_+;H)}$$

i.e.

$$2\operatorname{Re}\left(A\frac{d^2u}{dt^2}, A^2\frac{du}{dt}\right)_{L_2(R_+;H)} = 0. \tag{18}$$

For $u \in \overset{\circ}{W}{}_2^3(R_+;H)$ the following equalities are proved similarly

$$\operatorname{Re}\left(A\frac{d^2u}{dt^2}, A^3u\right) = -\left\|A^2\frac{du}{dt^2}\right\|_{L_2(R_+;H)} \tag{19}$$

$$2\operatorname{Re}\left(A\frac{du}{dt}, A^3u\right) = 0. \tag{20}$$

Taking into account equality (16)-(20) in (15), we have:

$$\left\|P_0u\right\|_{W_2^1(R_+;H)}^2 = \left\|\frac{d^3u}{dt^3}\right\|_{L_2(R_+;H)}^2 + \left(p^2+1-2q\right)\left\|A\frac{d^2u}{dt^2}\right\|_{L_2(R_+;H)} +$$

$$+\left(q^2+p^2-2q\right)\left\|A\frac{du}{dt}\right\|_{L_2(R_+;H)}^2 + q^2\left\|A^3u\right\|_{L_2(R_+;H)}^2 - p\left\|\varphi\right\|^2.$$

The lemma is proved

**Lemma 2.** *For $u \in \overset{\circ}{W}{}_2^3(R_+;H)$ it holds the equality*

$$\left\|F_1(d/dt;\gamma;A)\,u\right\|_{L_2(R_+;H)}^2 + \left(\alpha_{21}(\gamma)-p\right) =$$

$$= \left\|P_0u\right\|_{W_2^1(R_+;H)}^2 - \gamma\left\|A\frac{du}{dt}\right\|_{W_2^1(R_+;H)}^2. \tag{21}$$

**Proof.** For $u \in \overset{\circ}{W}{}_2^3(R_+;H)$ it holds the following equality

$$\left\|F_1(d/dt;\gamma;A)\,u\right\|_{L_2(R_+;H)}^2 =$$

$$= \left\|\frac{d^3u}{dt^3} + a_{2,1}(\gamma)\,A\frac{d^2u}{dt^2} + a_{1,1}(\gamma)\,A^2\frac{du}{dt} + a_{0,1}(\gamma)\,A^3u\right\|_{L_2(R_+;H)}^2 =$$

$$= \left\|\frac{d^3u}{dt^3}\right\|_{L_2(R_+;H)}^2 + a_{2,1}^2(\gamma)\left\|A\frac{d^2u}{dt^2}\right\|_{L_2(R_+;H)} +$$

$$+a_{1,1}^2\left\|A^2\frac{du}{dt}\right\|_{L_2(R_+;H)}^2+a_{0,1}^2\left\|A^3u\right\|_{L_2(R_+;H)}^2+$$

$$+2\operatorname{Re}a_{2,1}(\gamma)\left(\frac{d^3u}{dt^3},A\frac{d^2u}{dt^2}\right)_{L_2(R_+;H)}+2\operatorname{Re}a_{1,1}(\gamma)\left(\frac{d^3u}{dt^3},A^2\frac{du}{dt}\right)_{L_2(R_+;H)}+$$

$$+2\operatorname{Re}a_{0,1}(\gamma)\left(\frac{d^3u}{dt^3},A^3u\right)_{L_2(R_+;H)}+2a_{2,1}(\gamma)\,a_{1,1}(\gamma)\left(A\frac{d^2u}{dt^2},A^2\frac{du}{dt}\right)_{L_2(R_+;H)}+$$

$$+2a_{2,1}(\gamma)\,a_{0,1}(\gamma)\left(A\frac{d^2u}{dt^2},A^3u\right)_{L_2(R_+;H)}+2a_{1,1}(\gamma)\cdot a_{0,1}(\gamma)\left(A^2\frac{du}{dt},A^3u\right)_{L_2(R_+;H)}.$$

Taking into account $u(0)=u'(0)$ and (16)-(20), we get:

$$\left\|F_1(d/dt;\gamma;A)\,u\right\|_{L_2(R_+;H)}^2=\left\|\frac{d^3u}{dt^3}\right\|_{L_2(R_+;H)}^2+a_{21}^2(\gamma)\left\|A\frac{d^2u}{dt^2}\right\|_{L_2(R_+;H)}^2+$$

$$+a_{1,1}^2(\gamma)\left\|A^2\frac{du}{dt}\right\|_{L_2(R_+;H)}^2+a_{0,1}^2\left\|A^3u\right\|_{L_2(R_+;H)}^2-a_{2,1}(\gamma)\left\|\varphi_2\right\|^2-$$

$$-2a_{1,1}(\gamma)\left\|A\frac{d^2u}{dt^2}\right\|_{L_2(R_+;H)}-2a_{2,1}(\gamma)\,a_{0,1}(\gamma)\left\|A^2\frac{du}{dt}\right\|_{L_2(R_+;H)}^2=$$

$$=\left\|\frac{d^3u}{dt^3}\right\|_{L_2(R_+;H)}^2+\left(a_{2,1}(\gamma)-2a_{1,1}(\gamma)\right)\left\|A\frac{d^2u}{dt^2}\right\|_{L_2(R_+;H)}^2+$$

$$+\left(a_{11}^2(\gamma)-2a_{21}(\gamma)\,a_{1,0}(\gamma)\right)\left\|A^2\frac{du}{dt}\right\|_{L_2(R_+;H)}-a_{2,1}(\gamma)\left\|\varphi\right\|_{1/2}^2.$$

Taking into account $a_{21}(\gamma)=1+\sqrt{p^2-\gamma}$ $a_{1,1}(\gamma)=\sqrt{p^2-\gamma}+q$, $a_{1,0}(\gamma)=q$, we get:

$$\left\|F_1(d/dt;\gamma;A)\,u\right\|_{L_2(R_+;H)}^2=\left\|\frac{d^3u}{dt^3}\right\|_{L_2(R_+;H)}^2+\left(1+p^2-2q\right)\left\|A\frac{d^2u}{dt^2}\right\|_{L_2(R_+;H)}^2+$$

$$+\left(p^2+q^2-2q\right)\left\|A\frac{du}{dt}\right\|_{L_2(R_+;H)}^2+a_{2,1}(\gamma)\left\|\varphi\right\|_{1/2}^2-$$

$$-\gamma\left(\left\|A\frac{d^2u}{dt^2}\right\|_{L_2(R_+;H)}^2+\left\|A^2\frac{du}{dt}\right\|_{L_2(R_+;H)}^2\right).$$

Since $\left\|\frac{d^2u}{dt^2}\right\|_{L_2(R_+;H)}^2+\left\|A^2\frac{du}{dt}\right\|_{L_2(R_+;H)}^2=\left\|A\frac{du}{dt}\right\|_{W_2^1(R_+;H)}^2$ then taking into account the lemma in the last equality, we get:

$$\left\|F_1(d/dt;\gamma;A)\,u\right\|_{L_2(R_+;H)}^2+\left(a_{2,1}(\gamma)-p\right)\left\|\varphi\right\|_{1/2}^2=$$

$$=\left\|P_0u\right\|_{W_2^1(R_+;H)}^2-\gamma\left\|A\frac{du}{dt}\right\|_{W_2^1(R_+;H)}^2.$$

Similarly we prove

**Lemma 3.** *For* $u \in \overset{\circ}{W}{}_2^3 (R_+; H)$ *it holds the equality*

$$\|F_0 (d/dt; \beta; A) u\|^2_{L_2(R_+;H)} + (c_{2,1} (\beta) - \beta) \|\varphi\|^2_{1/2} =$$
$$= \|P_0 u\|^2_{W_2^1(R_+;H)} - \beta \left\|A^2 u\right\|^2_{W_2^1(R_+;H)}. \tag{22}$$

*Note that it follows from theorem 1 that in space* $\overset{\circ}{W}{}_2^3 (R_+; H)$ *the norms* $\|P_0 u\|_{W_2^1(R_+;H)}$ *and* $\|u\|_{W_2^3(R_+;H)}$ *are equivalent, therefore the following norms are finite*

$$N_1 = \sup_{0 \neq u \in \overset{\circ}{W}{}_2^3(R_+;H)} \left\|A \frac{du}{dt}\right\|_{W_2^1(R_+;H)} \cdot \|P_0 u\|^{-1}_{W_2^1(R_+;H)} \tag{23}$$

*and*

$$N_0 = \sup_{0 \neq u \in \overset{\circ}{W}{}_2^3(R_+;H)} \left\|A^2 u\right\|_{W_2^1(R_+;H)} \cdot \|P_0 u\|^{-1}_{W_2^1(R_+;H)} \tag{24}$$

Further, by obtaining the solvability conditions of problem (1), (2), the norms $N_1$ and $N_0$ are very important.

**Theorem 3.** *The norm*

$$N_1 = \begin{cases} \frac{1}{p}, & 0 < p \leq 1 \\ \left(\frac{1}{2p-1}\right)^{1/2}, & p > 1 \end{cases}.$$

**Proof.** Carrying out similar reasonings of the paper [6] we get that if the equation $(a_{2,1} (\gamma) - p) = 0$ (see formula (21)) has no solution from the interval $(0, \gamma_0) \left(\gamma_0 = p^2\right)$, then $N_1 = \gamma_0^{-1/2} = \frac{1}{p}$. If the equation $(a_{2,1} (\gamma) - p) = 0$ has a solution from the integral $(0, \gamma_0)$, then $N_1$ is the inverse of the square root of the least of these solutions. Therefore, we should solve the equation $a_{2,1} (\gamma) - p = 0$. Consequently,

$$1 + \sqrt{p^2 - \gamma} - p = 0 \tag{25}$$

or

$$\sqrt{p^2 - \gamma} = p - 1.$$

It is obvious that for $p \leq 1$ this equation has no solution from the interval $(0, p)$. Therefore $N_1 = \frac{1}{p}$, for $0 < p \leq 1$. And for $p > 1$ the equation (25) has the solution $\gamma_1 = 2p - 1 \in (0, p)$. Therefore $N_1 = \left(\frac{1}{2p-1}\right)^{1/2}$ for $p > 1$. The theorem is proved.

The following theoprem is proved in the same way.

**Theorem 4.** *The norm*

$$N_0 = \begin{cases} \beta_0^{-1/2}, p \leq 1; \quad p > 1, \quad q \leq p - \frac{1}{2} \\ \left(q^2 - (2q - 2p + 1)^2\right)^{\frac{1}{2}}, \quad p > 1, \quad q > p - \frac{1}{2} \end{cases}.$$

**Proof.** In order to find the number $N_0$ we should solve the equation $c_{11} (\beta) - p = 0$ (see formula (22)). Then

$$1 + \sqrt{2\sqrt{q^2 - \beta} + p^2 - 2q} = p$$

i.e.

$$\sqrt{2\sqrt{q^2-\beta}+p^2-2q}=p-1. \qquad (26)$$

Then for $p\leq 1$ the equation $c_{1,1}(\beta)-p=0$ has no solution from the interval $(0,\beta)$. Therefore for $p\leq 1$, $N_0=\beta_0^{-1/2}$. Let $p>1$. Then

$$2\sqrt{q^2-\beta}+p^2-2q=(p-1)^2, \quad 2\sqrt{q^2-\beta}=2q-2p+1.$$

Obviously, for $q\leq p-\frac{1}{2}\,(p>1)$ the equation (26) has no solution from the interval $(0,\beta)$. Therefore $N_0=\beta_0^{-1}$. If $q>p-\frac{1}{2}$, then for $q\leq\frac{p^2}{2}$ the equation (25) has the solution $\beta_1=q^2-(2q-2p+1)\in(0,\beta_0)$. Therefore $N_0=\beta_1^{-1/2}$, and for $q\geq p^2$ we get $p-\frac{1}{2}\leq q\leq p^2$.

Hence we have that $(p-1)^2<0$. And this is impossible.

Therefore, in this case, $N_0=\beta_0^{-1/2}$ as well. The theorem is proved.

Now prove the main theorem.

**Theorem 5.** *Let conditions 1)-3) be fulfilled, and*

$$q=N_1\max\left(\|A_1\|_{H_1\to H},\|A_1\|_{H_2\to H_1}\right)+N_0\max\left(\|A_2\|_{H_2\to H},\|A_2\|_{H_3\to H_1}\right)<1,$$

*where the numbers $N_1$ and $N_0$ are determined from theorems 3 and 4, respectively. Then problem (1) (2) is reqularly solvable in $W_2^3(R_+;H)$.*

**Proof.** By theorem 1, the operator $P_0$ isomorphically maps the space $\overset{\circ}{W}{}_2^3(R_+;H)$ onto $W_2^1(R_+;H)$. Then for any $\omega\in W_2^1(R_+;H)$ there exists $u\in\overset{\circ}{W}{}_2^3(R_+;H)$ such that $P_0^{-1}\omega=u$. Now write the problem (1), (2) in the form of the equation

$$Pu\equiv P_0u+P_1u=f, \quad u\in\overset{\circ}{W}{}_2^3(R_+;H), \quad f\in W_2^1(R_+;H).$$

After substitution of $u=P_0^{-1}\omega$ we get the equation $\left(E+P_1P_0^{-1}\right)\omega=f$ in $W_2^1(R_+;H)$. Since for any $\omega\in W_2^1(R_+;H)$

$$\left\|P_1P_0^{-1}\omega\right\|_{W_2^1(R_+;H)}=\|P_1u\|_{W_2^1(R_+;H)}=\left\|A_1\frac{du}{dt}+A_2u\right\|_{W_2^1(R_+;H)}\leq$$

$$\leq\left\|A_1\frac{du}{dt}\right\|_{W_2^1(R_+;H)}+\|A_2u\|_{W_2^1(R_+;H)}=$$

$$=\left(\left\|A_1\frac{d^2u}{dt^2}\right\|_{L_2(R_+;H)}+\left\|AA_1\frac{du}{dt}\right\|^2_{L_2(R_+;H)}\right)^{1/2}+$$

$$+\left(\left\|A_2\frac{du}{dt}\right\|^2_{L_2(R_+;H)}+\|AA_2u\|^2_{L_2(R_+;H)}\right)^{1/2}\leq\left(\left\|A_1A^{-1}\right\|^2_{H\to H}\left\|A\frac{d^2u}{dt^2}\right\|_{L_2(R_+;H)}+\right.$$

$$\left.+\left\|AA_1A^{-2}\right\|^2_{H\to H}\left\|A^2\frac{du}{dt}\right\|^2_{L_2(R_+;H)}\right)^{1/2}+\left(\left\|A_2A^{-2}\right\|^2_{H\to H}\left\|A^2u\right\|^2_{L_2(R_+;H)}+\right.$$

$$+\left\|AA_2A^{-3}\right\|^2\left\|A^3u\right\|^2_{L_2(R_+;H)}\Big)^{1/2}\leq \max\left(\left\|A_1A^{-1}\right\|_{H\to H},\left\|AA_1A^{-2}\right\|_{H\to H}\right)\times$$

$$\times\left(\left\|A\frac{d^2u}{dt^2}\right\|^2_{L_2(R_+;H)}+\left\|A^2\frac{du}{dt}\right\|^2_{L_2(R_+;H)}\right)^{1/2}+$$

$$+\max\left(\left\|A_2A^{-2}\right\|_{H\to H},\left\|AA_2A^{-3}\right\|_{H\to H}\right)\times$$

$$\times\left(\left\|A^2\frac{d^2u}{dt^2}\right\|^2_{L_2(R_+;H)}+\left\|A^3u\right\|^2_{L_2(R_+;H)}\right)^{1/2}=$$

$$=\max\left(\left\|A_1\right\|_{H_1\to H},\left\|A_1\right\|_{H_2\to H_1}\right)\times$$

$$\times\left\|A\frac{du}{dt}\right\|_{W_2^1(R_+;H)}+\max\left(\left\|A_2\right\|_{H\to H_2},\left\|A_2\right\|_{H_3\to H_1}\right)\cdot\left\|A^2u\right\|_{W_2^1(R_+;H)}.$$

Applying theorems 3 and 4, we get

$$\left\|PP_0^{-1}u\right\|_{W_2^1(R_+;H)}\leq$$

$$\leq\left(N_1\max\left(\left\|A_1\right\|_{H_1\to H},\left\|A_1\right\|_{H_2\to H_1}\right)+N_0\max\left(\left\|A_2\right\|_{H_2\to H},\left\|A_2\right\|_{H_3\to H_1}\right)\right)\times$$

$$\times\left\|P_0u\right\|_{W_2^1(R_+;H)}=q\left\|P_0u\right\|_{W_2^1(R_+;H)}=q\left\|\omega\right\|_{W_2^1(R_+;H)}.$$

Since by the theorem condition $q<1$, then the operator $E+P_1P_0^{-1}$ is invertible in $W_2^1(R_+;H)$. Then $\omega=\left(E+P_1P_0^{-1}\right)^{-1}f$, and $u=P_0^{-1}\left(E+P_1P_0^{-1}\right)^{-1}f$. Hence it follows that

$$\left\|u\right\|_{W_2^3(R_+;H)}\leq const\left\|f\right\|_{W_2^1(R_+;H)}.$$

The theorem is proved.

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