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GRAPHS OF CONGRUENCE SCHEMES OF ALGEBRAS HAVING k-ARY NEAR UNANIMITY TERMS

Abstract

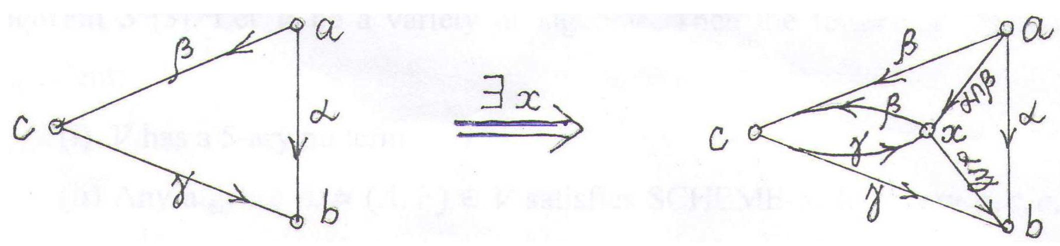
We present graphs of congruence schemes characterizing of algebras in varieties having (respectively) 4-ary, 5-ary and 6-ary near unanimity terms.

Here we present the graphs of some congruence schemes. These schemes characterize the algebras with k-ary near unanimity terms for values $k = 4, 5, 6$. The algebras with 3-ary near unanimity terms were characterized by a congruence scheme in the paper I. Chajda and S. Radeleczki [1, theorem 3.3]; for the sake of completeness we adduce this result also. A term function $m(x_1, ..., x_k)$ of algebra $\mathbb{A} =(A; F)$ is called k-ary near unanimity term (nu term) (see [2], p. 34) if the identities $m(y, x, x, ..., x) = m(x, y, x, ..., x) = ... = m(x, ..., x, y) = x$ hold for all $x, y \in A$. A variety $\mathcal{V}$ has k-ary nu term m, if in every algebra of $\mathcal{V}$ the term m is k-ary nu term.

In the paper [3] we characterized varieties with 4-ary (and 5-ary) nu terms. So, we below repeat these results without proofs and we present only corresponding graphs for congruence schemes.

Theorem 1 [1]. *Let $\mathcal{V}$ be a variety of algebras. Then the following assertions are equivalent: (1) $\mathcal{V}$ has a ternary nu term.*

(ii) Any algebra $\mathbb{A} = (A; F) \in \mathcal{V}$ satisfies SCHEME-3: for every $a, c, b \in A$ and any compatible reflexive relations $\alpha, \beta, \gamma \subseteq A \times A$ the implication below is satisfied (here $x := m(a, c, b)$)



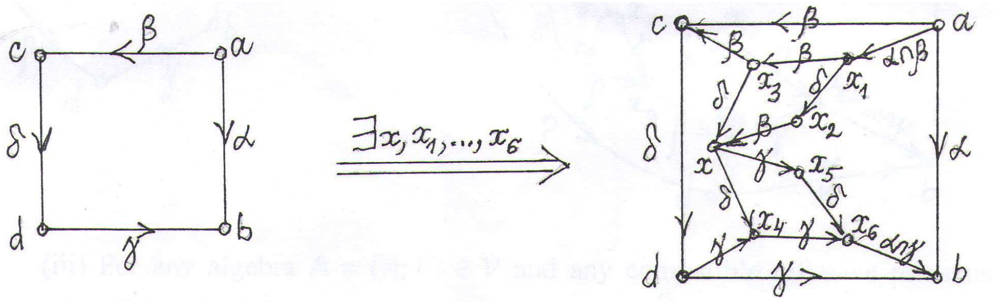
(iii) For any algebra $\mathbb{A} = (A; F) \in \mathcal{V}$ and any compatible reflexive relations $\alpha, \beta, \gamma \subseteq A \times A$ we have $\alpha \cap (\beta \circ \gamma) \subseteq (\alpha \cap \beta) \circ (\alpha \cap \gamma)$.

(iv) For any algebra $\mathbb{A} = (A; F) \in \mathcal{V}$ and any congruences $\alpha, \beta, \gamma \in Con\mathbb{A}$ we have $\alpha \cap (\beta \circ \gamma) = (\alpha \cap \beta) \circ (\alpha \cap \gamma)$.

Theorem 2 [3]. *Let $\mathcal{V}$ be a variety of algebras. Then the following assertions are equivalent:*

(i) $\mathcal{V}$ has a 4-ary nu term.

(ii) Any algebra $\mathbb{A} = (A; F) \in \mathcal{V}$ satisfies SCHEME-4: for every $a, c, e, b \in A$ and any compatible reflexive relations $\alpha, \beta, \delta, \gamma \subseteq A \times A$ the implication below is satisfied (here $x := m(a, c, d, b),\ x_1 := m(a, a, c, b),\ x_2 := m(a, a, e, b),\ x_3 :=$ $= m(a, c, c, b),\ x_4 := m(a, e, e, b),\ x_5 := m(a, c, b, b),\ x_6 := m(a, e, b, b))$



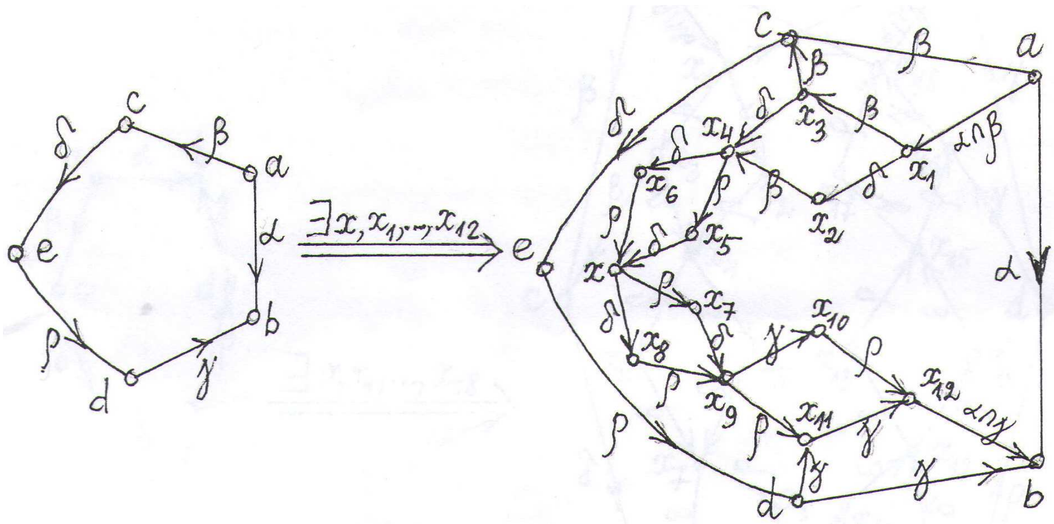
(iii) For any algebra $\mathbb{A} = (A; F) \in \mathcal{V}$ *and any compatible reflexive relations* $\alpha, \beta, \delta, \gamma \subseteq A \times A$ *we have* $\alpha \cap (\beta \circ \delta \circ \gamma) \subseteq [((\alpha \cap \beta) \circ (\beta \circ \delta \cap \delta \circ \beta) \circ (\delta \circ \gamma \cap \gamma \circ \delta)) \cap \alpha] \circ (\alpha \cap \gamma) \cap (\alpha \cap \beta) \circ [((\beta \circ \delta \cap \delta \circ \beta) \circ (\delta \circ \gamma \cap \gamma \circ \delta) \circ (\alpha \cap \gamma)) \cap \alpha]$.

(iv) For any algebra $\mathbb{A} = (A; F) \in \mathcal{V}$ *and any congruences* $\alpha, \beta, \delta, \gamma \in$ $\in Con\mathbb{A}$ *we have* $\alpha \cap (\beta \circ \delta \circ \gamma) = [((\alpha \cap \beta) \circ (\beta \circ \delta \cap \delta \circ \beta) \circ (\delta \circ \gamma \cap \gamma \circ \delta)) \cap$ $\cap \alpha] \circ (\alpha \cap \gamma) \cap (\alpha \cap \beta) \circ [((\beta \circ \delta \cap \delta \circ \beta) \circ (\delta \circ \gamma \cap \gamma \circ \delta) \circ (\alpha \cap \gamma)) \cap \alpha]$.

Theorem 3 [3]. *Let* $\mathcal{V}$ *be a variety of algebras. Then the following assertions are equivalent:*

(i) $\mathcal{V}$ *has a 5-ary nu term.*

(ii) Any algebra $\mathbb{A} = (A; F) \in \mathcal{V}$ *satisfies SCHEME-5: for every* $a, c, e, d, b \in A$ *and any compatible reflexive relations* $\alpha, \beta, \delta, \rho, \gamma \subseteq A \times A$ *the implication below is satisfied (here* $x := m(a, c, e, d, b)$, $x_1 := m(a, a, a, c, b)$, $x_2 := m(a, a, a, e, b)$, $x_3 := m(a, c, c, c, b)$, $x_4 := m(a, c, c, e, b)$, $x_5 := m(a, c, c, d, b)$, $x_6 := m(a, c, e, e, b)$, $x_7 := m(a, c, d, d, b)$, $x_8 := m(a, e, e, d, b)$, $x_9 := m(a, e, d, d, b)$, $x_{10} := m(a, e, b, b, b)$, $x_{11} := m(a, d, d, d, b)$, $x_{12} := m(a, d, b, b, b)$*)*



(iii) For any algebra $\mathbb{A} = (A; F) \in \mathcal{V}$ *and any compatible reflexive relations* $\alpha, \beta, \delta, \rho, \gamma \subseteq A \times A$ *we have*

$$\alpha \cap (\beta \circ \delta \circ \rho \circ \gamma) \subseteq [((\alpha \cap \beta) \circ (\beta \circ \delta \cap \delta \circ \beta) \circ (\delta \circ \rho \cap \rho \circ \delta) \circ$$
$$\circ (\delta \circ \rho \cap \rho \circ \delta) \circ (\gamma \circ \rho \cap \rho \circ \gamma)) \cap \alpha] \circ (\alpha \cap \gamma) \cap (\alpha \cap \beta) \circ [\alpha \cap ((\beta \circ \delta \cap \delta \circ \beta) \circ$$
$$\circ (\delta \circ \rho \cap \rho \circ \delta) \circ (\delta \circ \rho \cap \rho \circ \delta) \circ (\gamma \circ \rho \cap \rho \circ \gamma) \circ (\alpha \cap \gamma))].$$

(iv) For any algebra $\mathbb{A} = (A; F) \in \mathcal{V}$ *and any congruences* $\alpha, \beta, \delta, \rho, \gamma \in$ $\in C$ *on* $\mathbb{A}$ *we have*

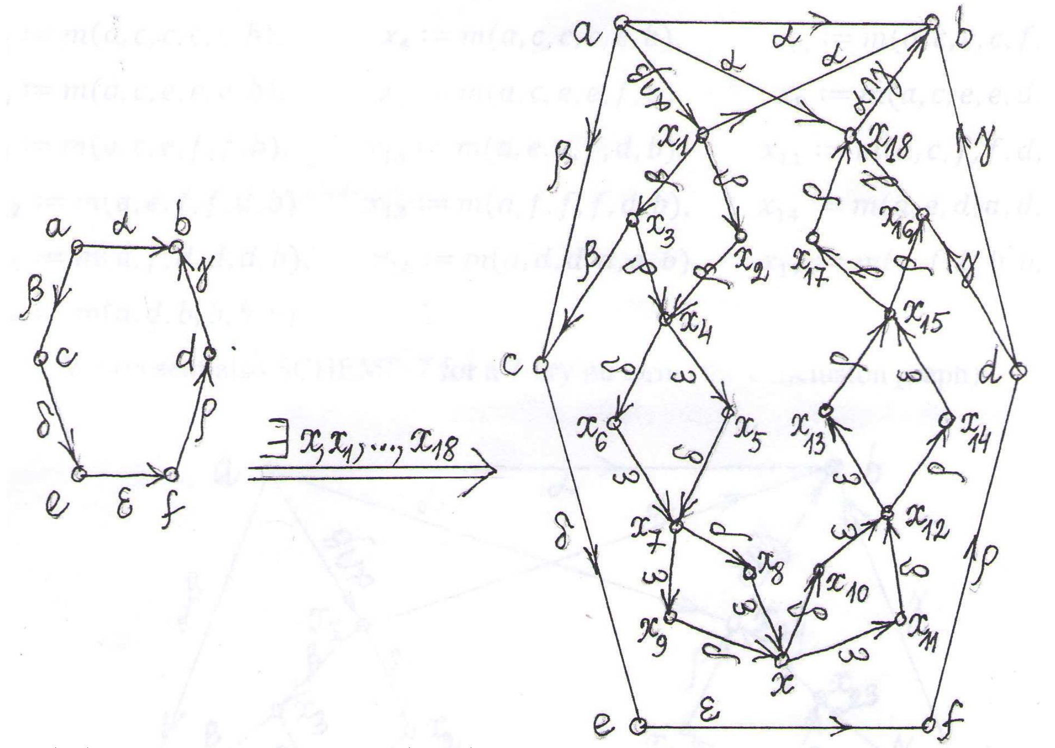
$$\alpha \cap (\beta \circ \delta \circ \rho \circ \gamma) = [((\alpha \cap \beta) \circ (\beta \circ \delta \cap \delta \circ \beta) \circ (\delta \circ \rho \cap \rho \circ \delta) \circ$$
$$\circ (\delta \circ \rho \cap \rho \circ \delta) \circ (\gamma \circ \rho \cap \rho \circ \gamma)) \cap \alpha] \circ (\alpha \cap \gamma) \cap (\alpha \cap \beta) \circ [\alpha \cap ((\beta \circ \delta \cap \delta \circ \beta) \circ$$
$$\circ (\delta \circ \rho \cap \rho \circ \delta) \circ (\delta \circ \rho \cap \rho \circ \delta) \circ (\gamma \circ \rho \cap \rho \circ \gamma) \circ (\alpha \cap \gamma))].$$

Now we present following

Theorem 4. *Let $\mathcal{V}$ be a variety of algebras. Then the following assertions are equivalent:*

(i) $\mathcal{V}$ has a 6-ary nu term.

(ii) Any algebra $\mathbb{A} = (A; F) \in \mathcal{V}$ satisfies SCHEME-6: for every $a, c, e, f, d, b \in \in A$ and any compatible reflexive relations $\alpha, \beta, \delta, \varepsilon, \rho, \gamma \subseteq A \times A$ the implication below is satisfied



(iii) For any algebra $\mathbb{A} = (A; F) \in \mathcal{V}$ and any compatible reflexive relations $\alpha, \beta, \delta, \varepsilon, \rho, \gamma \subseteq A \times A$ we have

$$\alpha \cap (\beta \circ \delta \circ \varepsilon \circ \rho \circ \gamma) \subseteq [((\alpha \cap \beta) \circ (\beta \circ \delta \cap \delta \circ \beta) \circ (\delta \circ \varepsilon \cap \varepsilon \circ \delta) \circ$$
$$\circ (\varepsilon \circ \rho \cap \rho \circ \varepsilon) \circ (\delta \circ \varepsilon \cap \varepsilon \circ \delta) \circ (\varepsilon \circ \rho \cap \rho \circ \varepsilon) \circ (\gamma \circ \rho \cap \rho \circ \gamma)) \cap \alpha] \circ$$
$$\circ (\alpha \cap \gamma) \cap (\alpha \cap \beta) \circ [\alpha \cap ((\beta \circ \delta \cap \delta \circ \beta) \circ (\delta \circ \varepsilon \cap \varepsilon \circ \delta) \circ (\varepsilon \circ \rho \cap \rho \circ \varepsilon) \circ$$
$$\circ (\delta \circ \varepsilon \cap \varepsilon \circ \delta) \circ (\varepsilon \circ \rho \cap \rho \circ \varepsilon) \circ (\gamma \circ \rho \cap \rho \circ \gamma) \circ (\alpha \cap \gamma))].$$

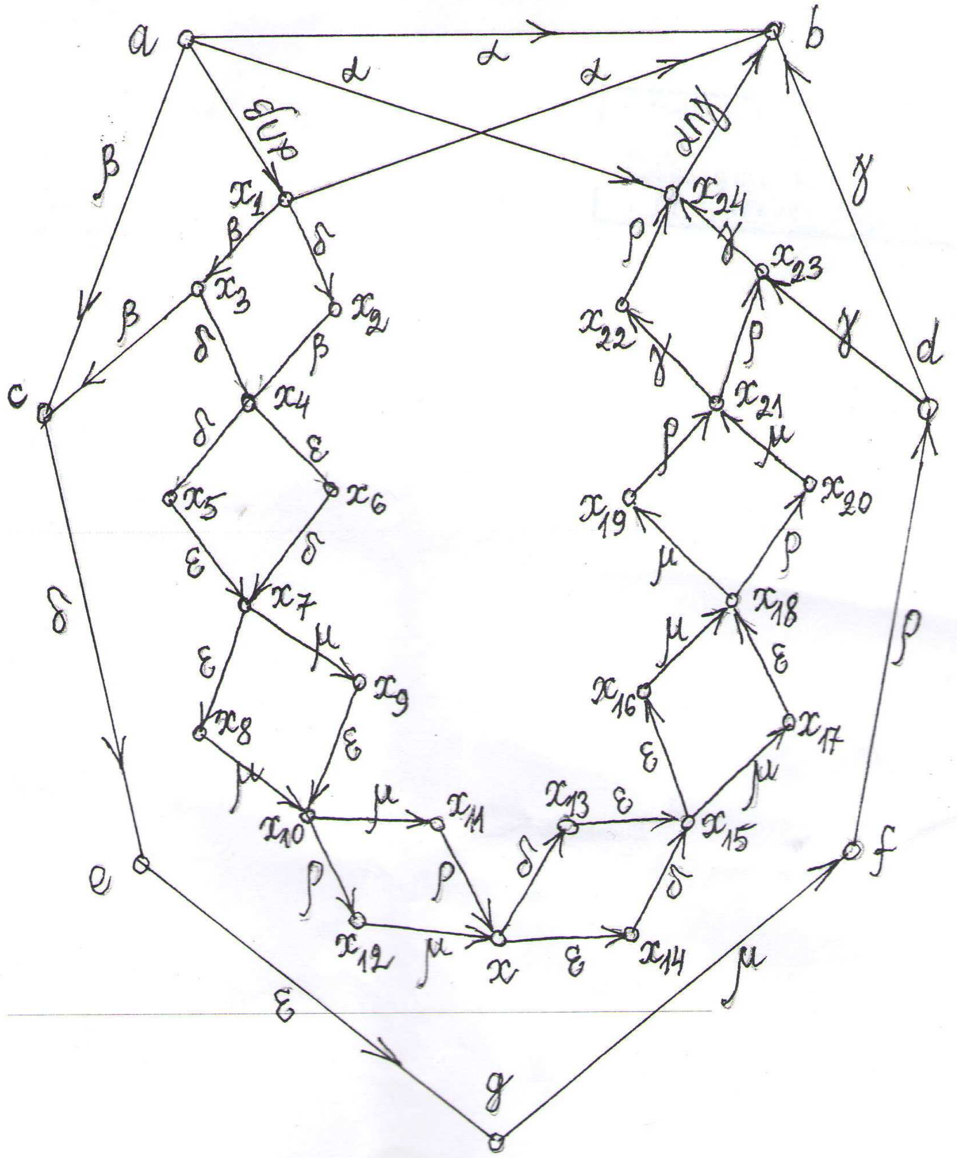
(iv) For any algebra $\mathbb{A} = (A; F) \in \mathcal{V}$ and any congruences $\alpha, \beta, \delta, \varepsilon, \rho, \gamma \in \in Con\mathbb{A}$ we have

$$\alpha \cap (\beta \circ \delta \circ \varepsilon \circ \rho \circ \gamma) = [((\alpha \cap \beta) \circ (\beta \circ \delta \cap \delta \circ \beta) \circ (\delta \circ \varepsilon \cap \varepsilon \circ \delta) \circ$$
$$\circ (\varepsilon \circ \rho \cap \rho \circ \varepsilon) \circ (\delta \circ \varepsilon \cap \varepsilon \circ \delta) \circ (\varepsilon \circ \rho \cap \rho \circ \varepsilon) \circ (\gamma \circ \rho \cap \rho \circ \gamma)) \cap \alpha] \circ (\alpha \cap \gamma) \cap$$
$$\cap (\alpha \cap \beta) \circ [\alpha \cap ((\beta \circ \delta \cap \delta \circ \beta) \circ (\delta \circ \varepsilon \cap \varepsilon \circ \delta) \circ (\varepsilon \circ \rho \cap \rho \circ \varepsilon) \circ (\delta \circ \varepsilon \cap \varepsilon \circ \delta) \circ$$
$$\circ (\varepsilon \circ \rho \cap \rho \circ \varepsilon) \circ (\gamma \circ \rho \cap \rho \circ \gamma) \circ (\alpha \cap \gamma))].$$

Proof. The proof is similar to the proof of Theorem 2 in [3]; only take here

$x := m(a, c, e, f, d, b)$, $x_1 := m(a, a, a, a, c, b)$, $x_2 := m(a, a, a, a, e, b)$,
$x_3 := m(a, c, c, c, c, b)$, $x_4 := m(a, c, c, c, e, b)$, $x_5 := m(a, c, c, c, f, b)$,
$x_6 := m(a, c, e, e, e, b)$, $x_7 := m(a, c, e, e, f, b)$, $x_8 := m(a, c, e, e, d, b)$,
$x_9 := m(a, c, e, f, f, b)$, $x_{10} := m(a, e, e, f, d, b)$, $x_{11} := m(a, c, f, f, d, b)$,
$x_{12} := m(a, e, f, f, d, b)$, $x_{13} := m(a, f, f, f, d, b)$, $x_{14} := m(a, e, d, d, d, b)$,
$x_{15} := m(a, f, d, d, d, b)$, $x_{16} := m(a, d, d, d, d, b)$, $x_{17} := m(a, f, b, b, b, b)$,
$x_{18} := m(a, d, b, b, b, b)$

We present also SCHEME-7 for a 7-ary nu term (the conclusion graph):



So, every SCHEME-k contains $2(k-3)$ rhombi inside it.

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