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ON A BOUNDARY VALUE PROBLEM FOR EQUATIONS WITH MULTIPLE CHARACTERISTIC

Abstract

In the paper, for a fourth order differential equation we study existence domains of fundamental system of solutions for large values of the spectral parameter in the case when the representations of solutions contain fractional powers of the parameter both in the exponent's index and in the multiplier of the exponent at the assumption that the appropriate principal characteristical polynomial has two different roots with definite multipliticities. A class of generalized regular solutions of boundary conditions for which it holds a formula of expansions in eigen functions, is distinguished.

Let's consider a boundary value problem on the interval [0,1]

$$y^{(2n)}+P_1(x,\lambda)\,y^{(2n-1)}+...+P_{2n}(x,\lambda)\,y=0, \tag{1}$$

$$U_i(y)=\sum_{j=0}^{2n-1}\alpha_{ij}(\lambda)\,y^{(i)}(0)+\beta_{ij}(\lambda)\,y^{(j)}(1)=0,\qquad i=\overline{1,2n}, \tag{2}$$

where

$$P_i(x,\lambda)=\sum_{j=0}^{i}P_{ij}(x)\,\lambda^j,\quad i=1,2n,\quad P_{ij}=const,\quad P_{ij}(x)\in C^{2n-i+j}[0,1],$$

$l<i$, $i=1,2n$, $\alpha_{ij}(\lambda)$, $\beta_{ij}(\lambda)$ are the polynomials of the spectral parameter $\lambda\in C$ and for all λ $rang\left(\alpha_{ij}(\lambda),\beta(\lambda)\right)_{i,j=1}^{2n}=2n$, it is assumed that the characteristic equation $\theta^{2n}+P_n\theta^{2n-1}+...+P_{2n,2n}=0$ has two different roots θ_1,θ_2 and each root has the multiplicity n and $\theta_1<0<\theta_2$.

In the paper [1], fundamental systems of the solutions of equation (1), containing fractional powers of the spectral parameter both in the exponent's index and in the multiplier of the exponent, and formulated in theorem 1 of this paper are found under certain conditions. In the same place, a class of boundary conditions for which expansion in eigen functions hold, are distinguished.

In the present paper, following the denotation of the paper [1], for concrete situations we have found the existence domains of these solutions, studied the classes of boundary conditions called generalized regular, for which the formula of expansion in eigen functions is valid.

Assume that the functions $F_k(x,\theta_k)=\sum\limits_{i=1}^{2n}P_{ij-1}(x)\,\theta_k^{2n-1}$, $k=\overline{1,2}$, $x\in[0,1]$ accept the values on the ray emanating from the origin of coordinates. Then all $\varphi_i^{(1)}(x)$, $k=\overline{1,2n}$ will be disposed on $2n$ rays for each k. Note on the plane $\lambda^{\frac{n_i-1}{n_i}}$ for the fixed $x_0\in[0,1]$ the points $\overline{\varphi}_1^{(1)}(x_0),...,\overline{\varphi}_{2n}^{(1)}(x_0)$. By $\overline{\varphi}_i^{(1)}(x_0)$ $i=\overline{1,2n}$ we denote the numbers complexly conjugated with $\overline{\varphi}_i^{(1)}$. Draw to the sides of the right $2n$ - angle the perpendiculars $l_1, ..., l_{2n}$. Denote by Ω_i the sector between l_1 and l_{i+1}. For definiteness we'll assume that the common boundary l_{i+1} of two sectors Ω_i and Ω_{i+1} belongs to Ω_{i+1}. Let the point ξ_i belong to the domain Ω_i. It is easy to verify that $\operatorname{Re}\xi_i\varphi_i^{(1)} = \left(\xi_i, \overline{\varphi}_i^{(1)}\right)$, where the scalar product of the vectors ξ_i and $\overline{\varphi}_i^{(1)}$ stands from the right. It is clear that $\operatorname{Re}\xi_i\varphi_i^{(1)} = |\xi_i| \left|\overline{\varphi}_i^{(1)}\right| \cos\alpha_i$, where α_i is an angle between the vectors ξ_i and $\overline{\varphi}_i^{(1)}$ determined by the formula $\cos\alpha_i = \frac{\xi_i, \overline{\varphi}_i^{(1)}}{|\xi_i|\ \overline{\varphi}_i^{(1)}}$. Choosing properly the minor coefficients, we can obtain the equality

$$\operatorname{Re}\psi_i(x_0,\lambda) - \operatorname{Re}\psi_j(x_0,\lambda) =$$

$$= 2|\lambda|^{\frac{n_j-1}{n_i}} \sin\frac{\alpha_i+\alpha_j}{2} \sin\frac{\alpha_j-\alpha_i}{2} n_i \sqrt{\left|\frac{F(x,\theta_k)}{\theta_2-\theta_1}\right|},$$

$$1 \le i, \quad j \le n_1, \quad n_1+1 \le i, \quad j \le 2n$$

and get that if $\alpha_i \le \alpha_j$, then $\operatorname{Re}\psi_i(x_0,\lambda) \ge \operatorname{Re}\psi_j(x_0,\lambda)$.

Lemma. *The plane λ (with a section on the positive part of a real axis) is covered with the sectors Ω_i' that while mapping $\lambda^{\frac{n_i-1}{n_j}}$ will pass to Ω_i.*

Let $\lambda \in \Omega_i'$. Then $\operatorname{Re}\lambda^{\frac{n_i-1}{n_j}}\psi_k^{(1)} \ge \operatorname{Re}\lambda^{\frac{n_i-1}{n_j}}\psi_i^{(1)}$. Fix j and l $\left(j,l=\overline{1,2n}\right)$.

Theorem. *In any sector of Ω_i', $i=\overline{1,2n}$ containing for the given j none of the curves determined by the equality $\operatorname{Re}\psi_i(x_0,\lambda) = \operatorname{Re}\psi_j(x_0,\lambda)$ there exist solutions for which as $|\lambda| \to \infty$, $\lambda \in \Omega_i'$ representations (4) from the paper [1] are valid.*

Furthermore, one of the boundaries of this sector may coincide with one of the curves of the given direction. If this curve is in the sector under consideration, it must be decomposed into two subsectors at the points of this curve.

Example. Consider the case $n=2$, $n_1=n_2=2$. Find the geometrical place of the points λ satisfying the equations

$$\operatorname{Re}\left[\lambda^{1/2}\sqrt{P_{21}(x_0)\theta_k^2 + P_{32}(x_0)\theta_k + P_{43}(x_0)}\right] = 0,$$

$$\operatorname{Re}\left[\lambda\theta_k \pm \lambda^{1/2}\sqrt{P_{21}(x_0)\theta_k^2 + P_{32}(x_0)\theta_k + P_{43}(x_0)}\right] = 0, \quad k=\overline{1,2}.$$

Let $P_{21}(x_0)\theta_k^2 + P_{32}(x_0)\theta_k + P_{43}(x_0) = \beta_1^{(k)} + i\beta_2^{(k)}$, $\lambda = \lambda_1 + i\lambda_2$, $\theta_n = \theta_{11}^{(k)} + i\theta_{12}^{(k)}$. The first two equations determine the straight lines: $\lambda_2 = -\frac{\beta_2^{(1)}}{\beta_1^{(1)}}\lambda_1$, $\lambda_2 = -\frac{\beta_2^{(2)}}{\beta_1^{(2)}}\lambda_1$. For the third equation we get:

$$4\left(\lambda_1\theta_{11}^{(1)} - \lambda_2\theta_{12}^{(1)}\right) - 4\left(\lambda_1\theta_{11}^{(1)} - \lambda_2\theta_{12}^{(1)}\right)^2 \left(\lambda_1\beta_1^{(1)} - \lambda_2\beta_2^{(1)}\right) - \left(\lambda_2\beta_1^{(1)} + \lambda_1\beta_2^{(1)}\right)^2 = 0.$$

Let: $\lambda_1\theta_{11}^{(1)} - \lambda_2\theta_{12}^{(1)} = p$, then $\frac{\lambda_2}{\lambda_1} = \frac{\theta_{11}^{(1)}}{\theta_{12}^{(1)}} + \frac{P}{\lambda_2\theta_{12}^{(1)}}$ and as $\lambda_2 \to \infty$, $\frac{\lambda_2}{\lambda_1} = \frac{\theta_{11}^{(1)}}{\theta_{12}^{(1)}}$. Make the substitution

$$t = \frac{-\theta_{12}^{(1)}\lambda_1 - \theta_{11}^{(1)}\lambda_2}{\sqrt{\theta_{11}^{(1)2} + \theta_{12}^{(1)2}}}, \quad u = \frac{\theta_{11}^{(1)}\lambda_1 - \theta_{12}^{(1)}\lambda_2}{\sqrt{\theta_{11}^{(1)2} + \theta_{12}^{(1)2}}}, \quad g = \sqrt{\theta_{11}^{(1)2} + \theta_{12}^{(1)2}},$$

and get

$$u^4 - \frac{t}{g} A u^2 - \frac{Bt^2}{g^2} = 0,$$

where

$$B = \frac{\left(\theta_{11}^{(1)}\beta_1^{(1)} + \theta_{12}^{(1)}\beta_2^{(1)}\right)^2}{4\left(\theta_{11}^{(1)2} + \theta_{12}^{(1)2}\right)^2}, \quad A = -\frac{\left(\theta_{12}^{(1)}\beta_1^{(1)} - \theta_{11}^{(1)}\beta_2^{(1)}\right)}{\theta_{11}^{(1)2} + \theta_{12}^{(1)2}}.$$

If $A^2 + 4B > 0$, we get two asymptotic parabolas. If $A^2 = 4B$ they become imaginary and merge (see [2, pp. 93-119]). If $A^2 < 4B$, the curve has no branches that go to infinity.

The points satisfying the fourth equation are determined similarly. Let $\beta_1^{(1)} = \theta_{11}^{(1)}$, $\beta_2^{(1)} = \theta_{12}^{(1)}$. Then after simple transformations, from the third equation we find:

$$u^4 - \frac{1}{4}\frac{t^2}{g^2} = 0.$$

And this determines two parabolas contrariwisely directed from the vertex, that merge with the curve when continue to infinity.

Now partition all the complex λ plane into the domains S_{ik}, S_{ki}, $i = \overline{1,2}$, $k = \overline{3,4}$ determined by the inequalities

$$S_{ik} = \left\{ \lambda : \operatorname{Re}\left[\lambda^{1/2}\int_0^x \left(\varphi_1^{(1)} - \varphi_i^{(1)}\right) dt\right] \le 0, \right.$$

$$\operatorname{Re}\left[\lambda^{1/2}\int_0^x \left(\varphi_2^{(1)} - \varphi_i^{(1)}\right) dt\right] \le 0,$$

$$\operatorname{Re}\left[\lambda^{1/2}\int_0^x \left(\varphi_3^{(1)} - \varphi_k^{(1)}\right) dt\right] \le 0, \ \operatorname{Re}\left[\lambda^{1/2}\int_0^x \left(\varphi_4^{(1)} - \varphi_k^{(1)}\right) dt\right] \le 0,$$

$$\left. \operatorname{Re}\left[\lambda\left(\theta_2 - \theta_1\right)x + \lambda^{1/2}\int_0^x \left(\varphi_k^{(1)} - \varphi_i^{(1)}\right) dt\right] \le 0 \right\}, \quad k \ne i$$

$$S_{ki} = \left\{ \lambda : \operatorname{Re}\left[\lambda^{1/2}\int_0^x \left(\varphi_1^{(1)} - \varphi_i^{(1)}\right) dt\right] \le 0, \right.$$

$$\operatorname{Re}\left[\lambda^{1/2}\int_0^x \left(\varphi_2^{(1)} - \varphi_i^{(1)}\right) dt\right] \le 0,$$

$$\operatorname{Re}\left[\lambda^{1/2}\int_0^x \left(\varphi_3^{(1)} - \varphi_k^{(1)}\right) dt\right] \le 0, \ \operatorname{Re}\left[\lambda^{1/2}\int_0^x \left(\varphi_4^{(1)} - \varphi_k^{(1)}\right) dt\right] \le 0,$$

$$\operatorname{Re}\left[\lambda\left(\theta_1-\theta_2\right)x+\lambda^{1/2}\int\limits_0^x\left(\varphi_i^{(1)}-\varphi_k^{(1)}\right)dt\right]\leq 0\right\}.$$

Denote $\Omega_1=S_{13}\cup S_{14}$, $\Omega_2=S_{23}\cup S_{24}$, $\Omega_3=S_{31}\cup S_{32}$, $\Omega_4=S_{41}\cup S_{42}$.

Theorem. *The asymptotic expansions of the solutions of equation (1) determined by formula (4) of [1] are valid at each vector of* Ω_i, $i=\overline{1,2}$.

Definition. *Boundary value problem (1)-(2) is said to be generalized regular if in reducing the left side of equality (5) of the paper [1] multiplied by* $z^{-n_m}\exp\left\{-P_m\left(z\right)\right\}$ *(here* $P_m\left(z\right)$ *has the highest real part) in the form*

$$f_{m(q)}\left(z\right)=\sum_{k=1}^{N_{1n(q)}}f_{n(q)}\left(z\right)\exp\left\{B_{kn(q)}z^{n(q)}\right\},$$

(where all $B_{kn(q)}$ *are different,* $f_{m(q)}\left(z\right)=\sum\limits_{k=1}^{m_{1n(q)}}Q_{km(q)}\left(z\right)\exp\left\{P_{km(q)}\left(z\right)\right\}$, $P_{km(q)}\left(z\right)$ *has the power not exceeding* $n-1\left(q-1\right)$*) the functions* $f_{kn(q)}\left(z\right)$ *corresponding to the extreme points of the segment* r *contain only one exponent, and the multiplies at the exponents have the same growth order* H'*, and of the order growth of the polynomials at exponents of intermediate members is no more than* H'.

These generalized regularity conditions are direct corollaries from the conditions of theorem 3 of the paper [1]. Therefore the statement of this theorem is valid for generalized regular problems.

References

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