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ON THE COMPLETENESS OF A SYSTEM OF ELEMENTARY SOLUTIONS FOR AN OPERATOR-DIFFERENTIAL EQUATION

Abstract

In this paper, we prove the completeness of the system of elementary solutions for a class of third order homogeneous operator-differential equations with multiple characteristics in the set of all solutions from the space $W_2^3(R_+;H)$.

Consider in a separable Hilbert space H the polynomial operator pencil

$$P(\lambda) = (-\lambda E + A)(\lambda E + A)^2 + \sum_{j=1}^{2} \lambda^{3-j} A_j, \tag{1}$$

where E is the identity operator, $A,\ A_1,\ A_2$ are linear operators, A is a self-adjoint positive-definite operator with compact inverse A^{-1}, and the operators $A_j A^{-j}$, $j = 1,2$, are bounded on H. Then it is obvious that the pencil $P(\lambda)$ has a discrete spectrum.

We associate to pencil (1) the boundary value problem of the form

$$P(d/dt)u(t) = 0,\ t \in R_+ = [0, +\infty), \tag{2}$$

$$\frac{d^s u(0)}{dt^s} = \varphi_s, \quad s = 0,1, \tag{3}$$

where $u(t) \in W_2^3(R_+;H)$ (see [1]), $\varphi_s \in H_{5/2-s},\ s = 0,1$. Here we denote by H_α the scale of Hilbert spaces generated by the operator A, i.e. $H_\alpha = D(A^\alpha)$, $\alpha \geq 0,\ (x,y)_{H_\alpha} = (A^\alpha x, A^\alpha y),\ x,y \in D(A^\alpha)$.

If $\lambda_n(\operatorname{Re}\lambda_n < 0)$ are the eigenvalues of the pencil $P(\lambda)$, $\psi_{0,n}, \psi_{1,n}, ..., \psi_{m,n}$ is the chain of their corresponding eigen and adjoint vectors [2], Then the vector-functions

$$u_{h,n}(t) = e^{\lambda_n t}\left(\psi_{h,n} + \frac{t}{1!}\psi_{h-1,n} + ... + \frac{t^h}{h!}\psi_{0,n}\right), \quad h = 0,1,...,m,$$

belong to $W_2^3(R_+;H)$ and satisfy equation (2). These vector functions are called elementary solutions of equation (2). By means of these solutions we define the vector

$$\widetilde{\psi}_{h,n} = \left\{\psi_{h,n}^{(0)}, \psi_{h,n}^{(1)}\right\} \in \widetilde{H} \equiv H_{5/2} \oplus H_{3/2},$$

where $\psi_{h,n}^{(s)} \equiv \dfrac{d^s}{dt^s} u_{h,n}(t)\,|_{t=0}$, $\quad s = 0,1,\ h = 0,1,...,m$. The system $\left\{\widetilde{\psi}_{h,n}\right\}_{n=1}^{\infty}$ will be called the derivative chain of eigen and adjoint vectors of the pencil $P(\lambda)$ generated by the boundary value problem of the form (2), (3).

In this paper, we give sufficient conditions under which problem (2), (3) has a unique solution in the space $W_2^3(R_+;H)$ for any $\varphi_s \in H_{5/2-s}$, $s=0,1$. Moreover, we prove the completeness of the system of elementary solutions of equation (2) in the set of all such solutions.

Definition. *Boundary value problem (2), (3) is called a regularly solvable if for any $\varphi_s \in H_{5/2-s}$, $s=0,1$ there exists a vector-function $u(t) \in W_2^3(R_+;H)$ satisfying the equation $P(d/dt)u(t)=0$ almost everywhere in R_+, the boundary conditions are satisfied in the sense of relations*

$$\lim_{t\to 0}\|u(t)-\varphi_0\|_{H_{5/2}}=0, \quad \lim_{t\to 0}\left\|\frac{du(t)}{dt}-\varphi_1\right\|_{H_{3/2}}=0$$

and the following inequality holds:

$$\|u\|_{W_2^3(R_+;H)} \le const\left(\|\varphi_0\|_{H_{5/2}}+\|\varphi_1\|_{H_{3/2}}\right).$$

Here $u(t)$ will be called a regular solution of boundary-value problem (2), (3).

The following theorem holds.

Theorem 1. *Let A be a self-adjoint positive-definite operator, the operators A_jA^{-j}, $j=1,2$, be bounded on H and the following inequality hold:*

$$\frac{1}{2}\left(\left\|A_1A^{-1}\right\|_{H\to H}+\left\|A_2A^{-2}\right\|_{H\to H}\right)<1.$$

Then boundary value problem (2), (3) is regularly solvable.

Proof. In the case $A_1=A_2=0$, it is easy to establish boundary value problem (2), (3), i.e. the problem

$$\left(-\frac{d}{dt}+A\right)\left(\frac{d}{dt}+A\right)^2 u(t)=0, \; t\in R_+=[0,+\infty), \tag{4}$$

$$u(0)=\varphi_0, \quad \frac{du(0)}{dt}=\varphi_1, \tag{5}$$

is regularly solvable. Really, since the general solution of equation (4) in the space $W_2^3(R_+;H)$ represented in the form

$$u_0(t)=e^{-tA}\zeta_0+tAe^{-tA}\zeta_1,$$

where $\zeta_k \in D(A^{5/2-k})$, $k=0,1$ (see [1, ch.1]), from condition (5) we obtain:

$$\begin{cases} u_0(0)=\zeta_0=\varphi_0, \\ \dfrac{du_0(0)}{dt}=-A\zeta_0+A\zeta_1=\varphi_1. \end{cases}$$

Hence we have

$$\zeta_1 = A^{-1}\varphi_1 + \varphi_0.$$

It is clear that

$$\|u_0(t)\|_{W_2^3(R_+;H)} = \left\|e^{-tA}\zeta_0 + tAe^{-tA}\zeta_1\right\|_{W_2^3(R_+;H)} \leq$$

$$\leq const\,\|\zeta_0\|_{H_{5/2}} + const\,\|\zeta_1\|_{H_{3/2}} \leq$$

$$\leq const\left(\|\varphi_0\|_{H_{5/2}} + \|\varphi_1\|_{H_{3/2}}\right),$$

i.e. problem (4), (5) is regularly solvable.

Continuing on, assuming at least one of A_j, $j = 1,2$, is non-zero, the regular solution of boundary value problem (2), (3) must be found in the form $u(t) = u_0(t) + v(t)$, where $u_0(t)$ is a regular solution of boundary value problem (4), (5), and $v(t) \in W_2^3(R_+;H)$. In this case, boundary value problem (2), (3) can be reduced to the following problem with respect to $v(t)$:

$$P(d/dt)v(t) = f(t), \tag{6}$$

$$v(0) = 0, \quad \frac{dv(0)}{dt} = 0, \tag{7}$$

where $f(t) \in L_2(R_+;H)$. Really, since

$$u_0(t) = e^{-tA}\zeta_0 + tAe^{-tA}\zeta_1,$$

thus

$$\zeta_0 = \varphi_0,\ \zeta_1 = A^{-1}\varphi_1 + \varphi_0,$$

then we have:

$$v(t) = u(t) - u_0(t),$$

$$v(0) = u(0) - u_0(0) = \varphi_0 - \varphi_0 = 0,$$

$$\frac{dv(0)}{dt} = \frac{du(0)}{dt} - \frac{du_0(0)}{dt} = \varphi_1 - \varphi_1 = 0.$$

In this case,

$$P(d/dt)v(t) = -P_0(d/dt)u_0(t) - P_1(d/dt)u_0(t),$$

$$v(0) = 0, \quad \frac{dv(0)}{dt} = 0.$$

Since $u_0(t)$ is a regular solution

$$P_0(d/dt)u_0(t) = 0,$$

then

$$P(d/dt)v(t) = -P_1(d/dt)u_0(t),$$
$$v(0) = 0, \quad \frac{dv(0)}{dt} = 0.$$

And since

$$f(t) = -P_1(d/dt)u_0(t) = -A_1\frac{d^2u_0(t)}{dt^2} - A_2\frac{du_0(t)}{dt} =$$
$$= A_1\left[A^2e^{-tA}\zeta_0 + A^2(-2E+tA)e^{-tA}\zeta_1\right] +$$
$$+A_2[-Ae^{-tA}\zeta_0 + A(E-tA)e^{-tA}\zeta_1] =$$
$$= [A_1A^{-1} - A_2A^{-2}]A^3e^{-tA}\zeta_0 +$$
$$+[A_1A^{-1}(-2E+tA) + A_2A^{-2}(E-tA)]A^3e^{-tA}\zeta_1,$$

then the function $f(t) \in L_2(R_+;H)$.

Subject to the conditions of the given theorem the regular solvability of problem (6), (7) was established in [3], that completes its proof. The theorem is proved.

Before turning to the basic question of the paper, we state the following assertion, which is easily proved by the basis of Lemma Keldysh (see [2]) on the expansion of the resolvent about the eigenvalues.

Lemma. *In order that, the system* $\left\{\widetilde{\psi}_{h,n}\right\}_{n=1}^{\infty}$ *be complete in the space* $\widetilde{H}$, *it is necessary and sufficient that for any vectors* $\xi_k \in H_{5/2-k}$, $k = 0,1$, *from the holomorphic vector-function*

$$R(\lambda) = \sum_{k=0}^{1}(A^{5/2-k}P^{-1}(\overline{\lambda}))^*\lambda^k A^{5/2-k}\xi_k$$

in the half-plane Π_- *follow that* $\xi_k = 0$, $k = 0,1$.

By $\sigma_\infty(H)$ we denote the set of compact operators acting on H.

It is known that, if $C \in \sigma_\infty(H)$, then $(C^*C)^{1/2}$ is a compact self-adjoint operator on H. The eigen values of the operator $(C^*C)^{1/2}$ will be called s-numbers of the operator C. We will enumerate the non-zero s-numbers of the operator C in decreasing order according to their multiplicity. Denote by

$$\sigma_p = \left\{C : C \in \sigma_\infty(H); \sum_{k=1}^{\infty} s_k^p(C) < \infty\right\}, \ 0 < p < \infty.$$

In Theorem 1 sufficient conditions are established under which boundary value problem (2), (3) has a unique solution in the space $W_2^3(R_+;H)$ for any $\varphi_s \in H_{5/2-s}$, $s = 0,1$. The set of all such solutions is denoted by $W(P)$. By the theorems on intermediate derivatives and on traces [1, ch.1] the set $W(P)$ is a closed subspace of the space $W_2^3(R_+;H)$. Now, in the space $W(P)$ we will prove the completeness of the system of elementary solutions of equation (2).

The following theorem holds.

Theorem 2. *Let the conditions of Theorem 1 be satisfied and one of the following conditions hold:*

1) $A^{-1} \in \sigma_p,\ 0 < p \le 1;$

2) $A^{-1} \in \sigma_p,\ 0 < p < \infty,\ A_j A^{-1} \in \sigma_\infty(H),\ j = 1, 2.$

then the system of elementary solutions of boundary value problem (2), (3) is complete in the space $W(P)$.

Proof. First, we will prove that under the conditions of the theorem the system $\left\{\widetilde{\psi}_{h,n}\right\}_{n=1}^{\infty}$ is complete in the space $\widetilde{H}$. We will prove by contradiction. If the system $\left\{\widetilde{\psi}_{h,n}\right\}_{n=1}^{\infty}$ is not complete in the space $\widetilde{H}$, then there is a non-zero vector $\xi = \{\xi_0, \xi_1\} \in \widetilde{H}$ such that $(\xi, \widetilde{\psi}_{h,n})_{\widetilde{H}} = 0,\ n = 1, 2, \ldots$. Then it follows from Keldysh lemma [2] that the vector-function $R(\lambda)$ is holomorphic in the half-plane Π_-. Under the conditions of the theorem (according to Theorem 1) boundary value problem (2), (3) is regularly solvable. If $u(t)$ is a regular solution of boundary value problem (2), (3), then it can be expressed in the form

$$u(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \widehat{u}(\lambda) e^{\lambda t} d\lambda, \tag{8}$$

where

$$\widehat{u}(\lambda) = P^{-1}(\lambda)\sum_{r=0}^{2} B_r u^{(2-r)}(0),$$

thus

$$B_0 = -E,\ B_1 = -\lambda E + Q,\ B_2 = -\lambda^2 E + \lambda Q + A_2 + A^2,\ Q = -A + A_1.$$

Further, as in [4], for $t > 0$ in (8) we can change the integration contour by

$$\Gamma_{\pm\theta} = \left\{\lambda : \lambda = r e^{\pm i\left(\frac{\pi}{2}+\theta\right)},\ r > 0\right\}.$$

As a result, for $t > 0$ we obtain:

$$\sum_{k=0}^{1}\left(\frac{d^k u(t)}{dt^k}, \xi_k\right)_{H_{5/2-k}} =$$

$$= \frac{1}{2\pi i}\int_{\Gamma_{\pm\theta}} \sum_{k=0}^{1}\left(A^{5/2-k}P^{-1}(\lambda)\lambda^k \sum_{r=0}^{2} B_r u^{(2-r)}(0), A^{5/2-k}\xi_k\right) e^{\lambda t} d\lambda =$$

$$= \frac{1}{2\pi i}\int_{\Gamma_{\pm\theta}} \sum_{r=0}^{2}(B_r u^{(2-r)}(0),\ R(\overline{\lambda}))e^{\lambda t} d\lambda = \frac{1}{2\pi i}\int_{\Gamma_{\pm\theta}} g(\lambda)e^{\lambda t} d\lambda,$$

where

$$g(\lambda)=\sum_{r=0}^{2}(B_r u^{(2-r)}(0),\ R(\overline{\lambda})).$$

Now, taking into account in case 1) the estimates of the resolvent of pencil (1) in the paper by [5], and in case 2) by using M.V. Keldysh theorem [2] with applying the Fragmen-Lindelof theorem, we obtain that $g(\lambda)$ is a polynomial. And since for $t>0$,

$$\frac{1}{2\pi i}\int\limits_{\Gamma_{\pm\theta}} g(\lambda)e^{\lambda t}d\lambda=0,$$

then, for $t>0$

$$\sum_{k=0}^{1}\left(\frac{d^k u(t)}{dt^k},\xi_k\right)_{H_{5/2-k}}=0.$$

Here passing to the limit as $t\to 0$, we have

$$\sum_{k=0}^{1}(\varphi_k,\xi_k)_{H_{5/2-k}}=0.$$

Since the choice of the vectors φ_k, $k=0,1$, is arbitrary, then $\xi_k=0,\ k=0,1$, and therefore $\xi=0$. We obtain a contradiction.

Further, by the theorem on traces [1, ch.1] for any function $u(t)\in W_2^3(R_+;H)$ the following estimate holds:

$$\left\|A^{5/2}u(0)\right\|_H+\left\|A^{3/2}\frac{du(0)}{dt}\right\|_H=$$

$$=\|u(0)\|_{H_{5/2}}+\left\|\frac{du(0)}{dt}\right\|_{H_{3/2}}\le const\,\|u\|_{W_2^3(R_+;H)}\,.$$

On the other hand, from the uniqueness of solutions of boundary value problem (2), (3) (see Theorem 1) we have:

$$\|u\|_{W_2^3(R_+;H)}\le const\left(\|\varphi_0\|_{H_{5/2}}+\|\varphi_1\|_{H_{3/2}}\right). \tag{9}$$

Since the system $\left\{\widetilde{\psi}_{h,n}\right\}_{n=1}^{\infty}$ is complete in the space $\widetilde{H}$, for a given $\varepsilon>0$ there exists the number N and the numbers $c_{h,n}^N$ such that

$$\left\|\varphi_0-\sum_{n=1}^{N}\sum_h c_{h,n}^N\psi_{h,n}^{(0)}\right\|_{H_{5/2}}<\varepsilon, \tag{10}$$

$$\left\|\varphi_1-\sum_{n=1}^{N}\sum_h c_{h,n}^N\psi_{h,n}^{(1)}\right\|_{H_{3/2}}<\varepsilon, \tag{11}$$

Since $\psi_{h,n}^{(s)} = \dfrac{d^s}{dt^s} u_{h,n}(t)\,|_{t=0}$ and $\varphi_s = \dfrac{d^s}{dt^s} u(t)\,|_{t=0}$, $s = 0, 1$, then for the solution

$$u(t) - \sum_{n=1}^{N} \sum_{h} c_{h,n}^{N} u_{h,n}(t)$$

in view of (9) we have:

$$\left\| u(t) - \sum_{n=1}^{N} \sum_{h} c_{h,n}^{N} u_{h,n}(t) \right\|_{W_2^3(R_+;H)} \leq$$

$$\leq const \left(\left\| \varphi_0 - \sum_{n=1}^{N} \sum_{h} c_{h,n}^{N} \psi_{h,n}^{(0)} \right\|_{H_{5/2}} + \left\| \varphi_1 - \sum_{n=1}^{N} \sum_{h} c_{h,n}^{N} \psi_{h,n}^{(1)} \right\|_{H_{3/2}} \right). \qquad (12)$$

Now, taking into account inequalities (10) and (11), then from inequality (12) we obtain:

$$\left\| u(t) - \sum_{n=1}^{N} \sum_{h} c_{h,n}^{N} u_{h,n}(t) \right\|_{W_2^3(R_+;H)} \leq \varepsilon const = \widetilde{\varepsilon}.$$

This means that the system of elementary solutions of the boundary value problem (2), (3) complete in the space of its regular solutions, i.e. in the space $W(P)$. The theorem is proved.

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