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SOLUTION OF THE INVERSE SCATTERING PROBLEM FOR THE STURM-LIOUVILLE EQUATION WITH A SPECTRAL PARAMETER IN DISCONTINUITY CONDITION

Abstract

In the paper, a total solution of the inverse scattering problem is given for Sturm-Liouville equation with a spectral parameter in the discontinuity condition in the absenese of a discrete spectrum.

It is known that the inverse scattering problem for the differential equation

$$-y'' - \lambda p(x) y + q(x) y = \lambda^2 y, \quad -\infty < x < \infty, \tag{1}$$

where $p(x)$ and $q(x)$ are real functions, $p(x)$ is absolutely continuous, and $p(x)$, $p'(x)$, $q(x)$ rather rapidly decrease as $|\lambda| \to \infty$, was investigated in the papers [1]-[3]. In the case $p(x) \equiv 0$, under the condition

$$\int_{-\infty}^{\infty} (1 + |x|) |q(x)| \, dx < \infty \tag{2}$$

the papers [4]-[6] deal with the solution of the inverse scattering problem.

In the present paper, we consider the inverse scattering problem for equation (1), when $p(x) = \beta\delta(x-a)$, where β is a real number, $\delta(x-a)$ is a Dirac function, and condition (2) is fulfilled. In this case, equation (1) may be reduced to the problem ([7])

$$-y'' + q(x) y = \lambda^2 y, \quad -\infty < x < \infty \tag{3}$$

$$y(a+0) = y(a-0), \tag{4}$$

$$y'(a+0) - y'(a-0) = \lambda\beta y(a). \tag{5}$$

As is known [8], problem (3)-(5) for all λ from the half-plane $\operatorname{Im}\lambda \geq 0$ has the solutions $e^+(x,\lambda)$, $e^-(x,\lambda)$ representable in the form

$$e^{\pm}(x,\lambda) = e_0^{\pm}(x,\lambda) \pm \int_x^{\pm\infty} K^{\pm}(x,t) e^{\pm i\lambda t} dt, \tag{6$_\pm$}$$

moreover, the kernels $K^{\pm}(x,t)$ satisfy the inequalities

$$\left|K^{\pm}(x,t)\right| \leq \frac{C}{2}\sigma^{\pm}\left(\frac{x+t}{2}\right) e^{C\sigma_{1^{\pm}}(x)}, \quad 0 < |x-a| < \pm(t-a), \tag{7$_\pm$}$$

$$\left|K^{\pm}(x,t)\right| \leq \left\{\frac{C}{2}\sigma^{\pm}\left(\frac{x+t}{2}\right) + \frac{|\beta|}{4}\sigma^{\pm}\left(\frac{x+2a-t}{2}\right)\right\} e^{C\sigma_1^{\pm}(x)}, \quad |t-a| < \pm(a-x),$$

where $C = 1 + \frac{|\beta|}{2}$, $\sigma^{\pm}(x) = \pm \int\limits_{x}^{\pm\infty} |q(\xi)|\, d\xi$, $\sigma_1^{\pm}(x) = \pm \int\limits_{x}^{\pm\infty} \sigma^{\pm}(\xi)\, d\xi$.

Furthermore, the functions $K^{\pm}(x,t)$ are continuous for $t \neq 2a - x, \quad x \neq a$, and the following relations are fulfilled:

$$K^{\pm}(x,t) = \pm\frac{1}{2}\int\limits_{x}^{\pm\infty} q(\xi)\, d\xi, \quad \pm x > \pm a,$$

$$K^{\pm}(x,x) = \pm\frac{1}{2}\left(1 + \frac{i\beta}{2}\right)\int\limits_{x}^{\pm\infty} q(\xi)\, d\xi, \quad \pm x < \pm a, \tag{8$_\pm$}$$

$$K^{\pm}(x, 2a - x + 0) - K^{\pm}(x, 2a - x - 0) =$$
$$= \mp\frac{i\beta}{2}\left\{\int\limits_{x}^{a} q(\xi)\, d\xi - \int\limits_{x}^{\pm\infty} q(\xi)\, d\xi\right\}, \quad \pm x < \pm a.$$

In formulae $(6)_{\pm}$, the functions $e_0^{\pm}(x,\lambda)$ are the Jost solutions of problem (3)-(5) for $q(x) \equiv 0$:

$$e_0^{\pm}(x,\lambda) = \begin{cases} e^{\pm i\lambda x}, & \pm x > \pm a \\ \left(1 + \frac{i\beta}{2}\right) e^{\pm i\lambda x} - \frac{i\beta}{2} e^{\pm i\lambda(2a - x)}, & \pm x < \pm a. \end{cases}$$

For $\lambda \in R/\{0\}$ the following relations hold:

$$\frac{1}{a(\lambda)} e^{\pm}(x,\lambda) = r^{\mp}(\lambda)\, e^{\mp}(x,\lambda) + \overline{e^{\mp}(x,\lambda)}, \tag{9$_\pm$}$$

where

$$r^{-}(\lambda) = \frac{b(\lambda)}{a(\lambda)}, \quad r^{+}(\lambda) = -\frac{\overline{b(\lambda)}}{a(\lambda)},$$

$$a(\lambda) = \frac{1}{2i\lambda} W\left[e^{+}(x,\lambda), e^{-}(x,\lambda)\right], \quad b(\lambda) = -\frac{1}{2i\lambda} W\left[e^{+}(x,\lambda), \overline{e^{-}(x,\lambda)}\right],$$

$W[y_1, y_2] = y_1' y_2 - y_1' y_2'$ is the wronskian of the functions y_1 and y_2.

The functions $r^{-}(\lambda)$, $r^{+}(\lambda)$ and $\frac{1}{a(\lambda)}$ are called left and right reflection factor and passage factor, respectively. The function $a(\lambda)$ admits analytic continuation to the half-plane $\operatorname{Im}\lambda > 0$ and may have there at most finite number of zeros. In future, we'll assume that the zeros also are absent, i.e.

$$a(\lambda) \neq 0 \quad for \quad \operatorname{Im}\lambda > 0, \tag{10}$$

and consequently, the discrete spectrum of problem (3)-(5) is absent.

In this case, the inverse scattering problem for problem (3)-(5) is in renewal of the function $q(x)$ by the left or right reflection factor and finding necessary and sufficient conditions to which should satisfy an arbitrarily taken function $r(\lambda)$ for it to be the right (left) reflection factor of some problem of the form (3)-(5) with a real coefficient satisfying condition (2).

In the following theorem we give the solution of the inverse scattering problem.

Theorem. *For the function* $r^{+}(\lambda), -\infty < \lambda < \infty$ *to be the right reflection factor of the problem of the form (3)-(5), without a discrete spectrum, with a real potential* $q(x)$ *satisfying inequality (2) and with a real number* β*, it is necessary and sufficient that the following conditions to be fulfilled:*

1) for real $\lambda \neq 0$*, the function* $r^{+}(\lambda)$ *is continuous,*

$$\left|r^{+}(\lambda)\right| \leq 1 - c\lambda^{2}\left(1+2^{2}\right)^{-1} \quad \text{and} \quad r^{+}(\lambda) - r_{0}^{+}(\lambda) = O\left(\frac{1}{\lambda}\right) \quad as \quad \lambda \to \pm\infty,$$

where

$$r_{0}^{+}(\lambda) = -\frac{i\beta}{2+i\beta}e^{-2i\lambda a};$$

2) the function $za(z)$*, where*

$$a(z) = \left(1+\frac{i\beta}{2}\right)e^{-\frac{1}{2\pi i}\int\limits_{-\infty}^{\pm\infty}\frac{\ln\ 1-\left|r^{+}(\lambda)\right|^{2}}{\lambda - z}d\lambda}$$

is continuous in the closed upper half-plane, and

$$\lim_{\lambda\to 0}\lambda a(\lambda)\left[r^{+}(\lambda)+1\right] = 0;$$

3) the functions

$$R^{\pm}(x) = \frac{1}{2\pi}\int\limits_{-\infty}^{\pm\infty}\left[r^{\pm}(\lambda) - r_{0}^{\pm}(\lambda)\right]e^{\pm i\lambda x}d\lambda,$$

where $r^{-}(\lambda) = \overline{-r^{+}(\lambda)}\cdot\frac{\overline{a}(\lambda)}{a(\lambda)}$*, are absolutely continuous on any segment not containing the point* $2a$*, the derivatives* $R^{+'}(\lambda)$ *and* $R^{-'}(\lambda)$ *for all* $\alpha > -\infty$ *and* $a' < \infty$ *satisfy the inequality*

$$\int\limits_{\alpha}^{\infty}(1+|x|)\left|R^{+'}(x)\right|dx < \infty, \quad \int\limits_{-\infty}^{\alpha'}(1+|x|)\left|R^{-'}(x)\right|dx < \infty;$$

4) the solutions $K^{\pm}(x,y)$ *of the equations*

$$R_{1}^{\pm}(x,y) + \overline{K^{\pm}(x,y)} \mp \frac{i\beta}{2+i\beta}K^{\pm}(x,2a-y) \pm$$

$$\pm\int\limits_{x}^{\pm\infty}K^{\pm}(x,t)\,R^{\pm}(t+y)\,dt = 0, \quad \pm y > \pm x, \tag{11$_\pm$}$$

where

$$R_{1}^{\pm}(x,y) = \begin{cases} R^{\pm}(x,y), & \pm x > \pm a \\ \left(1+\frac{i\beta}{2}\right)R^{\pm}(x+y) - \frac{i\beta}{2}R^{\pm}(2a-x+y), & \pm x < \pm a \end{cases},$$

satisfy the conditions

$$K^{\pm}(x,x)\big|_{a\mp 0}=\left(1+\frac{i\beta}{2}\right)K^{\pm}(x+y)\Big|_{a\mp 0}.$$

Proof. Necessity follows from the results of the papers [8]-[9]. Give the proof of sufficiency.

1. From conditions 1), 3) of the theorem it follows that the main equations $(11)_+$ and $(11)_-$ constructed by the function $r^+(\lambda)$, according to the paper [10] have unique solutions $K^+(x,y)$ and $K^-(x,y)$. In the expanded form the main equation $(11)_\pm$ looks as follows:

$$R^{\pm}(x+y)+\overline{K^{\pm}(x,y)}\pm\int_x^{\pm\infty}K^{\pm}(x,t)R^{\pm}(t+y)\,dt=0,\ \pm x>\pm a,\ \pm y>\pm x \quad (12)_\pm$$

$$\left(1+\frac{i\beta}{2}\right)R^{\pm}(x+y)-\frac{i\beta}{2}R^{\pm}(2a-x+y)+\overline{K^{\pm}(x,y)}\pm$$
$$\pm\int_x^{\pm\infty}K^{\pm}(x,t)R^{\pm}(t+y)\,dt=0,\ \pm x<\pm a,\ \pm y>\pm(2a-x). \quad (13)_\pm$$

$$\left(1+\frac{i\beta}{2}\right)R^{\pm}(x+y)-\frac{i\beta}{2}R^{\pm}(2a-x+y)+\overline{K^{\pm}(x,y)}\mp\frac{i\beta}{2+i\beta}K^{\pm}(x,2a-y)\pm$$
$$\pm\int_x^{\pm\infty}K^{\pm}(x,t)R^{\pm}(t+y)\,dt=0,\ \pm x<\pm a,\ \pm x<y<\pm(2a-x). \quad (14)_\pm$$

Assuming $y=2a-x\mp 0$ and $y=2a-x\pm 0$ in main equations $(14)_\pm$ and $(13)_\pm$, respectively, we have

$$\left(1+\frac{i\beta}{2}\right)R^{\pm}(2a\mp 0)-\frac{i\beta}{2}R^{\pm}(4a-2x\mp 0)+\overline{K^{\pm}(x,2a-x\mp 0)}\mp$$
$$\mp\frac{i\beta}{2+i\beta}K^{\pm}(x,x\pm 0)\pm\int_x^{\pm\infty}K^{\pm}(x,t)R^{\pm}(t+2a-x\mp 0)\,dt=0,\ \pm x<\pm a,$$
$$\left(1+\frac{i\beta}{2}\right)R^{\pm}(2a\pm 0)-\frac{i\beta}{2}R^{\pm}(4a-2x\pm 0)+\overline{K^{\pm}(x,2a-x\pm 0)}\pm$$
$$\pm\int_x^{\pm\infty}K^{\pm}(x,t)R^{\pm}(t+2a-x\pm 0)\,dt=0,\ \pm x<\pm a.$$

We subtract the second relation from the first one. As a result we get :

$$\left(1+\frac{i\beta}{2}\right)\left[R^{\pm}(2a\mp 0)-R^{\pm}(2a\pm 0)\right]+$$
$$+\overline{K^{\pm}(x,2a-x\mp 0)-K^{\pm}(x,2a-x\pm 0)}\mp$$

$$\mp\frac{i\beta}{2+i\beta}K^{\pm}(x,x\pm 0)\,,\ \ \pm x<\pm a.$$

Now assume (see $(8)_{\pm}$)

$$q^{\pm}(x)=\begin{cases}\mp\frac{4}{2+i\beta}\frac{dK^{\pm(x,x)}}{dx}= \\ =\pm\frac{4}{i\beta}\frac{d}{dx}\left[\overline{K^{\pm}(x,2a-x+0)-K^{\pm}(x,2a-x-0)}\right], \\ \pm x<\pm a,\quad \mp 2\frac{dK^{\pm(x,x)}}{dx},\quad \pm x>\pm a.\end{cases}\qquad (15)_{\pm}$$

2. Show that the functions $e^{+}(x,\lambda)$, $e^{-}(x,\lambda)$ constructed with the help of $K^{+}(x,t)$, $K^{-}(x,t)$ by formula $(6)_{+}$ and $(6)_{-}$ satisfy the equations

$$-e^{\pm''}(x,\lambda)+q^{\pm}(x)\,e^{\pm}(x,\lambda)=\lambda^{2}e^{\pm}(x,\lambda)\qquad (16)_{\pm}$$

and conditions

$$a^{\pm}(a+0,\lambda)=e^{\pm}(a-0,\lambda)\,,\qquad (17)_{\pm}$$

$$e^{\pm'}(a+0,\lambda)-e^{\pm'}(a-0,\lambda)=\lambda\beta e^{\pm}(a,\lambda)\,,\qquad (18)_{\pm}$$

moreover

$$\int\limits_{x'}^{\infty}(1+|x|)\left|q^{+}(x)\right|dx<\infty,\quad \int\limits_{-\infty}^{x''}(1+|x|)\left|q^{-}(x)\right|dx<\infty.\qquad (19)_{\pm}$$

At first suppose that the functions $R^{\pm}(x)$ are twice continuously differentiable, and for all $\alpha'>-\infty$, $\beta'<+\infty$

$$\int\limits_{\alpha'}^{\infty}(1+|x|)\left|R^{+''}(x)\right|dx<+\infty,\quad \int\limits_{-\infty}^{\beta'}(1+|x|)\left|R^{-''}(x)\right|dx<\infty.\qquad (20)$$

Then the solutions $K^{\pm}(x,y)$ of main equations $(11)_{\pm}$ are twice continuously differentiable for $y\neq 2a-x$ and $x\neq a$, and for each x all partial derivatives of first and second order are summable over y.

Consider the domain $\pm x<\pm a$, $\pm x<\pm y<\pm(2a-x)$. Then the main equations $(11)_{\pm}$ take the form of $(14)_{\pm}$. Differentiating these equations twice with respect to y and integrating by parts, we get

$$\left(1+\frac{i\beta}{2}\right)+R^{\pm''}(x+y)-\frac{i\beta}{2}R^{\pm''}(2a-x+y)+\overline{K_{yy}^{\pm''}(x,y)}\mp\frac{i\beta}{2+i\beta}K_{yy}^{\pm''}(x,2a-y)\mp$$

$$\mp K^{\pm}(x,x)\,R^{\pm'}(x+y)\mp K^{\pm}(x,t)\Big|_{t=2a-x-0}^{2a-x+0}\cdot R^{\pm'}(2a-x+y)\pm K_{t}^{\pm'}(x,t)\Big|_{t=x}\times$$

$$\times R^{\pm}(x+y)\pm K_{t}^{\pm'}(x,t)\Big|_{t=2a-x-0}^{2a-x+0}R^{\pm}(2a-x+y)\pm\int\limits_{x}^{\pm\infty}K_{tt}^{\pm''}(x,t)\,R^{\pm}(t+y)=0.$$

Further, differentiating equation $(14)_{\pm}$ twice respect to x, we have

$$\left(1+\frac{i\beta}{2}\right)R^{\pm''}(x+y)-\frac{i\beta}{2}R^{\pm''}(2a-x+y)+\overline{K_{xx}^{\pm''}(x,y)}\mp$$

$$\mp\frac{i\beta}{2+i\beta}K_{xx}^{\pm''}(x,2a-y)\mp K^{\pm}(x,x)\,R^{\pm}(x+y)\mp K^{\pm}(x,x)\cdot R^{\pm'}(x+y)\mp$$

$$\mp\left[K^{\pm}(x,t)\big|_{t=2a-x-0}^{2a-x+0}\right]R^{\pm}(2a-x+y)\pm$$

$$\pm\left[K^{\pm}(x,t)\big|_{t=2a-x-0}^{2a-x+0}\right]R^{\pm'}(2a-x+y)\mp K_{x}^{\pm'}(x,t)\Big|_{t=x}R^{\pm}(x+y)\pm$$

$$\pm\left[K_{x}^{\pm}(x,t)\big|_{t=2a-x-0}^{2a-x+0}\right]R^{\pm}(2a-x+y)\pm\int_{x}^{\pm\infty}K_{xx}^{\pm'}(x,t)\,R^{\pm}(t+y)\,dt=0.$$

Subtracting from the last equality the previous one, we get

$$\overline{K_{xx}^{\pm''}(x,y)}\mp\frac{i\beta}{2+i\beta}K_{xx}^{\pm''}(x,2a-y)-\overline{K_{yy}^{\pm''}(x,y)}\pm\frac{i\beta}{2+i\beta}K_{yy}^{\pm''}(x,2a-y)\mp$$

$$\mp2K^{\pm}(x,x)\,R^{\pm}(x+y)\mp2\left[K^{\pm}(x,t)\big|_{t=2a-x-0}^{2a-x+0}\right]'R^{\pm}(2a-x+y)\pm$$

$$\pm\int_{x}^{\pm\infty}\left(K_{xx}^{\pm''}(x,t)-K_{tt}^{\pm''}(x,t)\right)R^{\pm}(t+y)\,dt=0. \qquad (21)_{\pm}$$

From $(15)_{\pm}$ and main equation $(14)_{\pm}$

$$\mp2K^{\pm'}(x,x)\,R^{\pm}(x+y)\mp2\left[K^{\pm}(x,t)\big|_{t=2a-x-0}^{2a-x+0}\right]'R^{\pm}(2a-x+y)=$$

$$=\left(1+\frac{i\beta}{2}\right)q''+(x)\,R^{\pm}(x+y)-\frac{i\beta}{2}q^{\pm}(x)\,R^{\pm}(2a-x+y)=$$

$$=q^{\pm(x)}\left[\overline{-K^{\pm}(x,y)}\pm\frac{i\beta}{2}K^{\pm}(x,2a-y)\mp\int_{x}^{\pm\infty}K^{\pm}(x,t)\,R^{\pm}(t+y)\,dt\right]. \qquad (22)_{\pm}$$

From $(21)_{\pm}$ and $(22)_{\pm}$ it follows that the functions

$$h_{x}^{\pm}(y)=K_{xx}^{\pm''}(x,y)-q^{\pm}(x)\,K^{\pm}(x,y)-K_{yy}^{\pm''}(x,y)$$

satisfy the equations

$$\overline{h_{x}^{\pm}(y)}\pm\frac{i\beta}{2+i\beta}h_{x}^{\pm}(2a-y)\pm\int_{x}^{\pm\infty}h_{x}^{\pm}(t)\,R^{\pm}(t+y)\,dt=0,$$

i.e. the functions $h_{x}^{\pm}(t)$ are summable solutions of homogeneous equations corresponding to $(14)_{\pm}$. Behaving in the same way with equations $(12)_{\pm}$ and $(13)_{\pm}$, we get that the solutions of main equations $(11)_{\pm}$, according to [10] satisfy the equation

$$K_{xx}^{\pm''}(x,y)-q^{\pm}(x)\,K^{\pm}(x,y)-K_{yy}^{\pm''}(x,y)=0. \qquad (23)_{\pm}$$

By condition 4) of the theorem, from $(15)_{\pm}$ it follows that the functions $K^{\pm}(x,y)$ satisfy relations $(8)_{\pm}$. Under the made suppositions (20) it is easy to show that

$$\lim_{x+y\to\pm\infty}K_{x}^{\pm'}(x,y)=\lim_{x+y\to\pm\infty}K_{x}^{\pm'}(x,y)=0. \qquad (24)_{\pm}$$

Now, show that the functions $K^{\pm}(x,y)$ satisfy the conditions

$$K^{\pm}(a+0,y) = K^{\pm}(a-0,y), \quad \pm y > \pm a \tag{25$_\pm$}$$

$$K_x^{\pm'}(a+0,y) - K_x^{\pm'}(a-0,y) = i\beta K_y^{\pm'}(a,y), \quad \pm y > \pm a. \tag{26$_\pm$}$$

Assume $x = a \pm 0$ and $x = a \mp 0$ in main equations $(12)_{\pm}$ and $(14)_{\pm}$, respectively. From the obtained first relation subtract the second one. As a result we get that the differences $K^{\pm}(a+0,y) - K^{\pm}(a+0,y)$ are the solutions of homogeneous equations corresponding to equations $(11)_{\pm}$, for $x=0$. Therefore, by [10] we get $(25)_{\pm}$.

Prove that conditions $(26)_{\pm}$ are also fulfilled.

First of all note that for the solution of main equations, the following relations are valid:

$$K^{\pm}(x,2a-x\pm 0)\big|_{a\mp 0} = K^{\pm}(x,x)\big|_{a\pm 0},$$
$$K^{\pm}(x,2a-x\mp 0)\big|_{a\mp 0} = K^{\pm}(x,x)\big|_{a\mp 0}. \tag{27$_\pm$}$$

Indeed, in equations $(12)_{\pm}$ assume at first $y=x$, then $x = a \pm 0$, in equations $(13)_{\pm}$ at first $y = 2a - x \pm 0$, then $x = a \mp 0$. Further, from these equations, subtracting one from another one, by $(25)_{\pm}$ we get the first equality from $(27)_{\pm}$. Assuming once $y = 2a - x \mp 0$, $x = a \mp 0$ and $y = x \pm 0$, $x = a \mp 0$ another time in equations $(14)_{\pm}$ and subtracting one of the obtained equalities from another one it is easy to get the second relation from $(27)_{\pm}$.

Now differentiate equations $(12)_{\pm}$ and $(13)_{\pm}$ with respect to the variable x, and assume $x = a+0$ and $x = a-0$, respectively. As a result we have

$$R^{+'}(a+y) + \overline{K_x^{+'}(a+0,y)} - K^{+}(x,x)\Big|_{a+0} R^{+}(a+y) +$$
$$+\int_a^{+\infty} K_x^{+'}(a+0,t)\,R^{+}(t+y)\,dt = 0,$$

$$\left(1+\frac{i\beta}{2}\right)R^{+}(a+y) + \frac{i\beta}{2}R^{+'}(a+y) + \overline{K_x^{+'}(a-0,y)} - K^{+}(x,x)\Big|_{a-0} R^{+}(a+y) +$$
$$+\left[K^{+}(x,2a-x+0) - K^{+}(x,2a-x-0)\right]\Big|_{a-0} R^{+}(a+y) +$$
$$+\int_a^{\pm\infty} K_x^{+'}(a-0,t)\,R^{+}(t+y)\,dt = 0.$$

Subtracting one these equalities from another one, and taking into account $(27)_{\pm}$, we get

$$-i\beta R^{+}(a+y) + \overline{K_x^{+'}(a+0,y) - K_x^{+}(a-0,y)} +$$
$$+\left\{2K^{+}(x,x)|_{a-0} - 2K^{+}(x,x)|_{a+0}\right\} R^{+}(a+y) +$$
$$+\int_a^{\pm\infty}\left[K_x^{+'}(a+0,t) - K_x^{+'}(a-0,t)\right] R^{+}(t+y)\,dt = 0.$$

Further, assume $x=a$ in equation $(12)_{+}$, and differentiate with respect to y. Then integrating by parts, we get

$$R^{+'}(a+y)+\overline{K_y^{+'}(a,y)}-K^{+}(x,x)\Big|_{a+0}R^{+}(a+y)+$$

$$+\int_a^{+\infty}K_t^{+'}(a,t)\,R^{+}(t+y)\,dt=0.$$

Multiply these equalities by $-i\beta$ and substract the obtained relation from the equality next to last. As a result, by condition of the theorem we have

$$\overline{K_x^{+'}(a+0,y)-K_x^{+'}(a-0,y)-i\beta K_y^{+'}(a,y)}+$$

$$+\int_a^{\pm\infty}\left\{K_x^{+'}(a+0,t)-K_x^{+'}(a-0,t)-i\beta K_t'(a,y)\right\}R^{+}(t+y)\,dt=0.$$

Hence, according to the uniqueness theorem on the solution of main equation ([10]) we get relation $(26)_{\pm}$. Similaly, proceeding from equations $(12)_{-}$ and $(13)_{-}$, relation $(26)_{-}$ is established.

Thus, by fulfilling conditions (20), the solutions $K^{\pm}(x,y)$ of main equations $(11)_{\pm}$ satisfy equation $(23)_{\pm}$ by relations $(8)_{\pm}$, $(25)_{\pm}$, $(26)_{\pm}$ conditions $(24)_{\pm}$. Then for solving these problems we get integral equations from [8] which show that the functions $e^{\pm}(x,\lambda)$ constructed with the help of $K^{\pm}(x,t)$ by formulae $(6)_{\pm}$ satisfy equations $(16)_{\pm}$ and conditions $(17)_{\pm}$, $(18)_{\pm}$.

The case when only conditions 3) of the theorem are fulfilled, may be considered by means of the limit passage (see [3], p. 212).

Finally, show that conditions (19) are also fulfilled. Since for $\pm x>\pm a$ main equations $(11)_{\pm}$ have the form of $(12)_{\pm}$, i.e. similar to the case $\alpha=1$ form and conditions 3) of the theorem is the same as in the case $\alpha=1$, then it is easy show that relations (19) are valid if $x'\geq a$ and $x'\leq a$ (see [3], p. 209). It remains to show that $q^{+}(x)$ $(q^{-}(x))$ are summable in the interval (x',a) $((a,x''))$ for each $x'>-\infty$ $(x''<+\infty)$. If we use conditions 3) of the theorem, and integrability of partial derivatives $K_x^{\pm'}$, $K_t^{\pm'}$, these facts are easily established by means of the formula (equivalent to equation $(14)_{\pm}$)

$$K^{\pm}(x,y)=\left(1+\frac{\beta^2}{4}\right)\left[\overline{\varphi^{\pm}(x,y)}\mp\frac{i\beta}{2-i\beta}\varphi^{\pm}(x,2a-y)\right],$$

where

$$\varphi^{\pm}(x,y)=-\left(1+\frac{i\beta}{2}\right)R^{\pm}(x+y)+$$

$$+\frac{i\beta}{2}R^{\pm}(2a-x+y)\mp\int_x^{\pm\infty}K^{\pm}(x,y)\,R^{\pm}(t+y)\,dt.$$

3. Now for proving theorem, it suffices to show that for real $\lambda \neq 0$ the functions $e^+(x,\lambda)$ and $e^-(x,\lambda)$ are connected with the equalities

$$r^{\pm}(\lambda)\, e^{\pm}(x,\lambda) + \overline{e^{\pm}(x,\lambda)} = \frac{1}{a(\lambda)} e^{\mp}(x,\lambda). \qquad (28)_{\pm}$$

Indeed, from $(28)_{\pm}$, by $(16)_{\pm}$ it follows that:

$$q^+(x) = q^-(x) \overset{def}{=} q(x), \quad -\infty < x < +\infty,$$

and according to (19) we have:

$$\int_{-\infty}^{\infty} (1+|x|)\,|q(x)|\,dx < +\infty.$$

Show that then $r^+(\lambda)$ and $r^-(\lambda)$ are the right and left (respectively) reflection factors of the constructed problem (3)-(5).

Denote the right and left reflection factors of the constructed problem (3)-(5) by $\widetilde{r}^+(\lambda)$ and $\widetilde{r}^-(\lambda)$, respectively. The functions $e^+(x,\lambda)$ and $e^-(x,\lambda)$ will the Iost solutions of problem (3)-(5). Therefore, by the results of the direct scattering problem we can write the relations

$$\widetilde{r}^{\pm}(\lambda)\, e^{\pm}(x,\lambda) + \overline{e^{\pm}(x,\lambda)} = \frac{1}{\widetilde{a}(\lambda)} e^{\mp}(x,\lambda). \qquad (29)_{\pm}$$

From $(28)_{\pm}$ and $(29)_{\pm}$ we have

$$a(\lambda)\, r^+(\lambda)\, e^+(x,\lambda) + a(\lambda)\, \overline{e^+(x,\lambda)} = e^-(x,\lambda),$$

$$\widetilde{a}(\lambda)\, \widetilde{r}^+(\lambda)\, \widetilde{e}^+(x,\lambda) + \widetilde{a}(\lambda)\, \overline{e^+(x,\lambda)} = e^-(x,\lambda).$$

Subtracting one of these relations from another one, we have

$$\left\{a(\lambda)\, r^+(\lambda) - \widetilde{a}(\lambda)\, \widetilde{r}^+(\lambda)\right\} e^+(x,\lambda) + \left\{a(\lambda) - \widetilde{a}(\lambda)\right\} \overline{e^+(x,\lambda)} = 0.$$

Since $\lambda \neq 0$, the functions $e^+(x,\lambda)$ and $e^-(x,\lambda)$ are linearly independent, then it follows from the last identity that

$$a(\lambda)\, r^+(\lambda) - \widetilde{a}(\lambda)\, \widetilde{r}^+(\lambda) = 0, \quad a(\lambda) - \widetilde{a}(\lambda) = 0$$

i.e. $r^+(\lambda) = \widetilde{r}^+(\lambda)$, $a(\lambda) = \widetilde{a}(\lambda)$, similarly, from relations $(28)_-$ and $(29)_-$ we get $r^-(\lambda) = \widetilde{r}^-(\lambda)$.

Further, since the function $a(z)$ according to condition 2) of the theorem has no zeros in upper half-plane, then problem (3)-(5) has no discrete spectrum.

4. Now, prove relations $(28)_{\pm}$. Assume

$$\Phi^{\pm}(x,y) = R_1^{\pm}(x+y) \pm \int_x^{\pm\infty} K^{\pm}(x,t)\, R^{\pm}(t+y)\, dt,$$

where

$$R_1(x,y)=\begin{cases} R^{\pm}(x+y), & \pm x>\pm a,\\ \left(1+\frac{i\beta}{2}\right)R^{\pm}(x+y)-\frac{i\beta}{2}R^{\pm}(2a-x+y), & \pm x<\pm a.\end{cases}$$

Since $R^{\pm}(y)\in L_2(-\infty,+\infty)$, then for each fixed x $\Phi(x,y)\in L_2(-\infty,+\infty)$ we have

$$\lim_{N\to+\infty}\int_{-N}^{N}\Phi^{\pm}(x,y)\,e^{\mp\lambda y}dy=$$

$$=\left[r^{\pm}(\lambda)-r_0^{\pm}(\lambda)\right]\left[e_0^{\pm}(x,\lambda)\pm\int_x^{\pm\infty}K^{\pm}(x,t)\,e^{\pm\lambda t}dt\right]=$$

$$=\left[r^{\pm}(\lambda)-r_0^{\pm}(\lambda)\right]e^{\pm}(x,\lambda). \qquad (30)_{\pm}$$

On the other hand, from equations $(11)_{\pm}$

$$\Phi^{\pm}(x,y)=-\overline{K^{\pm}(x,y)}\pm\frac{i\beta}{2+i\beta}K^{\pm}(x,2a-y),\quad \pm y>\pm x.$$

Therefore

$$\lim_{N\to+\infty}\int_{-N}^{N}\Phi^{\pm}(x,y)\,e^{\mp\lambda y}dy=$$

$$=\lim_{N\to+\infty}\left\{\pm\int_{\mp N}^{x}\Phi^{\pm}(x,y)\,e^{\mp\lambda y}dy\right\}-\int_x^{\pm\infty}\overline{K^{\pm}(x,t)}e^{\mp\lambda y}dy+$$

$$+\frac{i\beta}{2+i\beta}\int_x^{\pm\infty}K^{\pm}(x,2a-y)\,e^{\mp\lambda y}dy=\lim_{N\to+\infty}\left\{\pm\int_{\mp N}^{x}\Phi^{\pm}(x,y)\,e^{\mp\lambda y}dy\right\}-$$

$$-\overline{e^{\pm}(x,\lambda)}+\overline{e_0^{\pm}(x,\lambda)}-r_0^{\pm}(\lambda)\left[e^{\pm}(x,\lambda)+e_0^{\pm}(x,\lambda)\right]. \qquad (31)_{\pm}$$

Comparing $(30)_{\pm}$ and $(31)_{\pm}$, and taking into account the formulae

$$r_0^{\pm}(\lambda)\,e_0^{\pm}(x,\lambda)+\overline{e_0^{\pm}(x,\lambda)}=\frac{2}{2+i\beta}e_0^{\mp}(x,\lambda),$$

we get

$$r^{\pm}(\lambda)\,e^{\pm}(x,\lambda)+\overline{e^{\pm}(x,\lambda)}=\frac{1}{a(\lambda)}h^{\mp}(x,\lambda), \qquad (32)_{\pm}$$

where

$$h^{\pm}(\lambda)=a(\lambda)\left[\frac{2}{2+i\beta}e_0^{\mp}(x,\lambda)+l.i.m.\left(\mp\int_{\mp N}^{x}\Phi^{\mp}(x,y)\,e^{\pm\lambda y}dy\right)\right]. \qquad (33)_{\pm}$$

Thus, it suffices to prove that $h^{\pm}(x,\lambda)=e^{\pm}(x,\lambda)$. If we use representations $(32)_{\pm}$ and $(33)_{\pm}$ for the functions $h^{\pm}(x,\lambda)$ and condition 2) of the theorem, then the proof of these equalities completely coincide with the proof similar to the statement of the case $\beta = 0$ (see. [3], pp. 277-278). Therefore we don't cite it.

The theorem is proved.

Remark. Condition 4) of the theorem is essential. The function

$$r^{+}(\lambda) = \frac{i\beta + \frac{\gamma}{i\lambda}}{2 + i\beta + \frac{\gamma}{i\lambda}} e^{-2i\lambda a}$$

for $\gamma \neq 0$ satisfies all the conditions of the theorem, except for 4). In this case

$$K^{\pm}(x,t) = \begin{cases} 0, \pm x > \pm a, \ \pm t > \pm x \vee \pm x < \pm a, \ \pm t > \pm (2a - x), \\ \frac{\gamma}{2}, \ \ \pm x < \pm a, \ \pm x < \pm t < \pm (2a - x), \end{cases}$$

consequently the Jost solutions satisfy equation (3) with $q(x) \equiv 0$ and condition (4), but condition (5) is not fulfilled. If $\gamma = 0$, then condition 4) is fulfilled, and in this case, the solution of the inverse problem exists: $r^{+}(\lambda) = r_0^{+}(\lambda)$ is the right reflection factor of problem (3)-(5) with the potential $q(x) = 0$.

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