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ON ASYMPTOTICS OF SOLUTION OF A BOUNDARY VALUE PROBLEM FOR A QUASILINEAR ELLIPTIC EQUATION DEGENERATING TO A PARABOLIC EQUATION

Abstract

In a rectangle we consider a boundary value problem for a second order quasilinear elliptic equation degenerating to a parabolic equation. Total asymptotics of the generalized solution of the problem under consideration is constructed and the remainder term is estimated.

A number of papers have been devoted to construction of the asymptotics of the solution of various boundary value problems for nonlinear elliptic equations with a small parameter at higher derivatives. Note some of them [1]-[9]. In [1]-[4], the input equations degererate to functional or ordinary differential equations. Boundary value problems for a quasilinear elliptic equation degenerating to a hyperbolic equation in a rectangular domain, in curvilinear trapezoid, in a semi-finite and finite strip were investigated in [5]-[9].

In the present paper in $D=\{(x,y)\,|0\leq x\leq a, 0\leq y\leq 1\}$ we consider the following boundary value problem

$$L_\varepsilon U\equiv -\varepsilon^p\left[\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}\right)^p+\frac{\partial}{\partial y}\left(\frac{\partial U}{\partial y}\right)^p\right]-\varepsilon\Delta U+$$

$$+\frac{\partial U}{\partial x}-\frac{\partial^2 U}{\partial y^2}+cU-f\left(x,y\right)=0, \tag{1}$$

$$U|_{x=0}=U|_{x=a}=0, (0\leq y\leq 1)\,;\quad U|_{y=0}=U|_{y=1}=0, (0\leq x\leq a) \tag{2}$$

where $\varepsilon>0$ is a small parameter, $p=2k+1$, k is an arbitrary natural number, $\Delta\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$, $c>0$ is a constant, $f\left(x,y\right)$ is a given smooth function.

Our goal is to construct asymptotic expansion of the generalized solution of problem (1), (2) from the class $\overset{0}{W}{}_2^1(D)$ following M.I. Vishik – L.A.Lyusternik's method [10]. For constructing asymptotics we conduct interative processes. In the first iterative process we'll look for the approximate solution of equation (1) in the form

$$W=W_0+\varepsilon W_1+...+\varepsilon^n W_n, \tag{3}$$

and the functions $W_i\left(x,y\right);\ i=0,1,...,n$ will be chosen so that

$$L_\varepsilon W=O\left(\varepsilon^{n+1}\right). \tag{4}$$

Substituting (3) into (4), expanding the nonlinear terms in powers of ε, and equating the terms with identical powers of ε, for determining $W_i; i=0,1,...,n$ we get the following recurrently associated equations:

$$\frac{\partial W_i}{\partial x}-\frac{\partial^2 W_i}{\partial y^2}+aW_i=f_i\left(x,y\right),\quad i=0,1,...,n, \tag{5}$$

where $f_0(x,y) = f(x,y)$, $f_i(x,y)$ are the known functions dependent on $W_0, W_1, ..., W_{i-1}$; $i = 1,2,...,n$. For example, the function $f_1(x,y)$ is of the form: $f_1(x,y) = \Delta W_0$.

Equations (5) will be solved under the following boundary conditions:

$$W_i|_{x=0} = 0, \ (0 \le y \le 1); \ \ W_i|_{y=0} = W_i|_{y=1} = 0 \ \ (0 \le x \le a); \ i = 0,1,...,n. \quad (6)$$

The following lemma is valid.

Lemma 1. *Let $f(x,y) \in C^{n+1,2n+6}(D)$, and the following condition be satisfied*

$$\frac{\partial^{2k} f(x,0)}{\partial y^{2k}} = \frac{\partial^{2k} f(x,1)}{\partial y^{2k}} = 0; \quad k = 0,1,...,n+2. \quad (7)$$

Then the solution of problem (5), (6) for $i = 0$ enters into the space $C^{n+2,2n+4}(D)$ and satisfies the relation

$$\frac{\partial^{i_1+2i_2} W_0(x,0)}{\partial x^{i_1} \partial y^{2i_2}} = \frac{\partial^{i_1+2i_2} W_0(x,1)}{\partial x^{i_1} \partial y^{2i_2}} = 0; \quad i_1 + i_2 \le n+2. \quad (8)$$

Proof. It is obvious that the solution of problem (5), (6) for $i = 0$ may be represented by the formula

$$W_0(x,y) = \sum_{k=1}^{\infty} \overline{W}_{0k}(x,y), \quad (9)$$

where $\overline{W}_{0k}(x,y)$ denotes the function

$$\overline{W}_{0k}(x,y) = \left[\int_0^x e^{-\left(c+k^2\pi^2\right)(x-\tau)} f_k(\tau) \right] \sin k\pi y, \quad (10)$$

moreover $f_k(x) = 2\int_0^1 f(x,\xi) \sin k\pi\xi d\xi$. Taking into account condition (7), we can get the estimation:

$$\left| f_k^{(\iota)}(x) \right| \le \frac{2M_{i,2n+4}}{k^{2n+4}\pi^{2n+4}}; \quad i = 0,1,...,n+1, \ x \in [0,a], \quad (11)$$

where $M_{i,2n+4} = \max\limits_{(x,y)\in D} \left| \frac{\partial^{i+2n+4} f(x,y)}{\partial x^{i+2n+4}} \right|$; $i = 0,1,...,n+1$. On the base of (11), it follows from (10) that

$$\left| \frac{\partial^i \overline{W}_{0k}(x,y)}{\partial x^{i_1} \partial y^{i_2}} \right| \le \frac{C}{k^{2n+4-2i_1-i_2}\pi^{2n+4-2i_1-i_2}}, C = const, (x,y) \in D. \quad (12)$$

Denoting $r = 2n+4-2i_1-i_2$, from (12) we get that the number series $\sum\limits_{k=1}^{\infty} \frac{1}{k^r}$ is majorant for the functional series $\sum\limits_{k=1}^{\infty} \frac{\partial^i \overline{W}_{0k}(x,y)}{\partial x^{i_1} \partial y^{i_2}}$. That was obtained by termwise differentiation of (9). And this number series converges for $r \ge 2$, i.e. for $2i_1 + i_2 \le$

$2n+4$. Hence it follows the belongness of W_0 to the space $C^{n+2,2n+4}(D)$ and validity of (8).

Lemma 1 is proved.

By lemma 1, the function $f_1(x,y)$ being the right side of equation (5) for $i=1$ satisfies condition (7) for $k=0,1,...,n+1$. Then by the same lemma 1, the function W_1, that is a solution of problem (5), (6) for $i=1$ will satisfy condition (8) for $i_1+i_2 \le n+1$. Continuing the process, we construct the functions $W_i; i=0,1,...,n$ entering into the right side of (3).

From (3) and (6) it follows that the constructed function W satisfies the following boundary conditions:

$$W|_{x=0}=0,\ (0\le y\le 1);\ \ W|_{y=0}=W|_{y=1}=0,\ (0\le x\le a). \tag{13}$$

This function, generally speaking, doesn't satisfy boundary condition from (2) for $x=a$.

Therefore we have construct a boundary layer type function near the boundary $x=a$ so that the obtained sum $W+V$ satisfies the boundary condition

$$(W+V)|_{x=a}=0. \tag{14}$$

Further more, by choosing V, the fulfilment of the equality

$$L_{\varepsilon,1}(W+V)-L_{\varepsilon,1}W=0\left(\varepsilon^{n+1}\right) \tag{15}$$

should also be provided. In (15) a new decomposition of the operator L_ε near the boundary $x=a$ is denoted by $L_{\varepsilon,1}$. In order to write a new decomposition of the operator L_ε near the boundary $x=a$, we make change of variables: $a-x=\varepsilon\tau$, $y=y$. Consider the auxiliary function $r=\sum\limits_{j=0}^{n+1} r_j(\tau,y)$, where $r_j(\tau,y)$ are some smooth functions. The expansion of $L_\varepsilon(r)$ in powers of ε in the coordinates (τ,y) has the form

$$L_{\varepsilon,1}r\equiv -\varepsilon^{-1}\left\{\frac{\partial}{\partial\tau}\left(\frac{\partial r_0}{\partial\tau}\right)^{2k+1}+\frac{\partial^2 r_0}{\partial\tau^2}+\frac{\partial r_0}{\partial\tau}+\right.$$

$$\left.+\sum_{j=1}^{n+1}\varepsilon^j\left[(2k+1)\frac{\partial}{\partial\tau}\left(\left(\frac{\partial r_0}{\partial\tau}\right)^{2k}\frac{\partial r_j}{\partial\tau}\right)+\frac{\partial^2 r_j}{\partial\tau^2}+\frac{\partial r_j}{\partial\tau}+H_j\right]+0\left(\varepsilon^{n+2}\right)\right\}, \tag{16}$$

where $H_j(r_0,r_1,...,r_{j-1})$ are the known functions dependent on r_0, $r_1,...,r_{j-1}$ and their first and second derivatives.

We look for a boundary layer type function V near the boundary $x=a$ in the form

$$V=V_0(\tau,y)+\varepsilon V_1(\tau,y)+...+\varepsilon^{n+1}V_{n+1}(\tau,y). \tag{17}$$

Expanding each function $W_i(a-\varepsilon\tau,y)$ at the point (a,y) in Taylor's formula, we get a new expansion of the function W in powers of ε in the coordinates (τ,y) in the form

$$W=\sum_{j=0}^{n+1}\varepsilon^j\omega_j(\tau,y)+0\left(\varepsilon^{n+2}\right). \tag{18}$$

Here $\omega_0 = \omega_0(a,y)$ is independent of τ, the remaining functions are determined from the formula $\omega_k = \sum\limits_{i+j=k} (-1) \frac{\partial^i W_j(a,y)}{\partial x^i} \tau^i; \;\; k = 1,2,...,n+1.$

Substituting expressions (17), (18) for the functions V, W into (15), and taking into account (16), we get the following equations for determining the functions $V_0, V_1, ..., V_{n+1}$:

$$\frac{\partial}{\partial \tau} \left(\frac{\partial V_0}{\partial \tau} \right)^{2k+1} + \frac{\partial^2 V_0}{\partial \tau^2} + \frac{\partial V_0}{\partial \tau} = 0, \tag{19}$$

$$(2k+1) \frac{\partial}{\partial \tau} \left[\left(\frac{\partial V_0}{\partial \tau} \right)^{2k} \frac{\partial V_j}{\partial \tau} \right] + \frac{\partial^2 V_j}{\partial \tau^2} + \frac{\partial V_j}{\partial \tau} = Q_j; \;\; j = 1,2,...,n+1. \tag{20}$$

Here Q_j are the known functions dependent on $\tau, y, V_0, V_1, ..., V_{j-1}, \omega_0, \omega_1, ..., \omega_j$ and their first and second derivatives. The formulae for Q_j may be written explicitly, but they are very bulky. Here we give the explicit form only of the function Q_1 : $Q_1 = -(2k+1) \frac{\partial}{\partial \tau} \left(\left(\frac{\partial V_0}{\partial \tau} \right)^{2k} \frac{\partial \omega_1}{\partial \tau} \right) - \frac{\partial^2 V_0}{\partial y^2} + CV_0.$

The boundary conditions for equations (19), (20) are obtained from the requirement that the sum $W+V$ satisfies the boundary condition

$$(W+V)|_{x=a} = 0. \tag{21}$$

Substituting the expressions for W from (3) and for V from (17) into (21), and taking into account that we look for V_j; $j = 0,1,...,n+1$ as a boundary layer type function, we have

$$V_0|_{\tau=0} = \varphi_0(y), \quad \lim_{\tau \to +\infty} V_0 = 0, \tag{22}$$

$$V_j|_{\tau=0} = \varphi_j(y), \quad \lim_{\tau \to +\infty} V_j = 0; \quad j = 1,2,...,n+1, \tag{23}$$

where $\varphi_i(y) = -W_i(1,y)$ for $i = 0,1,...,n$; $\varphi_{n+1} \equiv 0$.

The following lemma is valid.

Lemma 2. *For each $y \in [0,1]$ problem (19), (22) has a unique solution that is infinitely differentiable with respect to τ,* and with respect to y *has continuous derivative to the $(2n+4)$ -th order inclusively. The following estimation is valid:*

$$\left| \frac{\partial^i V_0(\tau, y)}{\partial \tau^{i_1} \partial y^{i_2}} \right| \leq g_i \left(|\varphi_0(y)|, |\varphi_0'(y)|, ..., \left| \varphi_0^{(i_2)}(y) \right| \right) \exp(-\tau), \tag{24}$$

where $i = i_1 + i_2$; $i_2 = 0,1,..,2n+4$; $g_i(t_1, t_2, ..., t_{i_2+1})$ are some known polynomials of own arguments with non-negative coefficients, moreover the free terms of these polynomials equal zero, and from the other coefficients even if one is not zero.

Proof. The existence and uniqueness of the solution of problem (19), (22) was proved in [11]. The solution of problem (19), (22) for $y=0$ and $y=1$ is defined by an identity zero, and for $y \in (0,1)$ the solution in the parametric form is represented by the following formulae:

$$\tau = \frac{2k+1}{2k} \left(q_0^{2k} - q^{2k} \right) + \ln \left| \frac{q_0}{q} \right|, \;\; V_0 = -q^{2k+1} - q. \tag{25}$$

Here q is a parameter, $q_0(y)$ is a real root of the algebraic equation

$$q_0^{2k+1} + q_0 + \varphi_0(y) = 0. \tag{26}$$

The smoothness of the solution of problem (19), (22) is also proved in [11]. Therefore, we'll only derive estimation (24). From the first equality of (25) we can get the estimation

$$|q| \le |q_0(y)| \exp\left[\frac{2k+1}{2k} q_0^{2k}(y)\right] \exp(-\tau). \tag{27}$$

Transforming equation (26), we have: $q_0(y) = \left[q_0^{2k}(y)+1\right]^{-1} \varphi_0(y)$, whence it follows that $|q_0(y)| \le |\varphi_0(y)|$. Hence we have that $\exp\left[\frac{2k+1}{2k} q_0^{2k}(y)\right]$ is bounded, i.e. $\exp\left[\frac{2k+1}{2k} q_0^{2k}(y)\right] \le C_0$. Consequently, from (27) we get the estimation

$$|q| \le C_0 |\varphi_0(y)| \exp(-\tau). \tag{28}$$

Taking into account (28) in the second equality of (25), we have

$$|V_0| \le C |\varphi_0(y)| \exp(-\tau), C > 0. \tag{29}$$

Recalling that the parametric form (25) of the solution of problem (19), (22) was obtained by means of substitution $\frac{\partial V_0}{\partial \tau} = q$, from (28) we get an estimation for $\frac{\partial V_0}{\partial \tau}$

$$\left|\frac{\partial V_0}{\partial \tau}\right| \le C_0 |\varphi_0(y)| \exp(-\tau). \tag{30}$$

We can represent the function $\frac{\partial^2 V_0}{\partial \tau^2}$ in the form

$$\frac{\partial^2 V_0}{\partial \tau^2} = -B^{-1}(\tau, y) \frac{\partial V_0}{\partial \tau}, \tag{31}$$

where $B(\tau, y)$ denotes the function

$$B(\tau, y) = (2k+1)\left(\frac{\partial V_0}{\partial \tau}\right)^{2k} + 1. \tag{32}$$

Taking into account $0 < B^{-1}(\tau, y) \le 1$, from (30) and (31) we get an estimation for $\frac{\partial^2 V_0}{\partial \tau^2}$. Differentiating sequentially the both hand sides of (31) with respect to τ and each time taking into account the estimations for the previous derivatives, we can get estimations for higher order derivatives $V_0(\tau, y)$ with respect to τ. These estimations will be of the form (30), i.e.

$$\left|\frac{\partial^i V_0}{\partial \tau^i}\right| \le C_0 |\varphi_0(y)| \exp(-\tau), \quad i = 2, 3, ... \tag{33}$$

Now prove estimations for the derivatives $V_0(\tau, y)$ with respect to y and for mixed derivatives. The function $\frac{\partial V_0}{\partial y} = z$ satisfies the equation in variation that is obtained from equation (19) by differentiating with respect to y:

$$\frac{\partial}{\partial \tau}\left[B(\tau, y) \frac{\partial z}{\partial \tau}\right] + \frac{\partial z}{\partial \tau} = 0. \tag{34}$$

From (22) we get that the function z should satisfy the boundary conditions

$$z|_{\tau=0} = \varphi_0'(y), \quad \lim_{\tau \to +\infty} z = 0. \tag{35}$$

The solution of problem (34) (35) has the form

$$z=\varphi_0'(y)\exp\left[-\int\limits_0^\tau B^{-1}(\xi,y)\,d\xi\right]. \tag{36}$$

Using (32) and estimation (30) in (36), we get the estimation

$$|z|=\left|\frac{\partial V_0}{\partial y}\right|\leq C\left|\varphi_0'(y)\right|\exp(-\tau). \tag{37}$$

From (36) it follows that $\frac{\partial z}{\partial\tau}=-B^{-1}(\tau,y)\,z$. Taking into account (37), hence we get an estimation for the mixed derivative

$$\left|\frac{\partial z}{\partial\tau}\right|=\left|\frac{\partial^2 V_0}{\partial y\partial\tau}\right|\leq C\left|\varphi_0'(y)\right|\exp(-\tau). \tag{38}$$

Now we can get an estimation for $\frac{\partial^2 V_0}{\partial y^2}$. Differentiating the both hand sides of (36) with respect to y, we have

$$\frac{\partial z}{\partial\tau}=-\left\{-\int\limits_0^\tau\left[B^{-1}(\xi,y)\right]_y' d\xi\right\}z+\varphi_0''(y)\exp\left[-\int\limits_0^\tau B^{-1}(\xi,y)\,d\xi\right]. \tag{39}$$

From (32) it follows that

$$\left[B^{-1}(\tau,y)\right]_y'=-(2k+1)(2k)B^{-2}(\tau,y)\left(\frac{\partial V_0}{\partial\tau}\right)^{2k-1}\frac{\partial^2 V_0}{\partial y\partial\tau}.$$

Obviously, $0<B^{-i}\leq 1$ for any natural number i. Knowing estimation (30) for $\frac{\partial V_0}{\partial\tau}$ and estimation (38) for $\frac{\partial^2 V_0}{\partial y\partial\tau}$, we estimate $\left[B^{-1}(\tau,y)\right]_y'$:

$$\left|\left[B^{-1}(\tau,y)\right]_y'\right|\leq C\left|\varphi_0(y)\right|\left|\varphi_0'(y)\right|\exp(-\tau). \tag{40}$$

Taking into account (37) and (40) in (39), we have

$$\left|\frac{\partial z}{\partial y}\right|=\left|\frac{\partial^2 V_0}{\partial y}\right|\leq\left[C_1\left|\varphi_0(y)\right|\left|\varphi_0'(y)\right|^2+C_2\left|\varphi_0''(y)\right|\right]\exp(-\tau). \tag{41}$$

Validity of estimation (24) for the subsequent derivatives $V_0(\tau,y)$ is proved in the same way.

Lemma 2 is proved.

From (8) it follows that the function $\varphi_0(y)$ and all its even derivatives vanish for $y=0$. Hence and from the estimations obtained in the proof of lemma 2 it follows that the function $V_0(\tau,y)$ and all its derivatives with respect to τ and all even derivatives with respect to y vanish for $y=0$ (see (30), (33), (41)).

Lemma 3. *Problems (20), (23) have unique solutions, the functions $V_j(\tau,y)$; $j=1,2,...,n+1$ with respect to τ are infinitely differentiable, and with respect to* *y have continuous derivatives to the* $(2n+2-2j)$*- th order inclusively. Therewith the following estimations are valid*

$$\left|\frac{\partial^i V_j(\tau,y)}{\partial\tau^{i_1}\partial y^{i_2}}\right| \leq \left[\sum_{S=0}^{i_2+j+1} |q_{js}(y)|\,\tau^s\right]\exp(-\tau). \tag{42}$$

where $i_2 = 0,1,...,2n+2-j;\ j=1,2,...,n+1, q_{js}(y)$ *are the known functions.*

Proof. In [11], the existence, uniqueness and smoothness of solutions of problems (20), (23) is proved, and the representation of these solutions is obtained in the following form:

$$V_j(\tau,y) = \left\{\varphi_j(y) - \int_0^\tau \left[B^{-1}(z,y)\exp(\partial(z,y))\int_z^{+\infty} Q_j(\xi,y)\,d\xi\right]dz\right\}\exp\left[-\nu(\tau,y)\right]. \tag{43}$$

Here $\nu(\tau,y)$ denotes the function

$$\nu(\tau,y) = \int_0^\tau B^{-1}(\xi,y)\,d\xi. \tag{44}$$

Substituting $j=1$ in (43), we get a formula for $V_1(\tau,y)$. Using the explicit expressions $Q_1(\tau,y)$, $\omega_1(\tau,y)$ and taking into account the known estimations for V_0, $\frac{\partial V_0}{\partial\tau}$, $\frac{\partial^2 V_0}{\partial y^2}$, we get

$$|Q_1(\tau,y)| \leq |q_1(y)|\exp(-\tau), \tag{45}$$

where $q_1(y)$ is a known function, moreover $q_1^{(2k)}(0) = q_1^{(2k)}(1) = 0;\ k=0,1,...,n+1$. Following (45), from (43) (for $j=1$) we can get the following estimation

$$|V_1(\tau,y)| \leq C\left(|\varphi_1(y)| + \tau\,|q_1(y)|\right)\exp(-\tau). \tag{46}$$

Differentiating the both sides of (43) (for $j=1$) with respect to τ, we get

$$\frac{\partial V_1}{\partial\tau} = -B^{-1}(\tau,y)\left[V_1 + \int_\tau^{+\infty} Q_1(\xi,y)\,d\xi\right]. \tag{47}$$

Using estimations (45) and (46) in (47), we get an estimation for $\frac{\partial V_1}{\partial\tau}$. The estimations for higher derivatives with respect to τ are obtained from the formula obtained by sequential differentiation of both hand sides of (47) and from the estimation for previous derivatives $V_1(\tau,y)$. Note that these estimations have the form $\left|\frac{\partial^i V_1(\tau,y)}{\partial\tau^i}\right| \leq (|q_1(y)| + |q_2(y)|\,\tau)\exp(-\tau)$; $i=1,2,...$, where $q_2(y)$ is a known function, moreover $q_2^{(2k)}(0) = q_2^{(2k)}(1) = 0;\ k=0,1,...,n+1$.

Now derive estimations for the derivatives $V_1(\tau,y)$ with respect to y and for mixed derivatives. We can determine the function $\frac{\partial V_1}{\partial y}$ as a solution of a boundary value problem in variations that is obtained from (20) (for $j=1$) by differentiating with respect to y. We can see that the function $\frac{\partial V_1}{\partial y}$ is also determined by formula (43), but in this formula the function $\varphi_j(y)$ should be replaced by $\varphi_1'(y)$, and the function $\int\limits_z^{+\infty} Q_j(\xi, y)\, d\xi$ by the following function: $\int\limits_z^{+\infty} Q'_{Iy}(\xi, y)\, d\xi + B'_y(z, y)\, \frac{\partial V_1(z,y)}{\partial z}$. Consequently, this time, by obtaining estimations, instead of (45) we have to use the estimation:

$$\left| \int\limits_z^{+\infty} Q_{Iy}(\xi, y)\, d\xi + B'_y(z, y)\, \frac{\partial V_1(z, y)}{\partial z} \right| \leq \left(|q_1(y)| + |q_2(y)|\, z\right) \exp(-z).$$

As a result, we get an estimation for $\frac{\partial V_1}{\partial y}$ in the form

$$\left| \frac{\partial V_1}{\partial y} \right| \leq \left(|q_1(y)| + |q_2(y)|\, \tau + |q_3(y)|\, \tau^2\right) \exp(-\tau). \tag{48}$$

If we differentiate the both hand sides of the formula for $\frac{\partial V_1}{\partial y}$ with respect to τ, we get an estimation of the form (48). It should be noted that at each differentiation of $V_1(\tau, y)$ with respect to y, the power of the polynomial with respect to τ, standing in the right side of the estimation increases by a unit. Estimation for $V_1(\tau, y)$ in the general case has the form

$$\left| \frac{\partial^i V_1(\tau, y)}{\partial \tau^{i_1} \partial y^{i_2}} \right| \leq \left(|q_{10}(y)| + |q_{11}(y)|\, \tau + ... + |q_{1 i_2+1}(y)|\, \tau^{i_2+1}\right) \exp(-\tau).$$

Continuing this process and each time taking into account the explicit form of the right hand side of the equation for V_j, we again get estimation (42).

Lemma 3. is proved.

Multiply all the functions V_j; $j = 0, 1, ..., n+1$ by a smoothing factor and leave previous denotation for the obtained new functions. At the expense of smoothing factors all the functions V_j; $j = 0, 1, ..., n+1$ vanish for $x = 0$. Therefore, hence and from (13) it follows that the constructed sum $W + V$, in addition to boundary condition (21) satisfies also the condition

$$(W + V)\,|_{x=0} = 0. \tag{49}$$

From the construction process it is known that all the functions $V_j(\tau, y)$; $j = 0, 1, ..., n+1$ vanish at $y = 0$ and $y = 1$. Hence and from (13) it follows that the sum $W + V$, in addition to conditions (21), (49) satisfies also the following boundary conditions:

$$(W + V)\,|_{y=0} = 0;\ (W + V)\,|_{y=1} = 0. \tag{50}$$

Thus, the sum that we have constructed $\widetilde{U} = W + V$ satisfies boundary conditions (21), (49) and (50). Having denoted $U - \widetilde{U} = z$, we have the following asymptotic expansion in small parameter of the solution of problem (1), (2):

$$U = \sum_{i=0}^{n} \varepsilon^i W_i + \sum_{j=0}^{n+1} \varepsilon^j V_j + z, \tag{51}$$

where z is a remainder term.

It holds the following.

Lemma 4. *For the remainder term z the following estimation is valid:*

$$\varepsilon^p \iint\limits_D \left[\left(\frac{\partial z}{\partial x} \right)^{p+1} + \left(\frac{\partial z}{\partial y} \right)^{p+1} \right] dxdy + \varepsilon \iint\limits_D \left[\left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 \right] dxdy +$$

$$+ \iint\limits_D \left(\frac{\partial z}{\partial y} \right)^2 dxdy + C_1 \iint\limits_D z^2 dxdy \leq C_2 \varepsilon^{2(n+1)}, \tag{52}$$

where $C_1 > 0$, $C_2 > 0$ are constants independent of ε.

Proof. Adding (4) and (15), we have:

$$L_\varepsilon \left(\widetilde{U} \right) = 0 \left(\varepsilon^{n+1} \right). \tag{53}$$

Subtracting equation (53) from (1), we get

$$-\varepsilon^p \left\{ \frac{\partial}{\partial x} \left[\left(\frac{\partial U}{\partial x} \right)^p - \left(\frac{\partial \widetilde{U}}{\partial x} \right)^p \right] + \frac{\partial}{\partial x} \left[\left(\frac{\partial U}{\partial x} \right)^p - \left(\frac{\partial \widetilde{U}}{\partial x} \right)^p \right] \right\} -$$

$$-\varepsilon \Delta z + \frac{\partial z}{\partial x} - \frac{\partial^2 z}{\partial y^2} + az = \varepsilon^{n+1} F(\varepsilon, x, y), \tag{54}$$

where $\|F(\varepsilon, x, y)\|_{L_2(p)} \leq C$ for any $\varepsilon \in [0, \varepsilon_0)$, moreover $C > 0$ is independent of ε.

From (2), (21), (49), (50) and (51) it follows that z satisfies the boundary conditions

$$z|_{x=0} = z|_{x=a} = 0, \quad z|_{y=0} = z|_{y=1} = 0. \tag{55}$$

Having multiplied the both sides of (54) by $z = U - \widetilde{U}$ and integrating by parts, allowing for boundary conditions (55) after certain transformations we get estimation (52).

Lemma 4 is proved.

Combining the obtained results, we get the following statement.

Theorem. *Let $f(x, y) \in C^{n+1, 2n+6}(D)$ and condition (7) be fulfilled. Then for the generalized solution of problem (1), (2) it holds asymptotic representation (51), where the functions W_i are determined by the first iterative process, V_j is a boundary layer type function near the boundary $x = 1$, and z is a remainder term, estimation (62) is valid for it.*

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