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ON APPROXIMATION OF FUNCTIONS BY MELLIN SINGULAR INTEGRALS

Abstract

The results on approximations of functions by singular integrals play an important part and have numerous applications in various fields of mathematics. A lot of papers were devoted (see e.g. [1]-[9] and the references) to the investigation of the problems on approximation of functions by singular integrals (including by Mellin singular integrals).

In the present paper we investigate approximate properties of Mellin's singular integral in the terms of mean oscillation of a locally summable function.

Introduction

Mellin convolution operators occupy an important place in theory of Mellin transformation. Such matters as the problems of theory of approximation of functions, investigation of boundary value problems for some differential equations require to study approximate properties of Mellin convolution operators with a kernel of Fejer type (see e.g. [1], [2]).

Mellin transformation and Mellin convolution are determined by the following formulae [6], respectively:

$$\hat{f}(s) = \int_0^\infty f(x)\, x^{s-1} dx,$$

$$(f * g)(x) = \int_0^\infty f\left(\frac{x}{y}\right) g(y) \frac{dy}{y}.$$

Mellin transformation and Mellin convolution are considered on a multiplicative group of positive real numbers, where the Haar's measure invariant with respect to shears, is of the form $\frac{dx}{x}$, and dx is a Lebesgue measure on $(-\infty, \infty)$.

Under certain conditions on the functions f and g, the following formula is valid

$$(f * g)\,\hat{}\,(s) = \hat{f}(s) \cdot \hat{g}(s).$$

In the present paper, we study approximate properties of Mellin'e singular integral with a kernel of Fejer type

$$\Phi_\varepsilon(f; x) = \frac{1}{\varepsilon} \int_0^\infty f\left(\frac{x}{t}\right) K\left(t^{\frac{1}{\varepsilon}}\right) \frac{dt}{t}$$

in the terms of mean oscillation of a locally summable function f, where K is a so called Mellin kernel of Fejer type (see 2).

A lot of papers [1]-[9] have been devoted to the investigation of the matters of approximation of functions by singular integrals (including Mellin singular integrals).

Necessary definitions, denotation are given, preliminary facts are cited in 1. The main result of this section is theorem 1.2. In the same place, it is proved (theorem 1.3.) that the main condition in theorem 1.2, generally speaking, may not be removed.

Approximate properties of Mellin singular integral with a kernel of Fejer type are studied in 2. In this section, theorem 2.1 is the main result. The results on upper bound of order of approximation of a function by Mellin singular integrals with a kernel of Fejer type in the terms of mean oscillation of a locally summable function (theorems 2.2 and 2.3) are obtained from this theorem.

1. Some definitions, denotation and preliminary facts

Let $R=(-\infty,+\infty)$, $R_+=(0,+\infty)$. If $E=R$ or $E=R_+$, then by $L_{loc}(E)$ we'll denote a totality of all functions locally summable on the set E. $L(X)=L(X;dx)$ is a set of all the functions summable on the set $X\subset R$ with respect to Lebesgue linear measure dx. Further, by $L\left(R_+;\frac{dx}{x}\right)$ we'll denote a set of all the functions f summable over the linear measure $\frac{dx}{x}$ on the set R_+.

Let $0<\tau\le 1$, $x\in R_+$, $I(x;\tau):=\left\{\rho\in R_+ : x\tau\le\rho\le\frac{x}{\tau}\right\}$, $f\in L_{loc}(R_+)$.

Introduce the following denotation.

$$f_{I(x;\tau)}:=\frac{1}{2\left|\ln\tau\right|}\int\limits_{x\tau}^{x\tau^{-1}} f(\rho)\,\frac{d\rho}{\rho},$$

$$\Omega^M\left(f,I(x;\tau)\right):=\frac{1}{2\left|\ln\tau\right|}\int\limits_{x\tau}^{x\tau^{-1}}\left|f(\rho)-f_{I(x;\tau)}\right|\frac{d\rho}{\rho}.$$

Call the quantity $\Omega^M\left(f,I(x;\tau)\right)$ a Mellin mean oscillation of the function f on the interval $I(x;\tau)$.

We introduce the following metric characteristics (see. [8])

$$m_f^M(x;\delta):=\sup\left\{\Omega^M\left(f,I(x;\tau)\right):\left|\ln\tau\right|\le\delta\right\},\quad x\in R_+,\delta\in R_+.$$

It is easy to see that the function $m_f^M(x;\delta)$ accepts only non-negative values and is monotonically increasing with respect to the argument $\delta\in(0,+\infty)$.

Theorem 1.1. *Let $f\in L_{loc}(R_+)$, $x_0\in R_+$, $0<\nu_1<\nu_2\le 1$. Then the following inequality is true:*

$$\left|f_{I(x_0;\nu_1)}-f_{I(x_0;\nu_2)}\right|\le\frac{2}{\ln 2}\left(m_f^M\left(x_0;\ln\frac{1}{\nu_1}\right)+\int\limits_{\ln\frac{1}{\nu_2}}^{\ln\frac{1}{\nu_1}}\frac{m_f^M(x_0;t)}{t}dt\right).\qquad(1.1)$$

Proof. We can show that if

$$x_0=e^{-y_0},(x_0\in R_+,y_0\in R),\quad\tau=e^{-r}\left(\tau\in(0,1),r\in R_+\right),$$

$$f^*(t):=f\left(e^{-t}\right)\ (t\in R),\ B(y_0;r):=\{t\in R:y_0-r\le t\le y_0+r\}=[y_0-r,y_0+r],$$

then the following equalities are true:

$$f_{I(x_0;\tau)}=\frac{1}{2r}\int\limits_{y_0-r}^{y_0+r} f*(t)\,dt=:f^*_{B(y_0:r)},$$

$$\Omega^{M}\left(f,I\left(x;\tau\right)\right)=\frac{1}{2r}\int\limits_{y_0-r}^{y_0+r}\left|f^{*}\left(t\right)-f^{*}_{B(y_0;r)}\right|dt.$$

Let $\xi=\ln\frac{1}{\nu_1}$, $\eta=\ln\frac{1}{\nu_2}$. Then $0<\eta<\xi<+\infty$ and there exists a non-negative integer p such that it holds the inequality $\dfrac{\xi}{2^{p+1}}<\eta\leq\dfrac{\xi}{2^{p}}$.

Then we have

$$\left|f_{I(x_0;\nu_1)}-f_{I(x_0;\nu_2)}\right|=\left|f^{*}_{B(y_0;\xi)}-f^{*}_{B(y_0;\eta)}\right|\leq$$

$$\leq\sum_{i=0}^{p-1}\left|f^{*}_{B\left(y_0;\frac{\xi}{2^i}\right)}-f^{*}_{B\left(y_0;\frac{\xi}{2^{i+1}}\right)}\right|+\left|f^{*}_{B(y_0;\eta)}-f^{*}_{B\left(y_0;\frac{\xi}{2^p}\right)}\right|, \tag{1.2}$$

If $i=0,1,...,p-1$, we get

$$\left|f^{*}_{B\left(y_0;\frac{\xi}{2^i}\right)}-f^{*}_{B\left(y_0;\frac{\xi}{2^{i+1}}\right)}\right|\leq\frac{1}{\left(\frac{\xi}{2^i}\right)}\int\limits_{B\left(y_0;\frac{\xi}{2^{i+1}}\right)}\left|f^{*}\left(t\right)-f^{*}_{B\left(y_0;\frac{\xi}{2^i}\right)}\right|dt\leq$$

$$\leq 2\cdot\frac{1}{2\left(\frac{\xi}{2^i}\right)}\int\limits_{B\left(y_0;\frac{\xi}{2^i}\right)}\left|f^{*}\left(t\right)-f^{*}_{B\left(y_0;\frac{\xi}{2^i}\right)}\right|dt=2\cdot\Omega^{M}\left(f;I\left(x_{.0};e^{-\frac{\xi}{2^i}}\right)\right). \tag{1.3}$$

From the definition of the function $m_f^M\left(x_0;\delta\right)$ it follows that

$$m_f^M\left(x_0;\delta\right)\geq\Omega^{M}\left(f;I\left(x_{.0};e^{-\delta}\right)\right). \tag{1.4}$$

Taking into account inequality (1.3), from inequality (1.2) we get

$$\left|f^{*}_{B\left(y_0;\frac{\xi}{2^i}\right)}-f^{*}_{B\left(y_0;\frac{\xi}{2^{i+1}}\right)}\right|\leq 2\cdot m_f^M\left(x_0;\frac{\xi}{2^i}\right), \tag{1.5}$$

where $i=0,1,...,p-1$.

Taking into attention the inequality $2\eta>\dfrac{\xi}{2^p}$, we have

$$\left|f^{*}_{B\left(y_0;\frac{\xi}{2^p}\right)}-f^{*}_{B(y_0;\eta)}\right|\leq\frac{1}{2\eta}\int\limits_{B(y_0;\eta)}\left|f^{*}\left(t\right)-f^{*}_{B\left(y_0;\frac{\xi}{2^p}\right)}\right|dt\leq$$

$$\leq 2\cdot\frac{1}{2\left(\frac{\xi}{2^p}\right)}\int\limits_{B\left(y_0;\frac{\xi}{2^p}\right)}\left|f^{*}\left(t\right)-f^{*}_{B\left(y_0;\frac{\xi}{2^p}\right)}\right|dt=2\cdot\Omega^{M}\left(f;I\left(x_{.0};e^{-\frac{\xi}{2^p}}\right)\right)\leq$$

$$\leq 2\cdot m_f^M\left(x_0;\frac{\xi}{2^p}\right). \tag{1.6}$$

From inequalities (1.2), (1.5) and (1.6) we get

$$\left|f_{I(x_0;\nu_1)}-f_{I(x_0;\nu_2)}\right|\leq 2\left(m_f^M\left(x_0;\frac{\xi}{2^p}\right)+\sum_{i=0}^{p-1}m_f^M\left(x_0;\frac{\xi}{2^i}\right)\right)=$$

$$= 2\left(m_f^M\left(x_0;\xi\right) + \sum_{i=1}^{p} m_f^M\left(x_0;\frac{\xi}{2^i}\right)\right). \qquad (1.7)$$

If $p \geq 1$, we have

$$\int_{\frac{\xi}{2^p}}^{\xi} \frac{m_f^M\left(x_0;t\right)}{t}dt = \sum_{i=1}^{p} \int_{\frac{\xi}{2^i}}^{\frac{\xi}{2^{i-1}}} \frac{m_f^M\left(x_0;t\right)}{t}dt \geq \sum_{i=1}^{p} m_f^M\left(x_0;\frac{\xi}{2^i}\right)\cdot \ln 2.$$

Furthermore,

$$\int_{\eta}^{\xi} \frac{m_f^M\left(x_0;t\right)}{t}dt \geq \int_{\frac{\xi}{2^p}}^{\xi} \frac{m_f^M\left(x_0;t\right)}{t}dt.$$

Taking into account the last relation, from inequality (1.7) we have

$$\left| f_{I(x_0;\nu_1)} - f_{I(x_0;\nu_2)}\right| \leq \frac{2}{\ln 2}\left(m_f^M\left(x_0;\ln\frac{1}{\nu_1}\right) + \int_{\ln\frac{1}{\nu_2}}^{\ln\frac{1}{\nu_1}} \frac{m_f^M\left(x_0;t\right)}{t}dt\right).$$

The theorem is proved.

Theorem 1.2. *Let $f \in L_{loc}\left(R_+\right)$, $x_0 \in R_+$ and the following condition be fulfilled:*

$$\int_0^1 \frac{m_f^M\left(x_0;t\right)}{t}dt < +\infty.$$

Then there exists the finite limit

$$d_f\left(x_0\right) := \lim_{\nu\to 1-0} f_{I(x_0,\nu)}$$

and it holds the inequality

$$\left| f_{I(x_0;\nu)} - d_f\left(x_0\right)\right| \leq \frac{2}{\ln 2}\left(m_f^M\left(x_0;\ln\frac{1}{\nu}\right) + \int_0^{\ln\frac{1}{\nu}} \frac{m_f^M\left(x_0;t\right)}{t}dt\right), \quad 0 < \nu \leq 1.$$

The proof follows from the previous theorem.

Remark. Note that if $f \in L_{loc}\left(R_+\right)$, then almost everywhere in R_+ there exists $d_f\left(x\right)$ and the equality $d_f\left(x\right) = f\left(x\right)$ is almost everywhere fulfilled.

Let $\varphi\left(r\right)$ be a non-negative, monotonically increasing on $(0,+\infty)$ function. Denote by $MO_\varphi^M\left(x_0\right)$, a class of all the functions $f \in L_{loc}\left(R_+\right)$ such that

$$m_f^M\left(x_0;r\right) = O\left(\varphi\left(r\right)\right), \quad r > 0.$$

Theorem 1.3. *Let $\varphi\left(r\right)$ be a non-negative, monotonically increasing on $(0,+\infty)$ function. If*

$$\int_0^1 \frac{\varphi\left(t\right)dt}{t} = +\infty,$$

then there exists a function $f \in MO_{\varphi}^{M}(1)$ *such that*

$$\lim_{\nu \to 1-0} f_{I(1;\nu)} = +\infty.$$

Proof. Consider the function

$$f(\rho) = \int_{|\ln \rho|}^{1} \frac{\varphi(u)}{u} du, \quad \rho \in R_{+}.$$

Then for $\nu \in (0,1)$ we have

$$f_{I(1;\nu)} = \frac{1}{2r} \int_{-r}^{r} f^{*}(t)\, dt = \frac{1}{2r} \int_{-r}^{r} \left(\int_{|t|}^{1} \frac{\varphi(u)}{u} du \right) dt,$$

where $r = \ln \frac{1}{\nu}$, $f^{*}(t) = f\left(e^{-t}\right)$.

Further, we get that if $0 < r \leq 1$, then

$$f_{I(1;\nu)} = \frac{1}{r} \int_{0}^{r} \left(\int_{t}^{1} \frac{\varphi(u)}{u} du \right) dt = \frac{1}{r} \int_{0}^{r} \varphi(x)\, dx + \int_{r}^{1} \frac{\varphi(x)}{x} dx,$$

but if $r > 1$, then

$$f_{I(1;\nu)} = \frac{1}{r} \int_{0}^{1} \left(\int_{t}^{1} \frac{\varphi(x)}{x} dx \right) dt - \frac{1}{r} \int_{1}^{r} \left(\int_{1}^{t} \frac{\varphi(x)}{x} dx \right) dt = \frac{1}{r} \int_{0}^{r} \varphi(x)\, dx + \int_{r}^{1} \frac{\varphi(x)}{x} dx.$$

So, for any $r \in (0, +\infty)$

$$f_{I(1;\nu)} = \frac{1}{r} \int_{0}^{r} \varphi(x)\, dx + \int_{r}^{1} \frac{\varphi(x)}{x} dx. \tag{1.8}$$

Taking into account this equality, we have

$$\Omega^{M}(f, I(1;\nu)) = \frac{1}{2r} \int_{-r}^{r} \left| f^{*}(t) - f^{*}_{B(0,r)} \right| dt = \frac{1}{2r} \int_{-r}^{r} \left| f^{*}(t) - f_{I(1,\nu)} \right| dt =$$

$$= \frac{1}{2r} \int_{-r}^{r} \left| \int_{|t|}^{1} \frac{\varphi(x)}{x} dx - \frac{1}{r} \int_{0}^{r} \varphi(x)\, dx - \int_{r}^{1} \frac{\varphi(x)}{x} dx \right| dt =$$

$$= \frac{1}{r} \int_{0}^{r} \left| \int_{t}^{r} \frac{\varphi(x)}{x} dx - \frac{1}{r} \int_{0}^{r} \varphi(x)\, dx \right| dt \leq \frac{1}{r} \int_{0}^{r} \left(\int_{t}^{r} \frac{\varphi(x)}{x} dx \right) dt +$$

$$+ \frac{1}{r} \int_{0}^{r} \varphi(x)\, dx = \frac{1}{r} \int_{0}^{r} \varphi(x)\, dx + \frac{1}{r} \int_{0}^{r} \varphi(x)\, dx \leq 2\varphi(r) = 2\varphi(|\ln \nu|).$$

Hence we get that $m_f^M(1;\delta) \le 2\varphi(\delta)$ $(\delta > 0)$, i.e. $f \in MO_\varphi^M(1)$.

By equality (1.8), the following inequality is true

$$f_{I(1;\nu)} \ge \int_r^1 \frac{\varphi(x)}{x} dx.$$

Hence, from the condition of the theorem it follows that

$$\lim_{\nu \to 1-0} f_{I(1;\nu)} = +\infty.$$

The theorem is proved.

2. Mellin singular integral

Let $K \in L\left(R_+; \frac{dx}{x}\right)$, and

$$\int_0^\infty K(x) \frac{dx}{x} = 1.$$

If we denote $K_\varepsilon(x) = \frac{1}{\varepsilon} K\left(x^{\frac{1}{\varepsilon}}\right)$, $\varepsilon > 0$, we have

$$\int_0^\infty K_\varepsilon(x) \frac{dx}{x} = \frac{1}{\varepsilon} \int_0^\infty K\left(x^{\frac{1}{\varepsilon}}\right) \frac{dx}{x} = \int_0^\infty K(u) \frac{du}{u} = 1.$$

The function of the form $K_\varepsilon(x)$ is called a Mellin kernel of Fejer type.

Consider a Mellin singular integral with a kernel of Fejer type (see [6]),

$$\Phi_\varepsilon(f;x) = \frac{1}{\varepsilon} \int_0^\infty f\left(\frac{x}{t}\right) K\left(t^{\frac{1}{\varepsilon}}\right) \frac{dt}{t},$$

where $f \in L_{loc}(R_+)$ is such that the integral exists everywhere in R_+. By means of change of variable we can show that

$$\Phi_\varepsilon(f;x) = \frac{1}{\varepsilon} \int_0^\infty f(u) K\left(\left(\frac{x}{u}\right)^{\frac{1}{\varepsilon}}\right) \frac{du}{u} = \int_0^\infty f\left(\frac{x}{u^\varepsilon}\right) K(u) \frac{du}{u}.$$

In particular, if $K(u) = \frac{1}{2} X_{I\left(1;\frac{1}{e}\right)}(u)$, where X_E is a characteristic function of the set E, then we can show that for this kernel

$$\Phi_\varepsilon(f;x) = f_{I(x;e^{-\varepsilon})}, \quad \textit{where} \quad x \in R_+, \ \ \varepsilon > 0.$$

Theorem 2.1. *Let $f \in L_{loc}(R_+)$, K be a kernel of Fejer type, $k(\tau) := \sup\{|K(t)| : |\ln t| \ge \tau\}$, $\tau > 0$, $k \in L(R_+)$, $x_0 \in R_+$, $\varepsilon > 0$. Then the following inequality is true:*

$$\left|\Phi_\varepsilon(f;x_0) - f_{I(x_0;e^{-\varepsilon})}\right| \le c(k) \left(m_f^M(x_0;\varepsilon) + \int_0^\infty k(t)\, m_f^M(x_0;4\varepsilon t)\, dt + \right.$$

$$+\int_0^{\varepsilon}\frac{m_f^M(x_0;t)}{t}\left(\int_0^{\frac{t}{4\varepsilon}}k(x)\,dx\right)dt+\int_{\varepsilon}^{\infty}\frac{m_f^M(x_0;t)}{t}\left(\int_{\frac{t}{4\varepsilon}}^{\infty}k(x)\,dx\right)dt\Bigg), \tag{2.1}$$

where $c(k)$ is a positive constant dependent only on the function k.

Proof. Let $x_0=e^{-y_0}\,(y_0\in R)$, $\Phi_\varepsilon^*(f;y_0)=\Phi_\varepsilon(f;e^{-y_0})=\Phi_\varepsilon(f;x_0)$. Then we have

$$\Phi_\varepsilon(f;x_0)=\frac{1}{\varepsilon}\int_{-\infty}^{\infty}K^*\left(\frac{y_0-t}{\varepsilon}\right)f^*(t)\,dt,\quad \varepsilon>0,$$

where $K^*(u):=K(e^{-u})$, $f^*(t)=f\left(e^{-t}\right)$. Furthermore, it is easy to see that

$$f_{I(x_0;e^{-\varepsilon})}=\frac{1}{2\varepsilon}\int_{y_0-\varepsilon}^{y_0+\varepsilon}f^*(t)\,dt=:f^*_{B(y_0;\varepsilon)}.$$

Taking into account the previous reasonings, we get

$$\left|\Phi_\varepsilon(f;x_0)-f_{I(x_0;e^{-\varepsilon})}\right|\le\frac{1}{\varepsilon}\int_{-\infty}^{\infty}\left|K^*\left(\frac{y_0-t}{\varepsilon}\right)\right|\left|f^*(t)-f^*_{B(y_0;\varepsilon)}\right|dt. \tag{2.2}$$

As $k(\tau)=\sup\{|K^*(t)|:|y|\ge\tau\}$, $\tau\in(0,+\infty)$, from (2.2) we have

$$\left|\Phi_\varepsilon(f;x_0)-f_{I(x_0;e^{-\varepsilon})}\right|\le\frac{1}{\varepsilon}\int_{-\infty}^{\infty}k\left(\left|\frac{y_0-t}{\varepsilon}\right|\right)\left|f^*(t)-f^*_{B(y_0;\varepsilon)}\right|dt,$$

where $x_0=e^{-y_0}$ $(x_0\in R_+,y_0\in R)$.

It is seen from the definition that $k(\tau)$ is a monotonically decreasing on $(0,+\infty)$ function.

Further we get

$$\left|\Phi_\varepsilon(f;x_0)-f_{I(x_0;e^{-\varepsilon})}\right|\le\sum_{n=-\infty}^{\infty}\frac{1}{\varepsilon}\int_{B(y_0;2^{n+1}\varepsilon)\setminus B(y_0;2^n\varepsilon)}k\left(\frac{|y_0-t|}{\varepsilon}\right)\left|f^*(t)-f^*_{B(y_0;\varepsilon)}\right|dt\le$$

$$\le\sum_{n=-\infty}^{\infty}\frac{1}{\varepsilon}\int_{2^n\varepsilon<|y_0-t|\le2^{n+1}\varepsilon}k\left(\frac{|y_0-t|}{\varepsilon}\right)\left|f^*(t)-f^*_{B(y_0;2^{n+1}\varepsilon)}\right|dt+$$

$$+\sum_{n=-\infty}^{\infty}\left|f^*_{B(y_0;2^{n+1}\varepsilon)}-f^*_{B(y_0;\varepsilon)}\right|\int_{2^n\varepsilon<|y_0-t|\le2^{n+1}\varepsilon}\frac{1}{\varepsilon}k\left(\frac{|y_0-t|}{\varepsilon}\right)dt=$$

$$=\sum_{n=-\infty}^{\infty}i_{1n}+\sum_{n=-\infty}^{\infty}i_{2n}. \tag{2.3}$$

We'll estimate each of the summands in the right side of relation (2.3).

If $n = 0, \pm 1, ...$, we have

$$i_{1n} \leq \frac{1}{\varepsilon} k\left(2^n\right) \int\limits_{2^n\varepsilon < |x-t| \leq 2^{n+1}\varepsilon} \left| f^*(t) - f^*_{B(y_0;2^{n+1}\varepsilon)} \right| dt \leq$$

$$\leq 2^{n+2} \cdot k\left(2^n\right) \cdot \frac{1}{2 \cdot 2^{n+1}\varepsilon} \int\limits_{B(y_0;2^{n+1}\varepsilon)} \left| f^*(t) - f^*_{B(y_0;2^{n+1}\varepsilon)} \right| dt =$$

$$= 2^{n+2} \cdot k\left(2^n\right) \cdot \Omega^M \left(f, I\left(x_0; e^{-2^{n+1}\varepsilon}\right)\right) \leq 2^{n+2} \cdot k\left(2^n\right) \cdot m_f^M \left(x_0; 2^{n+1}\varepsilon\right). \quad (2.4)$$

Now, consider the summands i_{2n}. If $n > -1$, we have

$$i_{2n} \leq \frac{1}{\varepsilon} k\left(2^n\right) 2^{n+1}\varepsilon \left| f^*_{B(y_0;2^{n+1}\varepsilon)} - f^*_{B(y_0;\varepsilon)} \right| = 2^{n+1} k\left(2^n\right) \left| f_{I\left(x_0;e^{-2^{n+1}\varepsilon}\right)} - f_{I(x_0;e^{-\varepsilon})} \right|.$$

Hence, by theorem 1.1, we get

$$i_{2n} \leq 2^{n+1} \cdot k\left(2^n\right) \cdot \frac{2}{\ln 2} \left(m_f^M \left(x_0; 2^{n+1}\varepsilon\right) + \int\limits_{\varepsilon}^{2^{n+1}\varepsilon} \frac{m_f^M (x_0; t)}{t} dt \right). \quad (2.5)$$

If $n = -1$, then $i_{2n} = 0$.

Finally, consider the case $n < -1$. Then we have

$$i_{2n} \leq \frac{1}{\varepsilon} k\left(2^n\right) 2^{n+1}\varepsilon \left| f^*_{B(y_0;\varepsilon)} - f^*_{B(y_0;2^{n+1}\varepsilon)} \right| = 2^{n+1} k\left(2^n\right) \left| f_{I(x_0;e^{-\varepsilon})} - f_{I\left(x_0;e^{-2^{n+1}\varepsilon}\right)} \right| \leq$$

$$\leq 2^{n+1} k\left(2^n\right) \cdot \frac{2}{\ln 2} \left(m_f^M (x_0; \varepsilon) + \int\limits_{2^{n+1}\varepsilon}^{\varepsilon} \frac{m_f^M (x_0; t)}{t} dt \right). \quad (2.6)$$

From relations (2.3)-(2.6) we have

$$\left| \Phi_\varepsilon (f; x_0) - f_{I(x_0;e^{-\varepsilon})} \right| \leq \sum_{n=-\infty}^{\infty} 2^{n+2} \cdot k\left(2^n\right) \cdot m_f^M \left(x_0; 2^{n+1}\varepsilon\right) +$$

$$+ \frac{1}{\ln 2} \sum_{n=-\infty}^{-2} 2^{n+2} \cdot k\left(2^n\right) \cdot m_f^M (x_0; \varepsilon) + \frac{1}{\ln 2} \sum_{n=-\infty}^{-2} 2^{n+2} \cdot k\left(2^n\right) \cdot \int\limits_{2^{n+1}\varepsilon}^{\varepsilon} \frac{m_f^M (x_0; t)}{t} dt +$$

$$+ \frac{1}{\ln 2} \sum_{n=0}^{\infty} 2^{n+2} \cdot k\left(2^n\right) \cdot m_f^M \left(x_0; 2^{n+1}\varepsilon\right) + \frac{1}{\ln 2} \sum_{n=0}^{\infty} 2^{n+2} \cdot k\left(2^n\right) \cdot \int\limits_{\varepsilon}^{2^{n+1}\varepsilon} \frac{m_f^M (x_0; t)}{t} dt. \quad (2.7)$$

We can show that if $\varphi(x)$ is a non-negative function monotonically increasing on the interval $(0, +\infty)$, the following inequalities are true

$$\sum_{n=-\infty}^{\infty} k\left(2^n\right) \cdot \varphi\left(2^{n+1}\varepsilon\right) \cdot 2^n \leq 2 \cdot \int\limits_0^\infty k(x)\, \varphi(4\varepsilon x)\, dx, \quad \varepsilon > 0; \quad (2.8)$$

$$\sum_{n=-\infty}^{-2} 2^n \cdot k\left(2^n\right) \leq 2 \cdot \int_0^{\frac{1}{4}} k\left(x\right) dx; \tag{2.9}$$

$$\sum_{n=0}^{\infty} k\left(2^n\right) \cdot \varphi\left(2^{n+1}\varepsilon\right) \cdot 2^n \leq 2 \cdot \int_{\frac{1}{2}}^{\infty} k\left(x\right) \varphi\left(4\varepsilon x\right) dx, \quad \varepsilon > 0; \tag{2.10}$$

Furthermore, if $\psi\left(x\right)$ is a non-negative function monotonically decreasing on the interval $(0, +\infty)$, the following inequality is true:

$$\sum_{n=-\infty}^{-2} k\left(2^n\right) \cdot \psi\left(2^{n+1}\varepsilon\right) \cdot 2^n \leq 2 \cdot \int_0^{\frac{1}{4}} k\left(x\right) \psi\left(2\varepsilon x\right) dx, \quad \varepsilon > 0. \tag{2.11}$$

By means of inequalities (2.8)-(2.11), from inequality (2.7) we get

$$\left|\Phi_\varepsilon\left(f; x_0\right) - f_{I\left(x_0; e^{-\varepsilon}\right)}\right| \leq c\left(k\right) \left(m_f^M\left(x_0; \varepsilon\right) + \int_0^{\infty} k\left(x\right) m_f^M\left(x_0; 4\varepsilon x\right) dx + \right.$$

$$\left. + \int_0^{\frac{1}{4}} k\left(x\right) \left(\int_{2\varepsilon x}^{\varepsilon} \frac{m_f^M\left(x_0; t\right)}{t} dt \right) dx + \int_{\frac{1}{2}}^{\infty} k\left(x\right) \left(\int_{\varepsilon}^{4\varepsilon x} \frac{m_f^M\left(x_0; t\right)}{t} dt \right) dx \right), \tag{2.12}$$

where $c\left(k\right) := \frac{8}{\ln 2} \cdot \max\left\{2, \int\limits_0^{\frac{1}{4}} k\left(x\right) dx\right\}$.

Changing the integration order in the integrals of inequality (2.12), after some elementary transformations we get inequality (2.1). The theorem is proved.

From theorems 1.2 and 2.1 we get .

Theorem 2.2. *Let $f \in L_{loc}\left(R_+\right)$ K, be a kernel of Fejer type, $k\left(\tau\right) := \sup\left\{\left|K\left(t\right)\right| : \left|\ln t\right| \geq \tau\right\}$, $\tau > 0$, $k \in L\left(R_+\right)$, $x_0 \in R_+$, $\varepsilon > 0$. Then under convergence of the integrals in the right side, the following inequality is true*

$$\left|\Phi_\varepsilon\left(f; x_0\right) - d_f\left(x_0\right)\right| \leq c\Big(m_f^M\left(x_0; \varepsilon\right) + \int_0^{\varepsilon} \frac{m_f^M\left(x_0; t\right)}{t} dt +$$

$$+ \int_0^{\infty} k\left(x\right) m_f^M\left(x_0; 4\varepsilon x\right) dx + \int_{\varepsilon}^{\infty} \frac{m_f^M\left(x_0; t\right)}{t} \Big(\int_{\frac{t}{4\varepsilon}}^{\infty} k(x) dx \Big) dt \Big),$$

where $c > 0$ is a constant dependent only on the function k.

The following theorem is also valid.

Theorem 2.3. *Let φ be a non-negative function monotonically increasing on $(0, +\infty)$, K and k be the same as in the previous theorem, and the following conditions be fulfilled:*

1) $\int\limits_0^{\varepsilon} \frac{\varphi(t)}{t} dt = O\left(\varphi\left(\varepsilon\right)\right)$, $\varepsilon > 0$;

2) $\int\limits_0^\infty k(x)\,\varphi(4\varepsilon x)\,dx = O(\varphi(\varepsilon))$, $\varepsilon > 0$;

3) $\int\limits_\varepsilon^\infty \frac{\varphi(t)}{t}\left(\int\limits_{\frac{t}{4\varepsilon}}^\infty k(x)\,dx\right)dt = O(\varphi(\varepsilon))$, $\varepsilon > 0$.

Then if $x_0 \in R_+$ and $f \in MO_\varphi^M(x_0)$, the following inequality is true

$$|\Phi_\varepsilon(f;x_0) - d_f(x_0)| \le c \cdot \|f\| \cdot \varphi(\varepsilon), \quad \varepsilon > 0,$$

where $\|f\| := \sup\left\{\frac{m_f^M(x_0;t)}{\varphi(t)} : t > 0\right\}$, and $c > 0$ is a constant dependent only on the function k.

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