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THE EXISTENCE OF AN ABSORBING SET FOR ONE MIXED PROBLEM WITH MEMORY OPERATOR

Abstract

In this work we consider the mixed problem for one semilinear hyperbolic equation with memory operator. We prove the existence of a bounded absorbing set for this problem.

Let $\Omega \subset R^N$ $(N \geq 1)$ be a bounded, connected set with a smooth boundary Γ. We consider the following problem:

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial}{\partial t}\left[u + F(u)\right] - \Delta u + |u|^p u = f \ in \ Q = \Omega \times (0,T), \tag{1}$$

$$u = 0 \ on \ \Gamma \times [0,T], \tag{2}$$

$$\left[u + F(u)\right]\big|_{t=0} = u^{(0)} + w^{(0)}, \ \left.\frac{\partial u}{\partial t}\right|_{t=0} = u^{(1)} \ in \ \Omega, \tag{3}$$

where $p > 0$ and F is a memory operator (at any instant t, $F(u)$ may depend not only on $u(t)$ but also on the previous evolution of u) which acts from $M\left(\Omega; C^0([0,T])\right)$ to $M\left(\Omega; C^0([0,T])\right)$. Here $M\left(\Omega; C^0([0,T])\right)$ is a space of strongly measurable functions $\Omega \to C^0([0,T])$. We assume that the operator F is applied at each point $x \in \Omega$ independently: the output $[F(u)](x,t)$ depends on $u(x,\cdot)|_{[0,t]}$, but not on $u(y,\cdot)|_{[0,t]}$ for any $y \neq x$.

We assume that

$$\begin{cases} \forall v_1, v_2 \in M\left(\Omega; C^0([0,T])\right), \forall t \in [0,T], \quad if \quad v_1 = v_2 \ \ in \ \ [0,t], \\ a.e. \ \ in \ \ \Omega, \quad then \ \ [F(v_1)](\cdot,t) = [F(v_2)](\cdot,t) \quad a.e. \ in \ \ \Omega, \end{cases} \tag{4}$$

$$\begin{cases} \forall \left\{v_n \in M\left(\Omega; C^0([0,T])\right)\right\}_{n\in N}, \quad if \ \ v_n \to v \ \ uniformly \ in \ \ [0,T], \\ a.e.in \quad then \ \ F(v_n) \to F(v) \ \ uniformly \ \ in \ \ [0,T], \ \ a.e. \ in \ \ \Omega, \end{cases} \tag{5}$$

$$\begin{cases} \exists L \in R^+, \ \exists g \in L^2(\Omega) : \forall v \in M\left(\Omega; C^0([0,T])\right), \\ \|[F(v)](x,\cdot)\|_{C^0([0,T])} \leq L\,\|v(x,\cdot)\|_{C^0([0,T])} + g(x) \quad a.e. \ \ in \ \ \Omega, \end{cases} \tag{6}$$

$$\begin{cases} \forall v \in M\left(\Omega; C^0([0,T])\right), \quad \forall [t_1,t_2] \subset [0,T], \\ if \ \ v(x,\cdot) \ \ is \ \ affine \quad in \quad [t_1,t_2] \quad a.e. \ \ in \ \ \Omega, \ \ then \\ \left\{[F(v)](x,t_2) - [F(v)](x,t_1)\right\} \cdot [v(x,t_2) - v(x,t_1)] \geq 0 \quad a.e. \ in \ \ \Omega, \end{cases} \tag{7}$$

$$\begin{cases} \exists L_1 \in R^+, \ \forall v \in M\left(\Omega; C^0([0,T])\right), \ \forall [t_1,t_2] \subset [0,T], \\ if \ \ v(x,\cdot) \ \ is \ \ affine \quad in \quad [t_1,t_2] \quad a.e. \ \ in \ \ \Omega, \ \ then \\ |[F(v)](x,t_2) - [F(v)](x,t_1)| \leq L_1 |v(x,t_2) - v(x,t_1)| \quad a.e. \ \ in \ \ \Omega, \end{cases} \tag{8}$$

$$\left\{\begin{array}{l} \exists 0<L_2<1:\forall v_1,v_2\in M\left(\Omega;W^{1,1}\left([0,T]\right)\right), \\ \left|\frac{\partial}{\partial t}\left[F\left(v_2\right)-F\left(v_1\right)\right]\right|\leq L_2\left|\frac{\partial}{\partial t}\left(v_2-v_1\right)\right|. \end{array}\right. \tag{9}$$

Let $V=H_0^1(\Omega)\times L^2(\Omega)\times L^2(\Omega)$ and

$$u^{(0)}\in H_0^1(\Omega),\quad w^{(0)}\in L^2(\Omega),\quad u^{(1)}\in L^2(\Omega), \tag{10}$$

$$h\in L^2(\Omega). \tag{11}$$

Definition. *A function $u\in L^2\left(0,T;H_0^1(\Omega)\right)\cap H^1\left(0,T;L^2(\Omega)\right)$ is said to be a solution of problem (1)-(3) if $F(u)\in L^2(Q)$ and*

$$\iint\limits_Q\left\{-\frac{\partial u}{\partial t}\frac{\partial v}{\partial t}-\left[u+F(u)\right]\frac{\partial v}{\partial t}+\nabla u\cdot\nabla v+|u|^p uv\right\}dxdt=$$

$$=\iint\limits_Q hvdxdt+\int\limits_\Omega\left[u^{(0)}(x)+w^{(0)}(x)+u^{(1)}(x)\right]v(x,0)\,dx$$

for every $v\in L^2\left(0,T;H_0^1(\Omega)\right)\cap H^1\left(0,T;L^2(\Omega)\right)$ $(v(\cdot,T)=0$ a.e. in $\Omega)$.

Well posedness of problem (1)-(3) without F was studied in the works of different authors (see, for example [2]). The corresponding problem for the parabolic equation without nonlinear term $|u|^p u$ was studied in [1].

The problem (1)-(3) has a unique solution under the conditions (4)-(11) (see [5]). We have proved the existence of a bounded absorbing set for this problem.

The problem (1)-(3) generates a semigroup $\{S(t)\}_{t\geq 0}$ in V by the formula

$$S(t)\left(u^{(0)},w^{(0)},u^{(1)}\right)=\left(u,F(u),u_t\right),$$

where u is a unique solution of this problem. We introduce the following functional

$$\Phi_\eta(y)=\frac{1}{2}\|q\|^2+\frac{1}{2}\|u_x\|^2-(h,u)+\frac{1}{p+2}\left(|u|^{p+2},1\right)+\eta\left[(u,q)+\frac{1}{2}\|u\|^2\right],$$

where $y=(u,F(u),q)$, η - any positive constant. We denote by $\|\cdot\|$ and $(\cdot,\cdot)$ the norm and scalar product in $L^2(\Omega)$.

We fix any $m\in N$, set $k=\frac{T}{m}$, $u_m^0=u^{(0)}$, $w_m^0=w^{(0)}$, $u_m^1=u^{(0)}+ku^{(1)}$, $u_m^{-1}=u^{(0)}-ku^{(1)}$, $u_m^n(x)=u_m(x,nk)$, $n=2,...,m$,

$$\Phi_{\eta m}^n=\Phi_\eta\left(u_m^n,w_m^n,\frac{u_m^n-u_m^{n-1}}{k}\right),\quad n=1,...,m\quad a.e.\ in\ \Omega,$$

$$w_m^n(x)=\left[F\left(u_m\right)\right](x,nk),\quad n=1,...,m\quad a.e.\ in\ \Omega,$$

$u_m(x,\cdot)$ =linear time interpolate of $u_m(x,nk)$, for $n=0,...,m$, a.e. in Ω and define $w_m(x,\cdot)$ similarly.

We consider the following problem

$$\frac{u_m^n-2u_m^{n-1}+u_m^{n-2}}{k^2}+\frac{u_m^n-u_m^{n-1}}{k}+$$

$$+\frac{w_m^n - w_m^{n-1}}{k} - \Delta u_m^n + |u_m^n|^p u_m^n = h \ \ in \ \ V', \tag{12}$$

$$u_m^0 = u^{(0)}, \ w_m^0 = w^{(0)}, \ \ u_m^1 = u^{(0)} + ku^{(1)}, \ u_m^{-1} = u^{(0)} - ku^{(1)} \tag{13}$$

and functional

$$\Phi_{\eta m}^n = \frac{1}{2}\left\|\frac{u_m^n - u_m^{n-1}}{k}\right\|^2 + \frac{1}{2}\|\nabla u_m^n\|^2 - (h, u_m^n) + \frac{1}{p+2}\left(|u_m^n|^{p+2}, 1\right) +$$

$$+\eta\left[\left(u_m^n, \frac{u_m^n - u_m^{n-1}}{k}\right) + \frac{1}{2}\|u_m^n\|^2\right], \tag{14}$$

$n = 1, 2, ..., m.$

Lemma. *Assume that (4)-(11) hold and let u be a solution of problem (1)-(3). Then there exists some $\delta > 0$ and a natural number m_1 such that for arbitrary $m > m_1$ holds the following inequality*

$$\frac{\Phi_{\eta m}^n - \Phi_{\eta m}^{n-1}}{k} + \delta\Phi_{\eta m}^n \le C, \quad n = 1, 2, ..., m, \tag{15}$$

where C is a positive constant independent of m.

Proof. By (12), (14) we have

$$\frac{\Phi_{\eta m}^n - \Phi_{\eta m}^{n-1}}{k} = \frac{1}{2k}\left(\frac{u_m^n - u_m^{n-1}}{k} - \frac{u_m^{n-1} - u_m^{n-2}}{k}, \frac{u_m^n - u_m^{n-1}}{k} + \frac{u_m^{n-1} - u_m^{n-2}}{k}\right) +$$

$$+\frac{1}{2k}\left(\nabla u_m^n - \nabla u_m^{n-1}, \nabla u_m^n + \nabla u_m^{n-1}\right) - \frac{1}{k}\left(h, u_m^n - u_m^{n-1}\right) +$$

$$+\frac{1}{(p+2)\,k}\left(|u_m^n|^{p+2} - \left|u_m^{n-1}\right|^{p+2}, 1\right) +$$

$$+\frac{\eta}{k}\left[\left(u_m^n, \frac{u_m^n - u_m^{n-1}}{k}\right) - \left(u_m^{n-1}, \frac{u_m^{n-1} - u_m^{n-2}}{k}\right) + \frac{1}{2}\left(u_m^n - u_m^{n-1}, u_m^n + u_m^{n-1}\right)\right] =$$

$$= \frac{1}{2}\left(\frac{u_m^n - 2u_m^{n-1} + u_m^{n-2}}{k^2}, 2\cdot\frac{u_m^n - u_m^{n-1}}{k} - \frac{u_m^n - u_m^{n-1}}{k} + \frac{u_m^{n-1} - u_m^{n-2}}{k}\right) +$$

$$+\frac{1}{2}\left(\frac{\nabla u_m^n - \nabla u_m^{n-1}}{k}, 2\nabla u_m^n - u_m^n + u_m^{n-1}\right) -$$

$$-\left(h, \frac{u_m^n - u_m^{n-1}}{k}\right) + \frac{1}{p+2}\left(\frac{|u_m^n|^{p+2} - \left|u_m^{n-1}\right|^{p+2}}{k}, 1\right) +$$

$$+\eta\left[\left(u_m^n, \frac{u_m^n - 2u_m^{n-1} + u_m^{n-2}}{k^2}\right) + \frac{1}{k}\left(u_m^n, \frac{u_m^{n-1} - u_m^{n-2}}{k}\right) -\right.$$

$$\left.-\frac{1}{k}\left(u_m^{n-1}, \frac{u_m^{n-1} - u_m^{n-2}}{k}\right) + \frac{1}{2}\left(\frac{u_m^n - u_m^{n-1}}{k}, u_m^n + u_m^{n-1}\right)\right] =$$

$$= \left(\frac{u_m^n - 2u_m^{n-1} + u_m^{n-2}}{k^2}, \frac{u_m^n - u_m^{n-1}}{k}\right) -$$

$$-\frac{1}{2}\left(\frac{u_m^n-2u_m^{n-1}+u_m^{n-2}}{k^2},\frac{u_m^n-2u_m^{n-1}+u_m^{n-2}}{k}\right)+\left(\nabla\left(\frac{u_m^n-u_m^{n-1}}{k}\right),\nabla u_m^n\right)-$$
$$-\frac{1}{2}\left(\nabla\left(\frac{u_m^n-u_m^{n-1}}{k}\right),\nabla\left(u_m^n-u_m^{n-1}\right)\right)-\left(h,\frac{u_m^n-u_m^{n-1}}{k}\right)+$$
$$+\frac{1}{p+2}\left(\frac{|u_m^n|^{p+2}-\left|u_m^{n-1}\right|^{p+2}}{k},1\right)+\eta\left(u_m^n,\frac{u_m^n-2u_m^{n-1}+u_m^{n-2}}{k^2}\right)+$$
$$+\eta\left(\frac{u_m^n-u_m^{n-1}}{k},\frac{u_m^{n-1}-u_m^{n-2}}{k}\right)+\frac{\eta}{2}\left(\frac{u_m^n-u_m^{n-1}}{k},2u_m^n-u_m^n+u_m^{n-1}\right)=$$
$$=\left(\frac{u_m^n-2u_m^{n-1}+u_m^{n-2}}{k^2},\frac{u_m^n-u_m^{n-1}}{k}\right)+$$
$$+\left(\nabla\left(\frac{u_m^n-u_m^{n-1}}{k}\right),\nabla u_m^n\right)-\left(h,\frac{u_m^n-u_m^{n-1}}{k}\right)+$$
$$+\left(|u_m^n|^p\,u_m^n,\frac{u_m^n-u_m^{n-1}}{k}\right)-\frac{1}{2k^3}\left\|u_m^n-2u_m^{n-1}+u_m^{n-2}\right\|^2-\frac{1}{2k}\left\|\nabla u_m^n-\nabla u_m^{n-1}\right\|^2+$$
$$+\eta\left(u_m^n,\frac{u_m^n-2u_m^{n-1}+u_m^{n-2}}{k^2}\right)+\eta\left(\frac{u_m^n-u_m^{n-1}}{k},u_m^n\right)+$$
$$+\eta\left(\frac{u_m^n-u_m^{n-1}}{k},\eta\frac{u_m^n-u_m^{n-1}}{k}-\frac{u_m^n-u_m^{n-1}}{k}+\frac{u_m^{n-1}-u_m^{n-2}}{k}\right)-\frac{\eta}{2k}\left\|u_m^n-u_m^{n-1}\right\|^2=$$
$$=-\left\|\frac{u_m^n-u_m^{n-1}}{k}\right\|^2-\left(\frac{w_m^n-w_m^{n-1}}{k},\frac{u_m^n-u_m^{n-1}}{k}\right)-\frac{1}{2k^3}\left\|u_m^n-2u_m^{n-1}+u_m^{n-2}\right\|^2-$$
$$-\frac{1}{2k}\left\|\nabla u_m^n-\nabla u_m^{n-1}\right\|^2-\eta\left(\frac{w_m^n-w_m^{n-1}}{k},u_m^n\right)-\eta\left\|\nabla u_m^n\right\|^2-\eta\left(|u_m^n|^{p+2},1\right)+$$
$$+\eta\left(h,u_m^n\right)+\eta\left\|\frac{u_m^n-u_m^{n-1}}{k}\right\|^2-\eta\left(\frac{u_m^n-u_m^{n-1}}{k},\frac{u_m^n-2u_m^{n-1}+u_m^{n-2}}{k}\right)-$$
$$-\frac{\eta}{2k}\left\|u_m^n-u_m^{n-1}\right\|^2. \tag{16}$$

Let

$$\delta<\eta. \tag{17}$$

Then by (7), (8) from (16) we obtain

$$\frac{\Phi_{\eta m}^n-\Phi_{\eta m}^{n-1}}{k}+\delta\Phi_{nm}^n=-\left\|\frac{u_m^n-u_m^{n-1}}{k}\right\|^2-\left(\frac{w_m^n-w_m^{n-1}}{k},\frac{u_m^n-u_m^{n-1}}{k}\right)-$$
$$-\frac{1}{2k^3}\left\|u_m^n-2u_m^{n-1}+u_m^{n-2}\right\|^2-\frac{1}{2k}\left\|\nabla u_m^n-\nabla u_m^{n-1}\right\|^2-\eta\left(\frac{w_m^n-w_m^{n-1}}{k},u_m^n\right)-$$
$$-\eta\left\|\nabla u_m^n\right\|^2-\eta\left(|u_m^n|^{p+2},1\right)+\eta\left(h,u_m^n\right)+\eta\left\|\frac{u_m^n-u_m^{n-1}}{k}\right\|^2-$$
$$-\eta\left(\frac{u_m^n-u_m^{n-1}}{k},\frac{u_m^n-2u_m^{n-1}+u_m^{n-2}}{k}\right)-\frac{\eta}{2k}\left\|u_m^n-u_m^{n-1}\right\|^2+$$

$$+\frac{\delta}{2}\left\|\frac{u_m^n-u_m^{n-1}}{k}\right\|^2+\frac{\delta}{2}\left\|\nabla u_m^n\right\|^2-\delta\left(h,u_m^n\right)+\frac{\delta}{p+2}\left(\left|u_m^n\right|^{p+2},1\right)+$$

$$+\delta\eta\left(u_m^n,\frac{u_m^n-u_m^{n-1}}{k}\right)+\frac{\delta\eta}{2}\left\|u_m^n\right\|^2\leq\left(-1+\eta+\frac{\delta}{2}\right)\left\|\frac{u_m^n-u_m^{n-1}}{k}\right\|^2+$$

$$+\left(-\frac{1}{2k^3}+\frac{\eta}{2k^2}\right)\left\|u_m^n-2u_m^{n-1}+u_m^{n-2}\right\|^2+\frac{\eta L_1\nu_0}{2}\left\|\frac{u_m^n-u_m^{n-1}}{k}\right\|^2+$$

$$+\frac{\eta}{2\nu_0}\left\|u_m^n\right\|^2+\left(-\eta+\frac{\delta}{2}\right)\left\|\nabla u_m^n\right\|^2+\left(-\eta+\frac{\delta}{p+2}\right)\left(\left|u_m^n\right|^{p+2},1\right)+$$

$$+(\eta-\delta)\left(h,u_m^n\right)+\frac{\eta}{2}\left\|\frac{u_m^n-u_m^{n-1}}{k}\right\|^2+\frac{\delta\eta}{2}\left\|u_m^n\right\|^2+\frac{\delta\eta}{2}\left\|\frac{u_m^n-u_m^{n-1}}{k}\right\|^2+$$

$$+\frac{\delta\eta}{2}\left\|u_m^n\right\|^2\leq\left(-1+\eta+\frac{\delta}{2}+\frac{\eta L_1\nu_0}{2}+\frac{\eta}{2}+\frac{\delta\eta}{2}\right)\left\|\frac{u_m^n-u_m^{n-1}}{k}\right\|^2+$$

$$+\frac{\eta k-1}{2k^3}\left\|u_m^n-2u_m^{n-1}+u_m^{n-2}\right\|^2+$$

$$+\left(\frac{\eta c_\Omega^2}{2\nu_0}-\eta+\frac{\delta}{2}+\frac{\nu(\eta-\delta)c_\Omega^2}{2}+\delta\eta c_\Omega^2\right)\left\|\nabla u_m^n\right\|^2+$$

$$+\left(-\eta+\frac{\delta}{p+2}\right)\left(\left|u_m^n\right|^{p+2},1\right)+\frac{\eta-\delta}{2\nu}\|h\|^2. \tag{18}$$

We choose the numbers ν_0, η, δ, ν, k such that

$$\begin{cases} -1+\eta+\frac{\delta}{2}+\frac{\eta L_1\nu_0}{2}+\frac{\eta}{2}+\frac{\delta\eta}{2}\leq 0, \\ \eta k-1\leq 0, \\ \frac{\eta C_\Omega^2}{2\nu_0}-\eta+\frac{\delta}{2}+\frac{\nu(\eta-\delta)c_\Omega^2}{2}+\delta\eta c_\Omega^2\leq 0, \\ -\eta+\frac{\delta}{p+2}\leq 0. \end{cases}$$

Combining last inequalities with (17) yields:

$$\nu_0>\frac{c_\Omega^2}{2},$$

$$\eta<\frac{2}{3+L_1\nu_0},k<\frac{1}{\eta},$$

$$\delta<\min\left\{\eta,\frac{2-\eta\left(3+L_1\nu_0\right)}{\eta+1},\frac{\eta\left(2\nu_0-c_\Omega^2\right)}{\nu_0\left(1+2\eta c_\Omega^2\right)}\right\},$$

$$\nu<\frac{\eta\left(2\nu_0-c_\Omega^2\right)-\delta\nu_0\left(1+2\eta c_\Omega^2\right)}{(\eta-\delta)c_\Omega^2\nu_0}.$$

Thus from (18) we obtain that for any $m > m_1$ it holds the following inequality

$$\frac{\Phi^n_{\eta m} - \Phi^{n-1}_{\eta m}}{k} + \delta \Phi^n_{\eta m} \le \frac{\eta - \delta}{2\nu} \|h\|^2, \quad n = 1, 2, ..., m,$$

where $m_1 = \frac{T}{k_1}, k_1 = \frac{1}{\eta}$.

Let $\|h\|^2 \le \overline{m}$. Then

$$\frac{\Phi^n_{\eta m} - \Phi^{n-1}_{\eta m}}{k} + \delta \Phi^n_{\eta m} \le C, \quad n = 1, 2, ..., m,$$

where $C = \frac{\eta - \delta}{2\nu} \overline{m}$.

Lemma is proved

Theorem. *Under the conditions (4)-(11) there exists an absorbing set $B_0 \subset V$ for the problem (1)-(3).*

Note that a bounded set $B_0 \subset V$ is said to be absorbing, if for arbitrary bounded set $B \subset V$ there exists $t_1(B)$ such that $S(t) B \subset B_0$ for all $t \ge t_1(B)$.

Proof. Let

$$B_0 = \left[\left\{ y = (u, F(u), q) \in V : \Phi_\eta(y) \le \frac{2C}{\delta} \right\} \right],$$

where $[M]$ denotes a closure of set M.

1. We prove at first that a set B_0 is bounded.

$$\Phi_\eta(y) = \frac{1}{2}\|q\|^2 + \frac{1}{2}\|u_x\|^2 - (h, u) + \frac{1}{p+2}\left(|u|^{p+2}, 1\right) + \eta\left[(u, q) + \frac{1}{2}\|u\|^2\right] \ge$$

$$\ge \frac{1}{2}\|q\|^2 + \frac{1}{2}\|u_x\|^2 - \frac{1}{2\nu_1}\|h\|^2 - \frac{\nu_1}{2}\|u\|^2 + \frac{1}{p+2}\left(|u|^{p+2}, 1\right) - \frac{\eta}{2\nu_2}\|u\|^2 -$$

$$-\frac{\eta\nu_2}{2}\|q\|^2 + \frac{\eta}{2}\|u\|^2 = \frac{1}{2}(1 - \eta\nu_2)\|q\|^2 + \frac{1}{2}\|u_x\|^2 +$$

$$+\frac{1}{2}\left(-\nu_1 - \frac{\eta}{\nu_2} + \eta\right)\|u\|^2 - \frac{1}{2\nu_1}\|h\|^2 + \frac{1}{p+2}\left(|u|^{p+2}, 1\right). \tag{19}$$

We choose ν_1, ν_2 such that

$$1 - \eta\nu_2 > 0, \quad -\nu_1 - \frac{\eta}{\nu_2} + \eta > 0,$$

that is

$$1 < \nu_2 < \frac{1}{\eta}, \quad \nu_1 < \eta\left(1 - \frac{1}{\nu_2}\right).$$

Let $\nu_3 = \frac{1}{2}\min\left\{1 - \eta\nu_2; \frac{1}{2}\left(-\nu_1 - \frac{\eta}{\nu_2} + \eta\right)\right\}$. Then by (19) we have

$$\Phi_\eta(y) \ge \nu_3\left(\|q\|^2 + \|u_x\|^2 + 2\|u\|^2\right) - \frac{1}{2\nu_1}\|h\|^2 =$$

$$= \nu_3\left(\|q\|^2 + \|u_x\|^2 + \|u\|^2\right) + \nu_3\|u\|^2 - \frac{1}{2\nu_1}\|h\|^2. \tag{20}$$

By the condition (6) we have

$$\|F(v)\|^2 \leq 2L^2 \|v\|^2 + 2\|g\|^2$$

or

$$\|v\|^2 \geq \frac{1}{2L^2} \|F(v)\|^2 - \frac{1}{L^2} \|g\|^2 .$$

Using the last inequality in (20) we obtain

$$\Phi_\eta(y) \geq \nu_3 \left(\|q\|^2 + \|u_x\|^2 + \|u\|^2 \right) + \frac{\nu_3}{2L^2} \|F(u)\|^2 - \frac{\nu_3}{L^2} \|g\|^2 - \frac{1}{2\nu_1} \|h\|^2 \geq$$

$$\geq \nu_4 \left(\|q\|^2 + \|u_x\|^2 + \|u\|^2 + \|F(u)\|^2 \right) - \frac{\nu_3}{L^2} \|g\|^2 - \frac{1}{2\nu_1} \|h\|^2 \geq$$

$$\geq \nu_4 \|y\|_V^2 - \frac{\nu_3}{L^2} m_1 - \frac{1}{2\nu_1} m, \tag{21}$$

where $\nu_4 = \min\left\{\nu_3; \frac{\nu_3}{L^2}\right\}$ and $\|g\|^2 \leq m_1$.

At last from (21) we obtain

$$\|y\|_V^2 \leq \frac{1}{\nu_4} \Phi_\eta(y) + \frac{1}{\nu_4} \left(\frac{\nu_3 m_1}{L^2} + \frac{m}{2\nu_1} \right) \leq \frac{1}{\nu_4} \cdot \frac{2C}{\delta} + \frac{1}{\nu_4} \left(\frac{\nu_3 m_1}{L^2} + \frac{m}{2\nu_1} \right),$$

that is B_0 is bounded.

2. Now we prove that B_0 is absorbing. We put an arbitrary bounded set $B \subset V : B = \{y \in V : \|y\|_V \leq \chi\}$. Let $y^0 = \left(u^{(0)}, w^{(0)}, u^{(1)}\right) \in B$. We must find $t_1(B) = t_1(\chi)$ such that $y = S(t) y^0 \in B_0$ for any $t \geq t_1(\chi)$. Since y is a solution of problem (1)-(3) with initial data y^0, then it holds inequality (15). By multiplying (15) by $e^{\delta nk}$ we have

$$\frac{\Phi_{\eta m}^n - \Phi_{\eta m}^{n-1}}{k} e^{\delta nk} + \delta \Phi_{\eta m}^n e^{\delta nk} \leq C e^{\delta nk}$$

or

$$\frac{\Phi_{\eta m}^n e^{\delta nk} - \Phi_{\eta m}^{n-1} e^{\delta(n-1)k}}{k} + \frac{\Phi_{\eta m}^{n-1} e^{\delta(n-1)k}}{k} - \frac{\Phi_{\eta m}^n e^{\delta nk}}{k} + \delta \Phi_{\eta m}^n e^{\delta nk} \leq C e^{\delta nk}$$

or

$$\frac{\Phi_{\eta m}^n e^{\delta nk} - \Phi_{\eta m}^{n-1} e^{\delta(n-1)k}}{k} - \Phi_{\eta m}^{n-1} \delta \frac{e^{\delta nk} - e^{\delta(n-1)k}}{\delta k} +$$

$$+ \delta \Phi_{\eta m}^n e^{\delta nk} \leq C e^{\delta k} e^{\delta(n-1)k}. \tag{22}$$

It is evident that

$$e^{\delta(n-1)k} = \frac{e^{\delta nk} - e^{\delta(n-1)k}}{\delta k} + \alpha(k),$$

where $\alpha(k) \to 0 \, (k \to 0)$.

By the last relation from (22) we have

$$\frac{\Phi_{\eta m}^n e^{\delta nk} - \Phi_{\eta m}^{n-1} e^{\delta(n-1)k}}{k} + \delta \left(\Phi_{\eta m}^n e^{\delta nk} - \Phi_{\eta m}^{n-1} e^{\delta(n-1)k} \right) + \delta \alpha(k) \Phi_{\eta m}^{n-1} \leq$$

$$\leq Ce^{\delta k}\left(\frac{e^{\delta nk}-e^{\delta(n-1)k}}{\delta k}+\alpha(k)\right)$$

or

$$(1+\delta k)\frac{\Phi_{\eta m}^{n}e^{\delta nk}-\Phi_{\eta m}^{n-1}e^{\delta(n-1)k}}{k}+\delta\alpha(k)\Phi_{\eta m}^{n-1}\leq$$

$$\leq Ce^{\delta k}\frac{e^{\delta nk}-e^{\delta(n-1)k}}{\delta k}+Ce^{\delta k}\alpha(k),$$

whence we obtain, that

$$\Phi_{\eta m}^{n}e^{\delta nk}-\Phi_{\eta m}^{n-1}e^{\delta(n-1)k}\leq\frac{C}{\delta}\left(e^{\delta nk}-e^{\delta(n-1)k}\right)+\frac{C-\delta\Phi_{\eta m}^{n-1}}{1+\delta k}k\alpha(k).$$

We sum the last inequality for $n=1,...,l$, for any $l\in\{1,...,m\}$. Then we have

$$\Phi_{\eta m}^{l}e^{\delta lk}-\Phi_{\eta m}^{0}\leq\frac{C}{\delta}\left(e^{\delta lk}-1\right)+\frac{k\alpha(k)}{1+\delta k}\sum_{n=1}^{l}\left(C-\delta\Phi_{\eta m}^{n-1}\right),$$

whence

$$\Phi_{\eta m}^{l}\leq\frac{C}{\delta}+\left(\Phi_{\eta m}^{0}-\frac{C}{\delta}+\frac{k\alpha(k)}{1+\delta k}\sum_{n=1}^{l}\left(C-\delta\Phi_{\eta m}^{n-1}\right)\right)e^{-\delta lk}.$$

Since $\left\|y^{0}\right\|_{V}\leq x$, then it is eident that $\Phi_{\eta m}^{0}\leq c(\chi)$, where $c(\chi)$ is a positive constant wich depends on χ. Therefore from the last inequality we have

$$\Phi_{\eta m}^{l}\leq\frac{C}{\delta}+\left(c(\chi)-\frac{C}{\delta}+\frac{k\alpha(k)}{1+\delta k}\sum_{n=1}^{l}\left(C-\delta\Phi_{\eta m}^{n-1}\right)\right)e^{-\delta lk}. \tag{23}$$

We choose l such that

$$\left(c(\chi)-\frac{C}{\delta}+\frac{k\alpha(k)}{1+\delta k}\sum_{n=1}^{l}\left(C-\delta\Phi_{\eta m}^{n-1}\right)\right)e^{-\delta lk}\leq\frac{C}{\delta} \tag{24}$$

or

$$\left(c(\chi)-\frac{C}{\delta}\right)e^{-\delta lk}\leq\frac{C}{\delta}-o(k).$$

Since $C=\frac{\eta-\delta}{2\nu}\overline{m}$, then we choose ν such that

$$c(\chi)-\frac{C}{\delta}\leq 0,$$

that is $\nu\leq\frac{\overline{m}}{2c(\chi)}\left(\frac{\nu}{\delta}-1\right)$.

Then (24) holds for any $l\in\{1,...,m\}$. Therefore from (23), (24) we obtain that

$$\Phi_{\eta m}^{l}\leq\frac{2C}{\delta}\quad for\ \ any\ \ l\in\{1,...,m\},$$

that is

$$\Phi_{\eta m}^{l}=\frac{1}{2}\left\|\frac{u_{m}^{l}-u_{m}^{l-1}}{k}\right\|^{2}+\frac{1}{2}\left\|\nabla u_{m}^{l}\right\|^{2}-\left(h,u_{m}^{l}\right)+\frac{1}{p+2}\left(\left|u_{m}^{l}\right|^{p+2},1\right)+$$

$$+\eta\left[\left(u_m^l,\frac{u_m^l-u_m^{l-1}}{k}\right)+\frac{1}{2}\left\|u_m^l\right\|^2\right]\leq\frac{2C}{\delta}\qquad(25)$$

for any $l\in\{1,...,m\}$.

Let $\widetilde{u}_m(x,t)=u_m^n(x)$, if $(n-1)k<t\leq nk, n=1,2,...,m$; a.e. in Ω and define $\widetilde{w}_m$ similarly. Then from (25) we have

$$\frac{1}{2}\left\|\frac{\partial u_m}{\partial t}\right\|^2+\frac{1}{2}\|\nabla\widetilde{u}_m\|^2-(h,\widetilde{u}_m)+\frac{1}{p+2}\left(|\widetilde{u}_m|^{p+2},1\right)+$$

$$+\eta\left[\left(\widetilde{u}_m,\frac{\partial u_m}{\partial t}\right)+\frac{1}{2}\|\widetilde{u}_m\|^2\right]\leq\frac{2C}{\delta}.\qquad(26)$$

Since as $m\to\infty$ (look [5])

$u_m\to u$ weakly star in $H^1\left(0,T;L^2(\Omega)\right)\cap L^\infty(0,T;V)$,

$\widetilde{u}_m\to u$ weakly star in $L^\infty(0,T;V)$,

then by taking $m\to\infty$ in the inequality (26) we have

$$\frac{1}{2}\left\|\frac{\partial u}{\partial t}\right\|^2+\frac{1}{2}\|\nabla u\|^2-(h,u)+\frac{1}{p+2}\left(|u|^{p+2},1\right)+$$

$$+\eta\left[\left(u,\frac{\partial u}{\partial t}\right)+\frac{1}{2}\|u\|^2\right]\leq\frac{2C}{\delta}$$

or

$$\Phi_\eta(y)\leq\frac{2C}{\delta},$$

that is

$$y\in B_0.$$

Theorem is proved

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