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ON IDENTITY FOR EIGENVALUES OF ONE BOUNDARY VALUE PROBLEM WITH EIGENVALUE DEPENDENT BOUNDARY CONDITION

Abstract

The regularized trace of on operator generated by a differential expression with an unbounded operator coefficient and eigenvalue dependent boundary condition is studied.

Let H be separable Hilbert space with scalar product $(\cdot,\cdot)$ and norm $\|\cdot\|$ in it. Let also $L_2 = L_2(H,(0,1)) \oplus H$, where $L_2(H,(0,1))$ is a Hilbert space of vector functions $y(t)$ for which $\int\limits_0^1 \|y(t)\|^2 dt < \infty$. The scalar product in L_2 is given as

$$(Y,Z)_{L_2} = \int\limits_0^1 (y(t),z(t))dt + (y_1,z_1), \tag{1}$$

where $Y = \{y(t), y_1\}$, $Z = \{z(t), z_1\}$, $y(t), z(t) \in L_2(H,(0,1))$ and $y_1, z_1 \in H$.

Consider in space $L_2(H,(0,1))$ the problem

$$-y''(t) + Ay(t) + q(t)y(t) = \lambda y(t), \tag{2}$$

$$y(0) = 0, \tag{3}$$

$$-y(1) = \lambda y'(1), \tag{4}$$

where $A = A^* > E$ (E is an identity operator in H) and $A^{-1} \in \sigma_\infty$.

Denote the eigenvalues and eigen-vectors of the operator by $\gamma_1 \le \gamma_2 \le ...$ and $\varphi_1, \varphi_2, ...$, respectively.

Suppose that the operator-valued function $q(t)$ is weakly measurable, $\|q(t)\|$ is bounded on [0,1] and satisfies also the conditions:

1. Second weak derivative of $q(t)$ exists on [0,1], $\left[q^{(l)}(t)\right]^* = q^{(l)}(t)$ and for each $t \in [0,1]$ $(l = 0,1,2)$ it holds $\sum\limits_{j=1}^{\infty} \left|(q^{(l)}(t)\varphi_j, \varphi_j)\right| < const.$
2. $q'(0) = q'(1) = 0$.
3. $\int\limits_0^1 (q(t)f,f)dt = 0$ for each $f \in H$.

For $q(t) \equiv 0$ one can associate with problem (2)-(4) in space L_2 the self-adjoint operator L_0 defined as

$$D(L_0) = \left\{Y = \{y(t), y_1\} \in L_2 / -y''(t) + Ay(t) \in L_2(H,(0,1)),\right.$$

$$y(0)=0, \quad u\ y_1=y'(1)\right\},$$

$$L_0(Y)=\left\{-y''(t)+Ay(t),-y(1)\right\}.$$

The operator corresponding to the case $q(t)\not\equiv 0$ denote by $L=L_0+Q$, where Q:$Q\left\{y(t),y'(1)\right\}=\left\{q(t)y(t),0\right\}$ is a bounded self-adjoint operator in L_2.

One can show that A^{-1} is compact in H if L_0^{-1} is compact in L_2.

Denote the eigenvalues of the operators L_0 and L by $\mu_1<\mu_2<...$ and $\lambda_1<\lambda_2<...$, respectively.

Note that in [1] the problem

$$-u''(x)+Au(x)=\lambda u(x), \quad x\in[0,1],$$

$$\lambda y'(1)+y(1)=0,$$

$$y(0)=0$$

is studied and the following asymptotic formula for the eigenvalues of L_0 is obtained:

$$\lambda_{k,n}\sim\gamma_k+\left(\pi n-\frac{\pi}{2}\right)^2, \quad n=\overline{1,\infty}.$$

By using this asymptoties similarly to the method of [2, lemma 2] one can prove the following lemma.

Lemma 1. *Let A^{-1} be compact in H. Suppose that eigenvalues of the operator A satisfy the relation $\gamma_k\sim a\cdot k^{\alpha}$, $a>0$, $\alpha>0$. Then*

$$\lambda_n\sim\mu_n\sim dn^{\delta},$$

where

$$\delta=\begin{cases}\dfrac{2\alpha}{\alpha+2}, & \alpha>2,\\ \dfrac{\alpha}{2}, & \alpha<2\\ 1, & \alpha=2,\end{cases}$$

$d>0$ is *some constant.*

Our aim in this paper is to prove the formula for the regularized trace of the operator L.

Differential operator equations with eigenvalue dependent boundary conditions are studied, for example, in [1-5].

In [5] we get the trace formula for the operator in $L_2(H,(0,\pi))\oplus H$ generated by the differential expression

$$ly\equiv-y''(t)+Ay(t)+q(t)$$

and boundary conditions

$$y(0)=0,$$

$$y'(\pi)-\lambda y(\pi)=0.$$

By using lemma 1, similar to [7, lemma 2, theorem 2] one can show that

$$\lim_{m\to\infty}\sum_{n=1}^{n_m}(\lambda_n-\mu_n-(Q\psi_n,\psi_n))=0, \tag{5}$$

where $\{n_m\}$ is some subsequence of natural numbers, $\{\psi_n\}$ are orthonormal eigenfunctions of the operator L_0.

The orthonormal eigen-vectors of the operator L_0 are of the form

$$\psi_n=\sqrt{\frac{4x_{m_n,k_n}}{2x_{m_n,k_n}-\sin 2x_{m_n,k_n}+4x_{m_n,k_n}^3\cos^2 x_{m_n,k_n}}}\times$$

$$\times\left\{\sin x_{m_n,k_n}t\varphi_k,\, x_{m_n,k_n}\cos x_{m_n,k_n}\varphi_k\right\}, \tag{6}$$

where $\{\varphi_k\}$ are eigen-vectors of the operator A.

Call $\lim\limits_{m\to\infty}\sum\limits_{n=1}^{n_m}(\lambda_n-\mu_n)$ the regularized trace of the operator L, since as it will be shown later, its value is independent of the choice of the subsequence $\{n_m\}$.

Prove the following lemma.

Lemma 2. *Provided that for operator-valued function $q(t)$ hold the conditions 1)-3), then*

$$\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\left|\int_0^1\frac{2x_{m,k}\cos 2x_{m,k}t\ f_k(t)dt}{2x_{m,k}-\sin 2x_{m,k}+4x_{m,k}^3\cos^2 x_{m,k}}\right|<\infty, \tag{7}$$

where $f_k(t)=(q(t)\varphi_k,\varphi_k)$.

Proof. Observe that, for large m

$$\frac{2x_{m,k}}{2x_{m,k}-\sin 2x_{m,k}+4x_{m,k}^3\cos^2 x_{m,k}}=\frac{1}{1-\dfrac{\sin 2x_{m,k}}{2x_{m,k}}+2x_{m,k}^2\cos^2 x_{m,k}}<$$

$$<\frac{1}{1-\dfrac{\sin 2x_{m,k}}{2x_{m,k}}}=1+O\left(\frac{1}{x_{m,k}}\right). \tag{8}$$

By integrating twice by parts, and using the condition 2), we get

$$\int_0^1\cos 2x_{m,k}t\ f_k(t)dt=\frac{1}{2x_{m,k}}\sin 2x_{m,k}f_k(1)-\frac{1}{(2x_{m,k})^2}\int_0^1\cos 2x_{m,k}t\ f_k''(t)dt$$

In virtue of asymptotics $x_{m,k}=\pi m+\dfrac{\pi}{2}+O\left(\dfrac{1}{m^3}\right)$, relationship (8), also the condition 1) we get that the considered series is absolutely convergent, that completes the proof of the statement of the lemma.

Now by using lemma 2, prove the following theorem.

Theorem 1. *Let the conditions of lemma 1 hold. Provided that for the operator-valued function $q(t)$ hold the conditions 1)-3), then the formula*

$$\lim_{m\to\infty}\sum_{n=1}^{n_m}(\lambda_n-\mu_n)=\sum_{k=1}^{\infty}\frac{f_k(0)-f_k(1)}{4} \tag{9}$$

is true.

Proof. By taking into consideration lemma 2 and condition 3) in (5), we get

$$\lim_{m\to\infty}\sum_{n=1}^{n_m}(\lambda_n-\mu_n)=\lim_{m\to\infty}\sum_{n=1}^{n_m}(Q\psi_n,\psi_n)=$$

$$=\lim_{m\to\infty}\sum_{n=1}^{n_m}\int_0^1\frac{4x_{m_n,k_n}\sin^2 x_{m_n,k_n}t\ f_{k_n}(t)}{2x_{m_n,k_n}-\sin 2x_{m_n,k_n}+4x_{m_n,k_n}^3\cos^2 x_{m_n,k_n}}dt=$$

$$=-\sum_{m=1}^{\infty}\sum_{m=1}^{\infty}\int_0^1\frac{2x_{m,k}\cos 2x_{m,k}t\ f_k(t)dt}{2x_{m,k}-\sin 2x_{m,k}+4x_{m,k}^3\cos^2 x_{m,k}} \tag{10}$$

First find the sum

$$\sum_{m=1}^{\infty}\int_0^1\frac{2x_{m,k}\cos 2x_{m,k}t\ f_k(t)dt}{2x_{m,k}-\sin 2x_{m,k}+4x_{m,k}^3\cos^2 x_{m,k}} \tag{11}$$

For the fixed k investigate the asymptotic behavior of the function

$$T_N(t)=\sum_{m=1}^{N}\frac{2x_{m,k}\cos 2x_{m,k}t}{2x_{m,k}-\sin 2x_{m,k}+4x_{m,k}^3\cos^2 x_{m,k}}.$$

Express the m-th term of the sum $T_N(t)$ as a residue at the pole $x_{m,k}$ of some function of complex variable z :

$$g(z)=\frac{z\cos 2zt}{\left(\frac{tgz}{z}+z^2+\gamma_k\right)z^2\cos^2 z}$$

This function has simple poles at the points $x_{m,k}$, $\left(m+\frac{1}{2}\right)\pi$, and $z=0$.

We have

$$\underset{z=x_{m,k}}{res}\, g(z)=\frac{x_{m,k}\cos 2x_{m,k}t}{\left(\frac{tgz}{z}+z^2+\gamma_k\right)'_{z=x_{m,k}}x_{m,k}^2\cos^2 x_{m,k}}=$$

$$=\frac{2x_{m,k}\cos 2x_{m,k}t}{2x_{m,k}-\sin 2x_{m,k}+4x_{m,k}^3\cos^2 x_{m,k}}$$

Find the residue at $\left(m+\dfrac{1}{2}\right)\pi$:

$$\underset{z=\left(m+\frac{1}{2}\right)\pi}{res} g(z)=\underset{z=\left(\frac{1}{2}+m\right)\pi}{res}\frac{z\cos 2zt}{\left(\dfrac{\sin z}{z}+z^2\cos z+\gamma_k\cos z\right)z^2\cos z}=$$

$$=\frac{\left(\dfrac{\pi}{2}+\pi m\right)\cos\left(\pi+2\pi m\right)t}{\dfrac{\sin\left(\dfrac{\pi}{2}+\pi m\right)}{\dfrac{\pi}{2}+\pi m}\left(\dfrac{\pi}{2}+\pi m\right)^2\left(-\sin\left(\dfrac{\pi}{2}+\pi m\right)\right)}=-\cos(2m+1)\pi t.$$

Take as a contour of integration the rectangle with vertices at $\pm iB$, $A_N\pm iB$, which bypass the origin on the right hand side of imaginary axis. Further B will go to infinity and take $A_N=\pi N$. For such choice of A_N we have $x_{N,k}<A_N<x_{N+1,k}$.

Since $g(z)$ is an odd function of z, then integral vanishes along the contour on imaginary axis.

If $z=u+i\vartheta$, then for large ϑ and for $u\geq 0$ $\quad g(z)$ is of order $O\left(\dfrac{e^{2|\vartheta(t-1)|}}{|\vartheta|^3}\right)$ that is why for the given value of A_N the integrals along upper and lower sides of the contour also go to zero as $B\to\infty$.

Hence, we get the formula

$$T_N(t)+S_N(t)=\frac{1}{2\pi i}\lim_{B\to\infty}\int\limits_{A_N-iB}^{A_N+iB}g(z)dz+$$
$$+\frac{1}{2\pi i}\lim_{r\to 0}\int\limits_{\substack{|z|=r\\ -\frac{\pi}{2}<\varphi<\frac{\pi}{2}}}g(z)dz,\qquad(12)$$

where

$$S_N(t)=-\sum_{m=1}^{N}\cos(2m+1)\pi t.$$

As $N\to\infty$

$$\lim_{B\to\infty}\frac{1}{2\pi i}\int\limits_{A_N-iB}^{A_N+iB}g(z)dz\sim\frac{1}{2\pi i}\int\limits_{A_N-i\infty}^{A_N+i\infty}\frac{\cos 2zt}{\cos^2 z}dz=$$

$$=\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\frac{\cos(2\pi Nt+2ti\vartheta)}{(A_N+i\vartheta)^3(1+\cos(2A_N+2i\vartheta))}d\vartheta=$$

$$=\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\frac{\cos 2\pi Nt\cos 2i\vartheta t-\sin 2\pi Nt\sin 2i\vartheta t}{(A_N+i\vartheta)^3(1+\cos 2i\vartheta)}d\vartheta=$$

$$
=\frac{1}{2\pi}\cos 2\pi Nt\int\limits_{-\infty}^{+\infty}\frac{ch\ 2\vartheta t}{(A_N+i\vartheta)^3(1+\cos 2i\vartheta)}d\vartheta+
$$

$$
+\frac{1}{2\pi i}\sin 2\pi Nt\int\limits_{-\infty}^{+\infty}\frac{sh\ 2\vartheta t}{(A_N+i\vartheta)^3(1+\cos 2i\vartheta)}d\vartheta. \tag{13}
$$

Denote the integrals on the right hand side of (13) by I_1 and I_2, respectively. Then,

$$
|I_1|<\int\limits_{-\infty}^{+\infty}\frac{d\vartheta}{\sqrt{\left(A_N^2+\vartheta^2\right)^3}}=2\int\limits_{0}^{+\infty}\frac{d\vartheta}{\sqrt{\left(A_N^2+\vartheta^2\right)^3}}<
$$

$$
<\frac{2}{A_N}\int\limits_{0}^{+\infty}\frac{d\vartheta}{A_N^2+\vartheta^2}=\frac{2}{A_N^2}arctg\frac{\vartheta}{A_N}\mid_0^\infty=\frac{const}{A_N^2} \tag{14}
$$

The similar estimate is obtained also for I_2. So, by using (13) and (14) in (12), we get

$$
\lim_{N\to\infty}\int\limits_0^1 T_N(t)f_k(t)dt=-\lim_{N\to\infty}\int\limits_0^1 S_N(t)f_k(t)dt+
$$

$$
+\frac{1}{2\pi i}\lim_{r\to\infty}\int\limits_0^1 f_k(t)dt\int\limits_{\substack{|z|=r\\ -\frac{\pi}{2}<\varphi<\frac{\pi}{2}}} g(z)dz\ dt \tag{15}
$$

By condition 3) the second term in the right hand side of (15) can be written as

$$
\lim_{r\to 0}\int\limits_0^1 f_k(t)\int\limits_{\substack{|z|=r\\ -\frac{\pi}{2}<\varphi<\frac{\pi}{2}}}\frac{z(\cos 2zt-1)}{\left(\frac{tgz}{z}+z^2+\gamma_k\right)z^2\cos^2 z}dz\ dt=
$$

$$
=-\lim_{r\to 0}\int\limits_0^1 f_k(t)\int\limits_{\substack{|z|=r\\ -\frac{\pi}{2}<\varphi<\frac{\pi}{2}}}\frac{2\ z\sin^2 zt}{\left(\frac{tgz}{z}+z^2+\gamma_k\right)z^2\cos^2 z}dz\ dt.
$$

Since the numerator of the integrand for small z is of order $O(z^3)$, and the denominator is of order $O(z^2)$, then the last one goes to zero.

So, by substitution $\pi t=z$ we have

$$
\lim_{N\to\infty}\int\limits_0^1 T_N(t)f_k(t)dt=-\lim_{N\to\infty}\int\limits_0^1 S_N(t)f_k(t)dt=
$$

$$
=\sum_{m=0}^{\infty}\int\limits_0^1\cos(2m-1)\pi\ t\ f_k(t)dt=\frac{1}{\pi}\sum_{m=0}^{\infty}\int\limits_0^1\cos(2m+1)\pi\ t\ f_k(t)d\ \pi t=
$$

$$=\frac{1}{4}\sum_{m=0}^{\infty}\int_0^{\pi}\cos(2m+1)z\ f_k\left(\frac{z}{\pi}\right)dz=\frac{1}{4}\frac{2}{\pi}\left[\sum_{m=0}^{\infty}\cos m\cdot 0\int_0^{\pi}\cos m\ z\ f_k\left(\frac{z}{\pi}\right)dz-\right.$$

$$\left.-\cos m\cdot\pi\int_0^{\pi}\cos m\ z\ f_k\left(\frac{z}{\pi}\right)dz\right]=\frac{f_k(0)-f_k(1)}{4} \tag{16}$$

From (10) and (16) it follows that

$$\lim_{m\to\infty}\sum_{n=1}^{n_m}(\lambda_n-\mu_n)=\sum_{k=1}^{\infty}\frac{f_k(0)-f_k(1)}{4}. \tag{17}$$

If, in particular, $q(t)\in\sigma_1$,(σ_1 is a space of compact operators singular numbers of which form convergent series (see [7], p. 121) then (17) takes the form

$$\lim_{m\to\infty}\sum_{n=1}^{n_m}(\lambda_n-\mu_n)=\frac{trq(0)-trq(1)}{4}.$$

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